Conference @ YITP
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トポロジカルストリングとAGT予想

Topological Strings & AGT Relations

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based on [Maruyoshi-M.T. arXiv:1006.4505]
[M.T. arXiv:1007.2524]

1. AGT conjecture and Surface Operators

Today I will study "AGT relation" between gauge theory and 2-dim CFT. This mysterious relation has attracted attention recently.

2D Liouville CFT ⟷ **$\mathcal{N}=2$ superconformal quiver gauge theories**

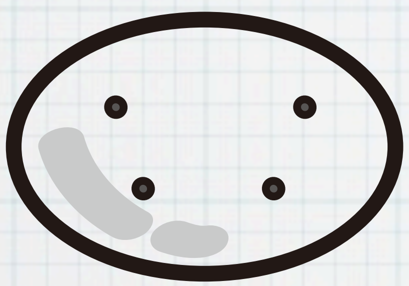
[Alday-Gaiotto-Tachikawa, '09]

$$\mathcal{B}_{\text{Liouville}}(q) = Z_{\text{Nekrasov}}(q)$$

Free-field CFT vs U(1)-theory

Let us consider the linear dilaton CFT with the background charge Q

$$\langle \phi(z_1)\phi(z_2)\rangle = -\ln(z_i - z_j)$$



$$\langle \prod_{i=1}^{4} e^{\alpha_i \phi(z_i)} \rangle = \prod_{i<j} (z_i - z_j)^{-\alpha_i \alpha_j}$$

We choose $z_{1,\cdots,4} = 0,\ q,\ 1,\ \infty$

$$\mathcal{B}_{\text{free CFT}} = (1-q)^{-\alpha_2 \alpha_3}$$

$$m_1 m_2$$

Some algebra leads to the following expansion

$$(1-q)^{-m_1 m_2} = \sum_{Y} q^{|Y|} \prod_{(i,j)\in Y} \frac{\prod_{f=1,2}\phi(m_f,(i,j))}{E(0,Y,(i,j))^2}$$

$$\phi(m,(i,j)) = m + i - j$$

$$E(a,Y,(i,j)) = a + ({Y_j}^T - i + 1) + (Y_i - j)$$

(where we use

$$\sum_{Y} \chi_Y(x)\chi_Y(y) = \exp \sum_n n x_n y_n$$
)

This is the Nekrasov partition function for the “U(1) gauge theory with 2-flavors”!

$$\mathcal{B}_{\text{free CFT}} = Z_{U(1),N_f=2}$$

surface operator from Liouville theory [Alday-Gaoitto-Gukov-Tachikawa-Verlinde, '09]

In the chapter we give a rough sketch of an important extension of the AGT relation. The point is that the AGT relation, or Liouville/gauge-theory coppespondence, also holds for extended objects, such as surface operators.

- **probe for the S-duality (geometric Langrange duality)**
  [Gukov-Witten, '06]

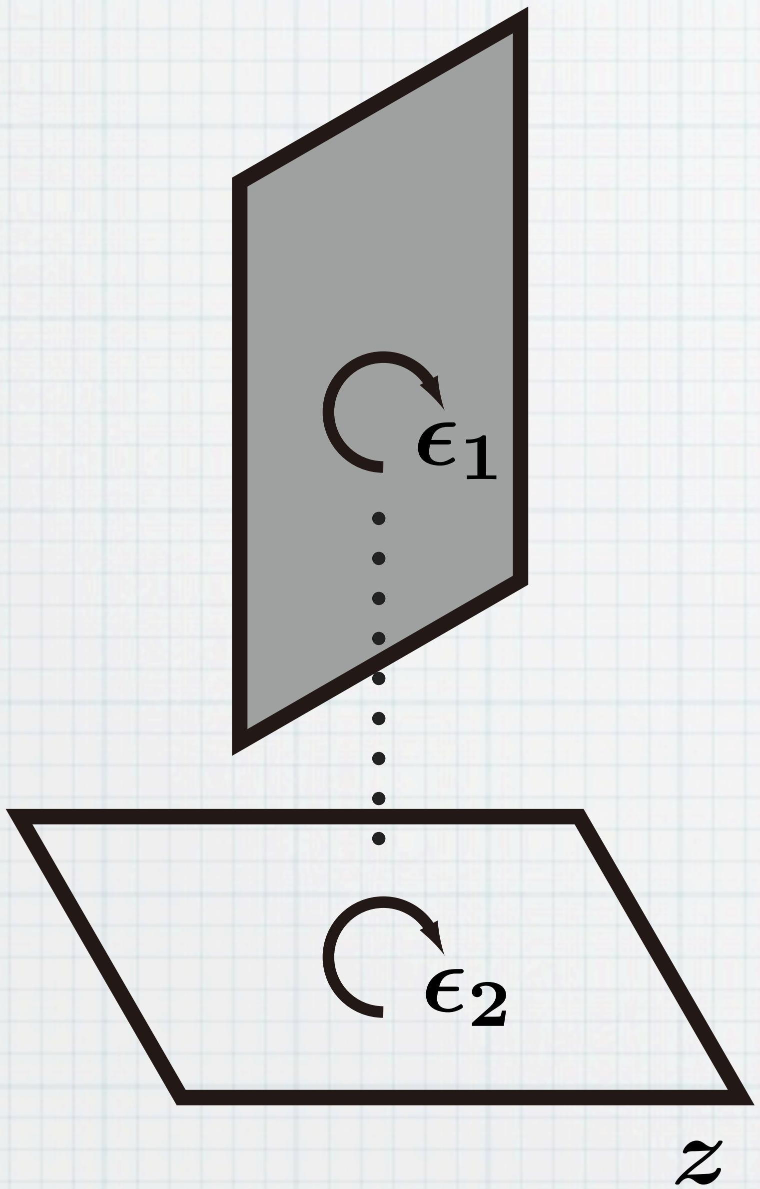
- **ramified Donaldson-Witten inv.**
  [Kronheimer-Mrowka, '93,'95] [Braverman, '04]
  [Alday-Tachikawa, '10]

- **generator of 'tHooft-Wilson operator**

- **order parameter for phases of gauge theory**

surface operator

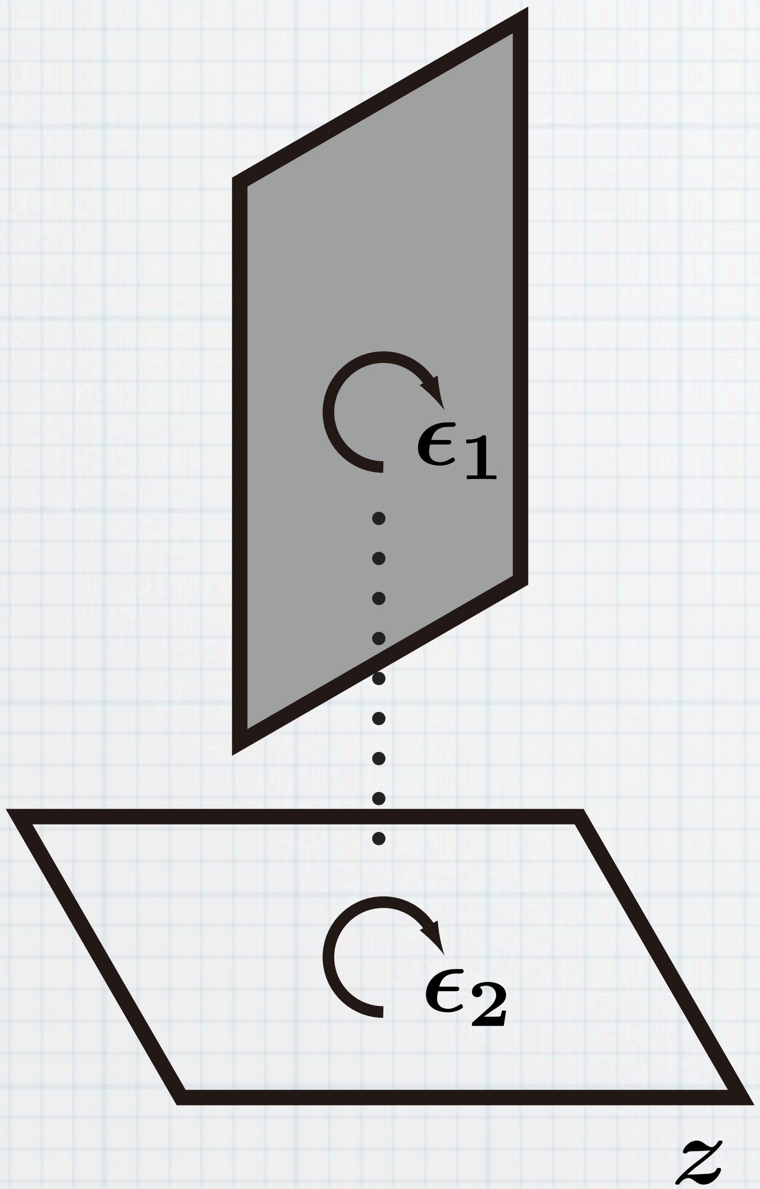
$\mathcal{N}=(2,2)$ sigma model coupled the 4D bulk gauge field



A surface operator is characterized by the subgroup of the gauge group which maintains the surface operator

$$\mathbb{L} \in G$$

surface operator



A surface operator creates the singularity near the locus of it's insertion:

$$A \sim \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \frac{dz}{z}$$

$$\longrightarrow \begin{cases} m = \int_{\mathbb{C}_{z=0}} F \\ k = \int_{\mathbb{C}^2} F \wedge F \end{cases}$$

Instantons in the presence of such a are operator labelled by these two topological numbers.

$$\Psi(z) = \sum_{k,m} q^k z^m \int_{\mathcal{M}_{N,k,m}} dV_{\epsilon_{1,2}}$$

AGGTV found that the partition function in the presence of the elementary surface operator is dual to the conformal block with the degenerate field insertion:

[Alday-Gaoitto-Gukov-Tachikawa-Verlinde, '09]

surface operator $\longleftrightarrow$ insertion of degenerate field $\Phi_{2,1}$

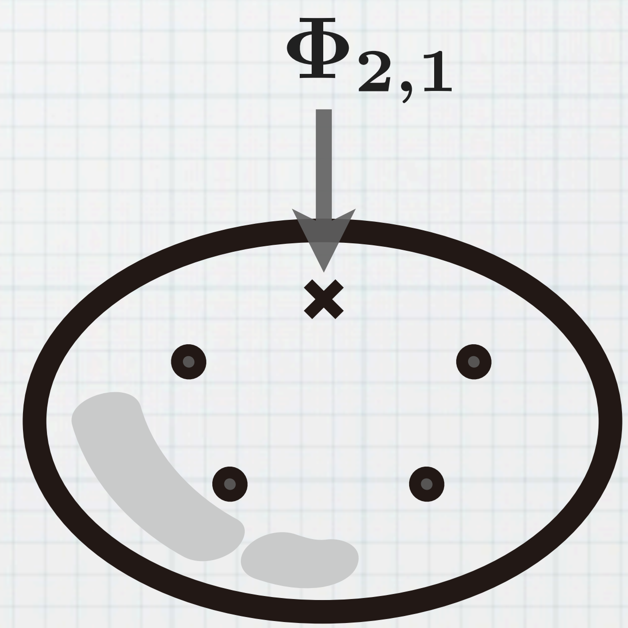
$$\Psi(z) = \exp \sqrt{\frac{\epsilon_1}{\epsilon_2}} \mathcal{G}(z) + \cdots \qquad \langle \Delta_1 | V_2(1) V_3(q) \Phi_{2,1}(z) | \Delta_4 \rangle$$

WKB $\downarrow$

$$\mathcal{G}(z) = \int^z \sqrt{\phi_2(z')} dz'$$

$\downarrow$

$$x^2 = \phi_2(z)$$ SW curve (Gaiotto form)

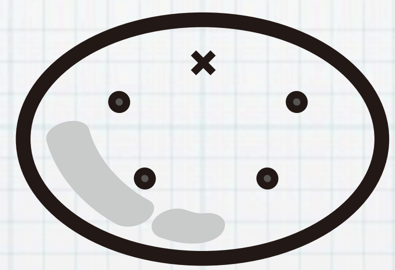


2. Geometric Engineering of Surface Operators

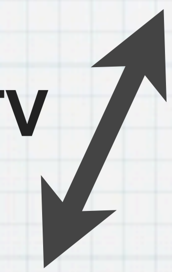
[M.T, arXiv:1007.2524]

In the chapter we report our work in progress. Let us consider the the following proposals for stringy derivations of surface operators

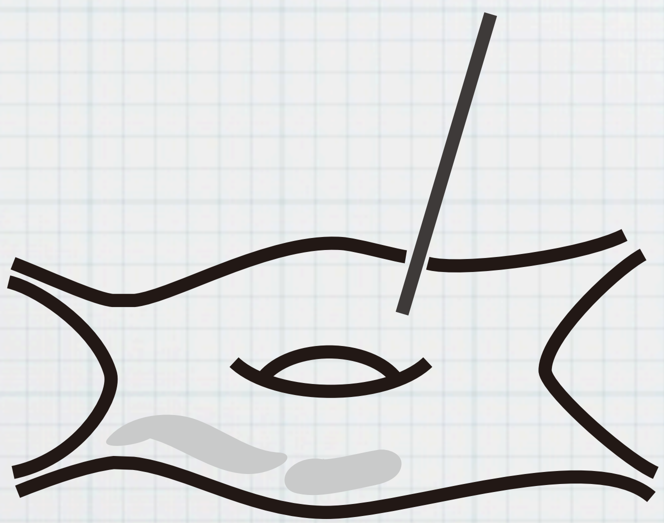
insertion of degenerate field $\Phi_{2,1}$



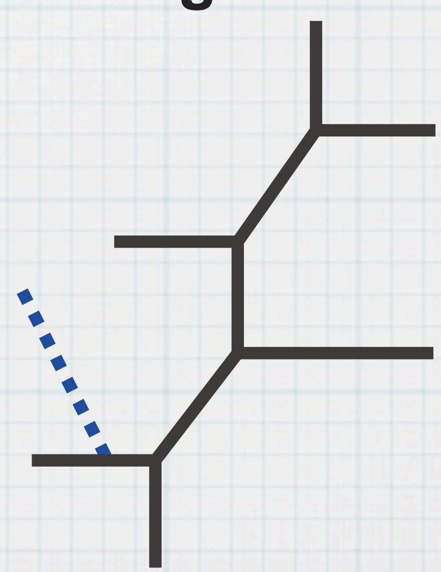
AGGTV



M2-brane construction



engineering via toric brane



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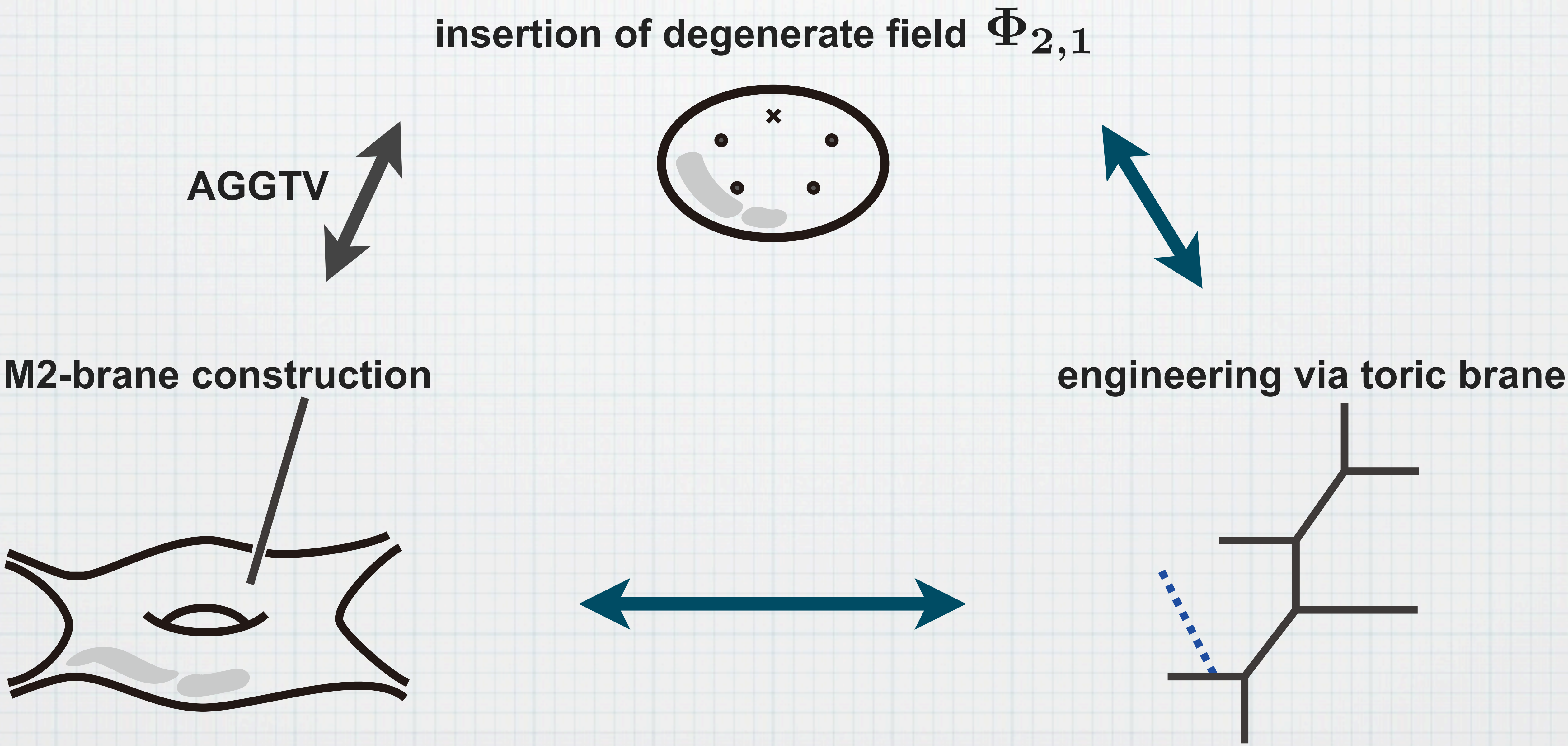
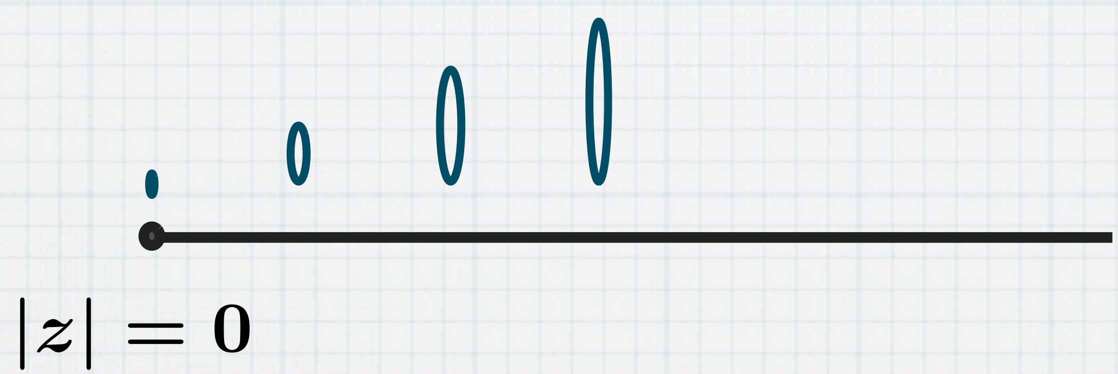


toric Calabi-Yau three-folds

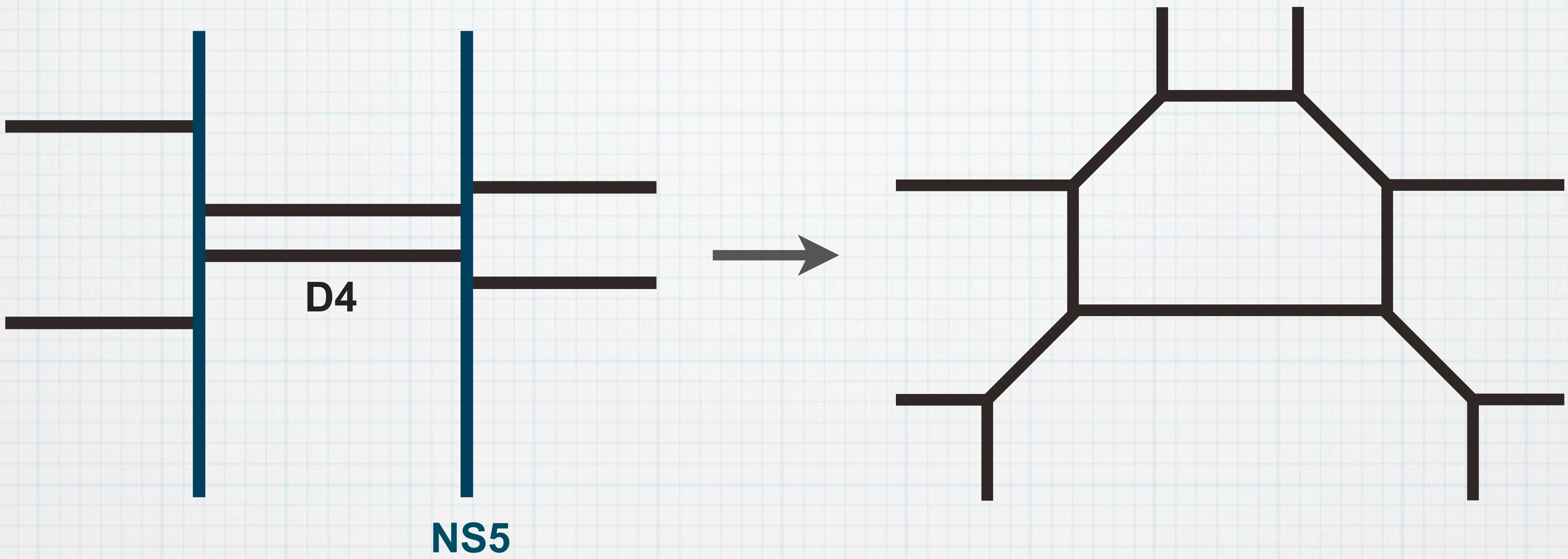
* $\mathbb{C}^1$

$$z = |z| e^{i\theta} \qquad\qquad S^1(\theta) \longrightarrow \mathbb{R}_{\geq 0}(|z|)$$



Any toric manifold is labelled by such a skeleton diagram which encodes the degeneration locus of the torus fiber.

* SU(N) geometry



The low-energy effective action for Type IIA superstring on the Calabi-Yau is equal to the Seiberg-Witten solution for SU(2) gauge theory with 4-flavors. Thus the Calabi-Yau engineers the gauge theory.

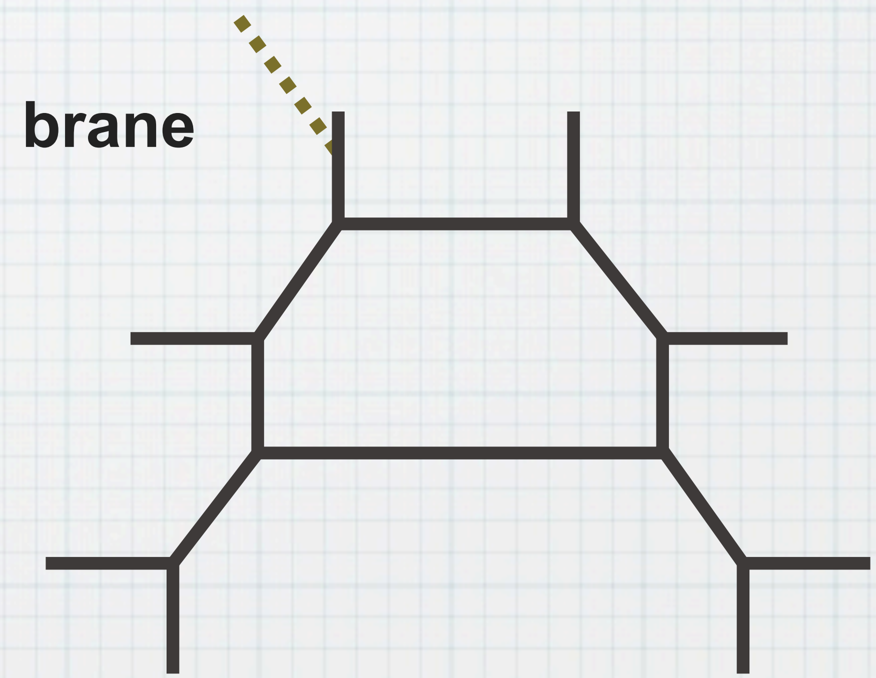
geometric engineering : The closed string topological string partition function of a certain toric CY is equal to Nekrasov partition function.

↓

[Kozcaz-Pasquetti-Wyllard, '10]

insertion of degenerate field $\Phi_{2,1}$ ⟷ open topological string amplitude on certain toric CY

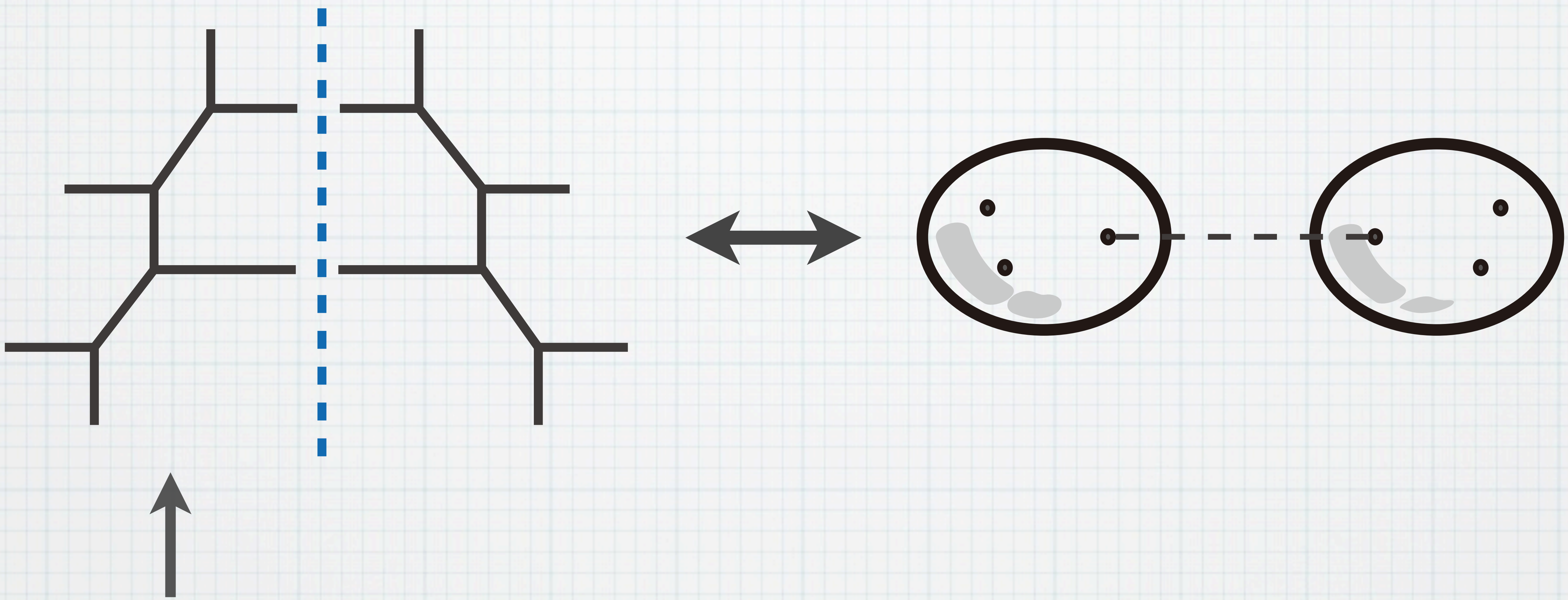
$$\Psi(z) = \exp \sqrt{\frac{\epsilon_1}{\epsilon_2}} \mathcal{G}(z) + \cdots$$ ⟷



We will derive the correspondence by utilizing the string dualities!

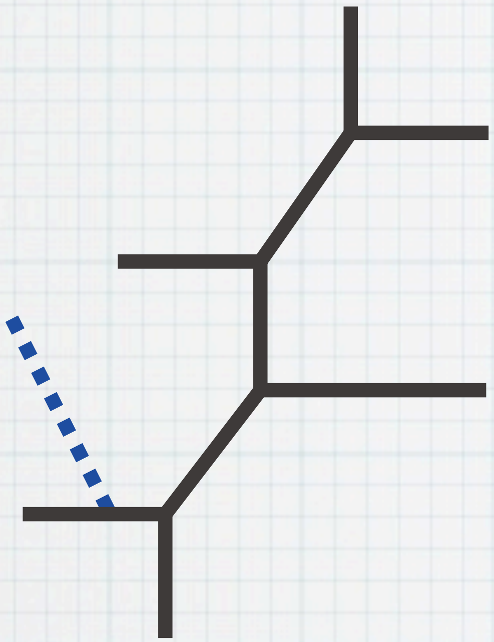
simple example

The degeneration of the Riemann surface in CFT side corresponds to the following optical-theorem-like computation methods.



4 hypermultiplets theory : $\mathcal{T}_{3,0}(A_1)$

Recall the Gaiotto's theory, which is that the 3-punctured sphere describes 4-hypermultiplets of gauge theory. So, the partition function for the half-toric-geometry will give the patririon function of it. Let us study this half-geometry for example.



topological vertex

$$\Phi(z) = \sum_n z^n \prod_{k=1}^{n} \frac{(k\hbar + m_1)(k\hbar + m_1 + m_2 + m_3)}{k\hbar(k\hbar + m_1 + m_3)}$$

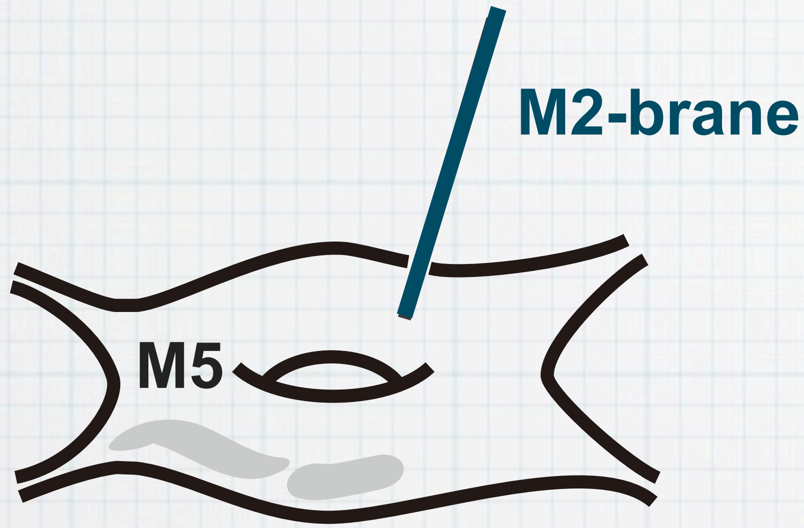
$$G(z) = \int^z \frac{\sqrt{\alpha_1^2 z(z-1) + \alpha_2^2)1 - z) + \alpha_3^2 z}}{z(1-z)}$$

* This result reproduces the correct Seiberg-Witten curve $x^2 = \phi_2(z)$

c.f. [Schiappa-Wyllard, '09]

“Proof” of the AGGTV-conjecture

*** M2-brane construction**



	0	1	2	3	4	5	6	7	8	9	10
M5	○	○	○	○			Σ		×	×	×
M2	○	○	×	×	×	×	×	×	○	×	×

$$\Sigma \; : \; H(x,y) = 0.$$

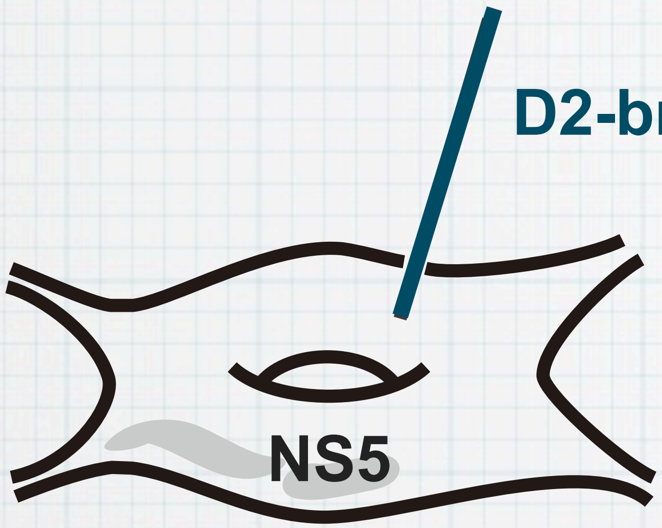
↓ dim. reduction along x^{10}

*** IIA superstring with N NS5-branes on Σ and a transverse D2-brane**

Next, let us use the following duality:

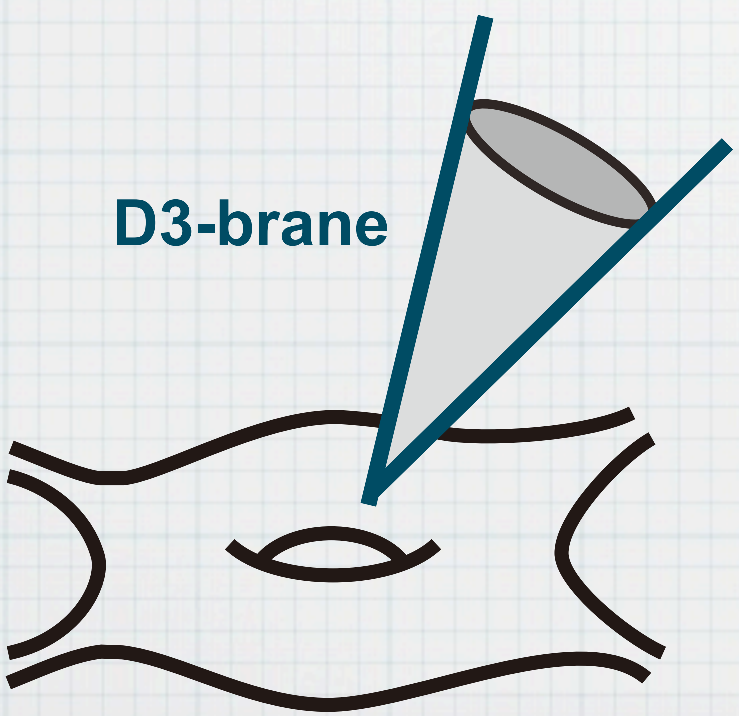
> N NS5-branes (IIA) are T-dual to A_{N-1} -singularity

* IIA superstring with N NS5-branes on Σ and a transverse D2-brane



T-duality along a circle transverse to Σ

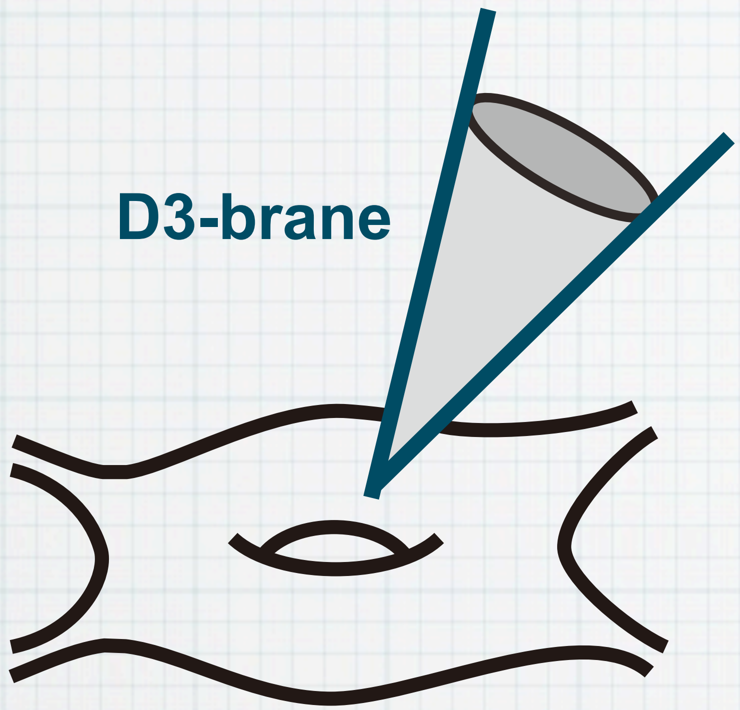
* IIB superstring on the Calabi-Yau $uv + H(x, y) = 0$



and D3 brane on

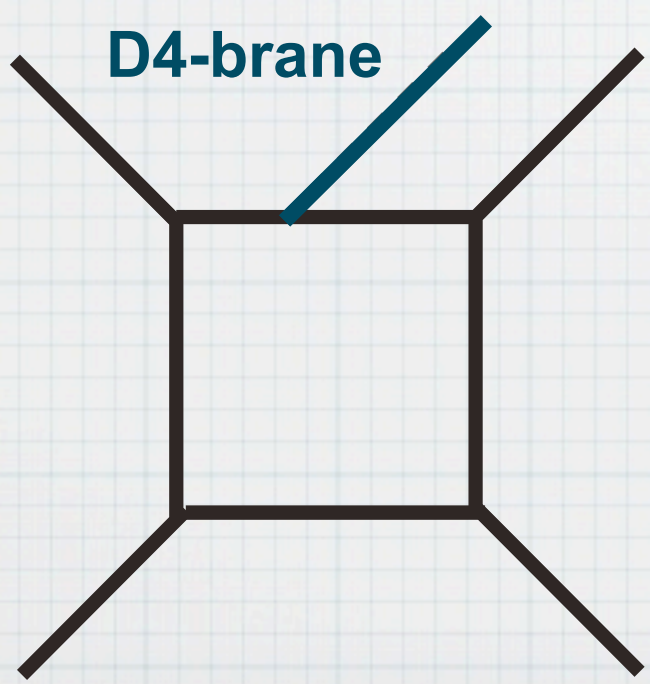
$$u = 0, \quad v = \text{arbitrary}, \quad H(x_0, y_0) = 0.$$

* IIB superstring on the Calabi-Yau with a D3-brane



mirror symmetry [Aganagic-Vafa, ’01]

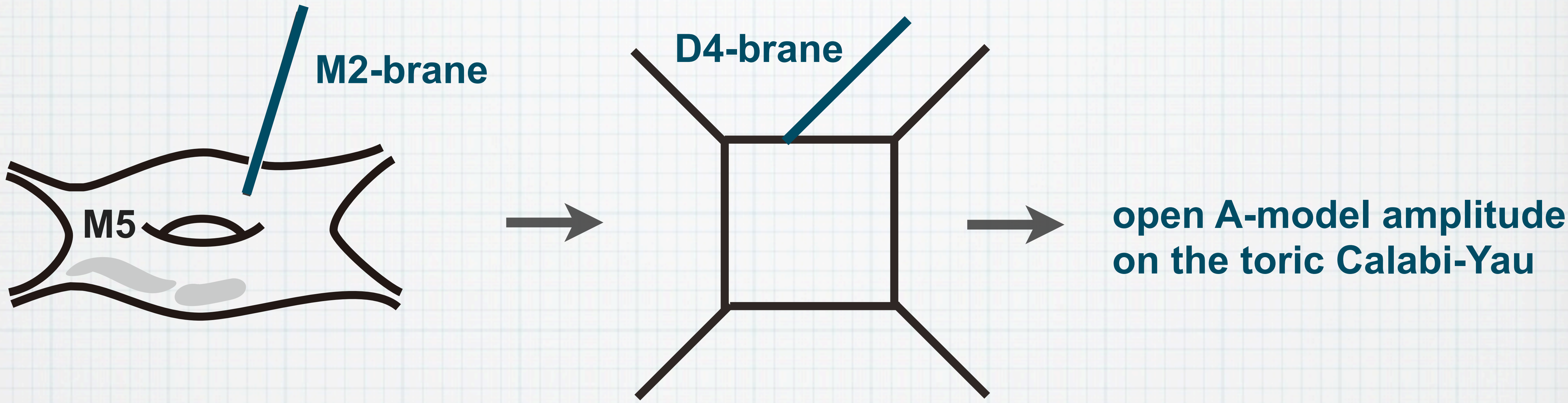
* IIA superstring on the toric Calabi-Yau and the toric D4 brane



This is precisely the situation which open topological string amplitude engineers!!

[Ooguri-Vafa, ’99]

In this way we have proven the proposal of [Kozcaz-Pasquetti-Wyllard, ‘10]



Moreover, by employing the derivation of [Dijkgraaf-Vafa, ’09], we can show that such a brane is described by the following degenerate field!

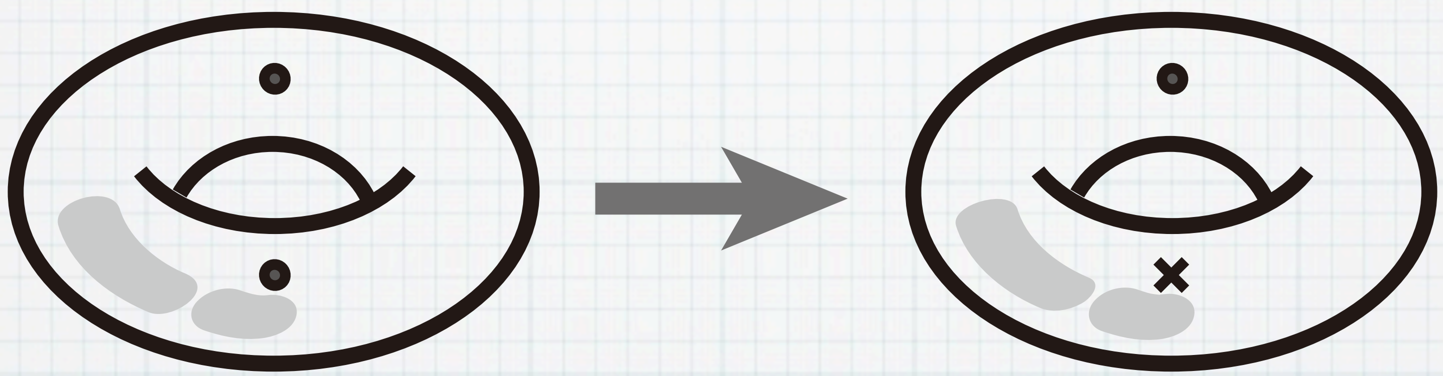
$$V(q) = e^{-b\phi(q)}$$

Thus string dualities also verify the AGGTV conjecture!

3. AGGTV relation via Bubbling Calabi-Yau

[Maruyoshi-M.T. arXiv:1006.4505][M.T, arXiv:1007.2524]

[Alday-Gaoitto-Gukov-Tachikawa-Verlinde, ‘09] implies



$\mathcal{T}_{2,1}$ $\mathcal{T}_{1,1}$

$\hat{A}_1$ quiver gauge theory $\mathcal{N}=2^*$ gauge theory

$a,\ a',\ m,\ m'$ $a,\ m$

$$a' = a + \frac{\epsilon_2}{2}$$
$$m' = -\frac{\epsilon_2}{2}$$

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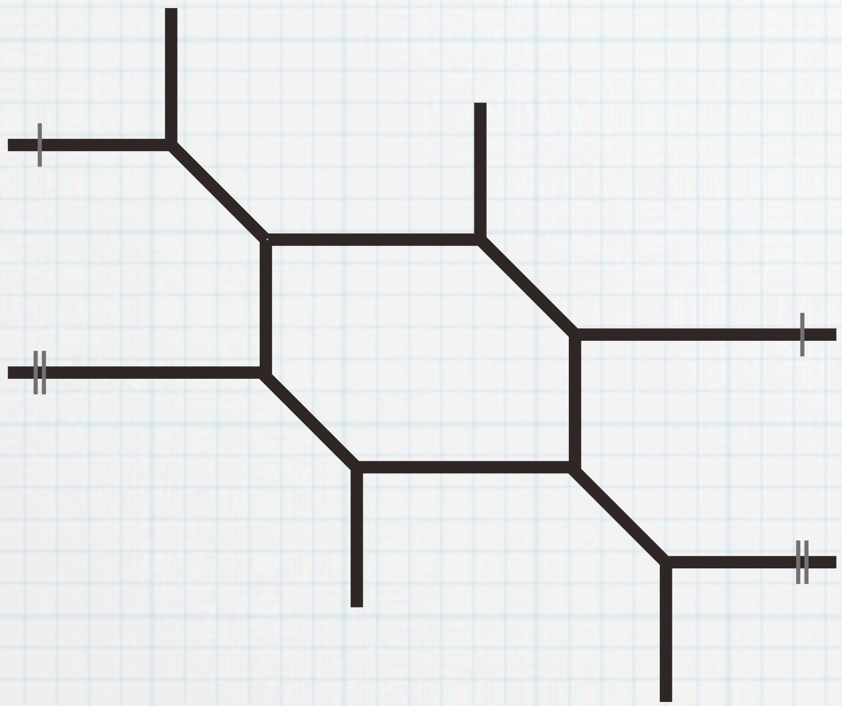
In the stringy realization, what will happen in the degeneration of parameter?

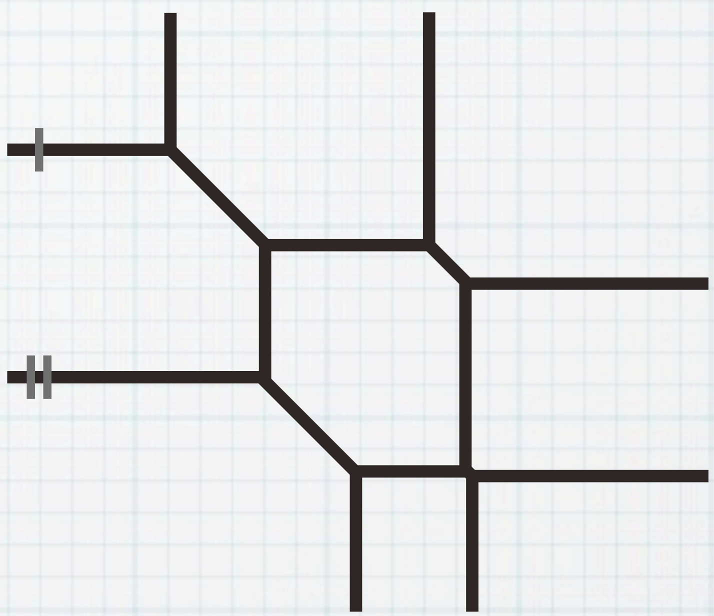
$\hat{A}_1$ quiver gauge theory

$\mathcal{N} = 2^*$ gauge theory
with surface operator



$$a' = a + \frac{\epsilon_2}{2}$$
$$m' = -\frac{\epsilon_2}{2}$$

→



Relation to the toric brane construction??

→ Bubbing of Calabi-Yau !! [Okuda-Gomiz, '07]

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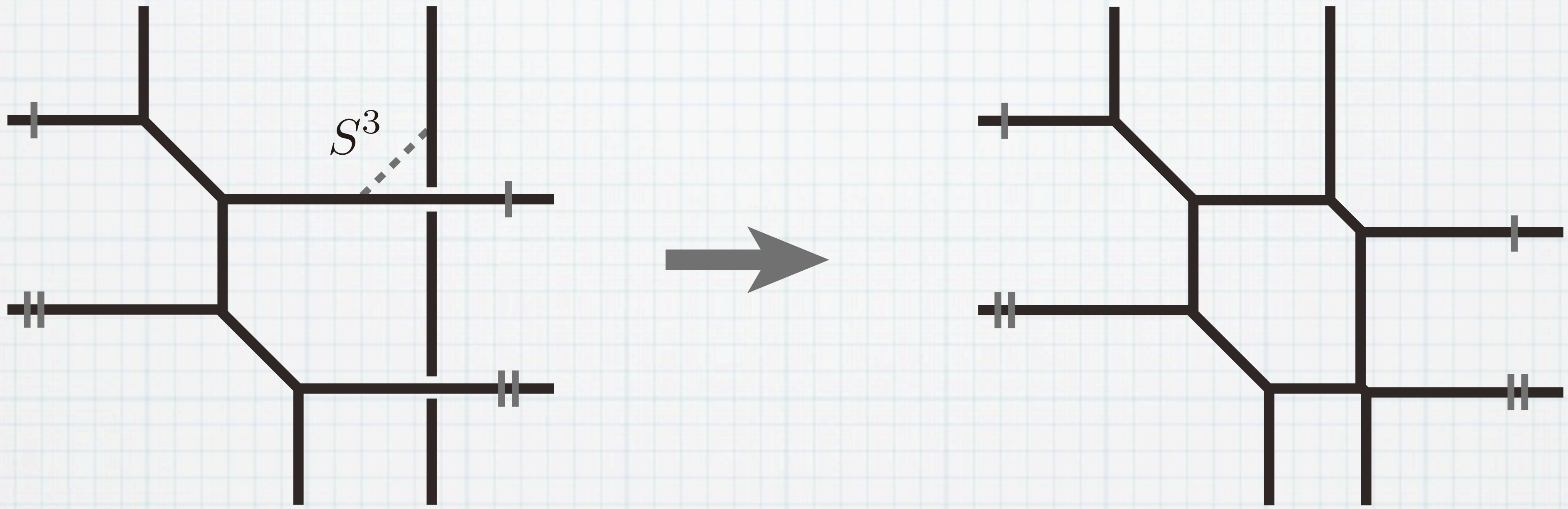
Toric barnes melts into the background geometry via geometric transition (open/closed duality)



In this way the AGGTV conjecture includes a gauge theory version of open/closed duality

decoupling limit

By taking the decoupling limit of the hypermlutiplet $m \to \infty$, we obtain the ramified instanton partition function for the SU(2) pure SYM

$$\Psi_{\text{pure SYM}}(\Lambda, \lambda) = \sum_{\vec{Y}, \vec{W}} \lambda^{|\vec{Y}|}(\Lambda^4 \lambda^{-1})^{|\vec{W}|} z_{\text{vect}}(a, \vec{Y}) z_{\text{vect}}(b = a - \epsilon_2/2, \vec{W})$$
$$\times z_{\text{bifund}}(b = a + \epsilon_2/2, \vec{W}; a, \vec{Y}; m_2 = -\epsilon_2/2)$$

Meanwhile, Maruyoshi and M.T. proposed the following CFT expression for the partition function by using irregular conformal block

[Maruyoshi-M.T., '10]

$$\Psi_{\text{pure SYM}} = \sum_{n=0}^{\infty} \sum_{Y} \Lambda^{4|Y|} \lambda^{n-|Y|} \beta^{Y}_{1^n} Q^{-1}_{\Delta}([1^{|Y|}]; Y)$$

where $\Phi_{2,1}(z)|\Delta, Y\rangle = \sum_{Y'} z^{|Y'|-|Y|+\delta} \beta^{Y}_{Y'} |\Delta', Y'\rangle$

Do they match? It is a ramified version of Gaiotto-Marshakov-Mironov-Morozov conjecture.

SUMMARY

- We develop AGT relation in order to describes surface operators

- We can prove the AGT relation in the presence of the surface operators by employing string dualities.

- Topological vertex is useful to study AGT relation.

- Understanding of the AGGTV relation involves geometric transition.

Fin