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Quarter BPS classified by Brauer algebra

Yusuke Kimura (Univ. of Oviedo)

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The problem of AdS/CFT

Map between string states and gauge invariant operators

$$\left\langle O_\alpha(x)^\dagger O_\beta(y)\right\rangle = \frac{c(1/N)\delta_{\alpha\beta}}{(x-y)^{2\Delta_\alpha(g,1/N)}}$$

4D $\mathcal{N}$=4 SYM (CFT)

Scaling dimension of local operator = Energy (in global time) of string state

$$\Delta(\lambda, N) = E(R/l_s, g_s)$$

$$4\pi\lambda / N = g_s$$

$$\sqrt{\lambda} = R^2/\alpha'$$

The operator with the definite scaling dimension is a linear combination of the naive operators [operator mixing].

$$\hat{D}\tilde{O}_\alpha = \Delta_{\beta\alpha}\tilde{O}_\beta$$

$$\left\langle \tilde{O}_\alpha(x)^\dagger \tilde{O}_\beta(y)\right\rangle = \frac{1}{(x-y)^{2\Delta_0}}\left(S_{\alpha\beta} + T_{\alpha\beta}\log|\Lambda x|\right)$$

$$\Delta = TS^{-1}$$

Definite scaling dimension = Eigenstate of the dilataton operator.

Simplification at large *N* (planar theory)

Only single-traces (i.e. only the flavour structure should be taken care of.)

$$tr(XYYYYXX) \leftrightarrow \left| \downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow \right\rangle$$

Dilatation operator = Hamiltonian of integrable system

$$D^{(1-loop)}_{planar} = \frac{\lambda}{8\pi^2} \sum_i \left(1 - P_{i,i+1}\right)$$

Diagonalisation of the hamiltonian of integrable system solves the mixing problem.

$$\Delta(\lambda, N) = E(R/l_s, g_s)$$

<u>What to do if N is not big enough?</u>

Operator mixing in the non-planar theory

holomorphic gauge inv. ops. built from two complex matrices, *X*, *Y*.

(This sector is closed in all order perturbation theory.)

$$D^{(1-loop)}_{non-planar} = -2tr\left([X,Y][\partial_X,\partial_Y]\right)$$

$$X = \Phi_1 + i\Phi_2$$
$$Y = \Phi_3 + i\Phi_4$$
$$Z = \Phi_5 + i\Phi_6$$

--Example -----

$$O_1 = tr(XXYY)$$
$$O_2 = tr(XYXY)$$
$$O_3 = tr(XX)tr(YY)$$
$$O_4 = tr(XY)tr(XY)$$

multi-trace

$$O_1' = O_1 - O_2 \qquad O_3' = O_2 - NO_4$$
$$O_2' = O_1 + \frac{N}{2}O_4 \qquad O_4' = O_3 + 2O_4$$

$$DO_1' = 12NO_1'$$
$$DO_a' = 0 \qquad (a = 2,3,4)$$

We will need a good method to organise gauge invariant operators of single traces and multi-traces.

Recall some important facts of the 1/2 BPS primary.

$$O_R(X) = tr_n\left(p_R X^{\otimes n}\right)$$

Schur polynomial

$$p_R = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma$$

$$X^{i_1}_{j_1} X^{i_2}_{j_2} \cdots X^{i_n}_{j_n}$$

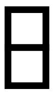
The upper indices are transformed as the product of the fundamental rep. of *GL(N)*.

$$\left\langle O_R(X)^\dagger O_S(X)\right\rangle \propto \delta_{RS}$$

[Corley, Jevicki, Ramgoolam 01]

$$\left\langle X^i_j[x] X^{\dagger k}_{\ l}[y]\right\rangle_0 = \delta^i_l \delta^k_j \frac{1}{(x-y)^2}$$

$$p_{[2]} = \frac{1}{2}(1+s) \qquad \frac{1}{2}\left(X^{i_1}_{j_1}X^{i_2}_{j_2} + X^{i_2}_{j_1}X^{i_1}_{j_2}\right) \Rightarrow \frac{1}{2}\left(trXtrX + tr(XX)\right) = O_{[2]}$$



$$p_{[1,1]} = \frac{1}{2}(1-s) \qquad \frac{1}{2}\left(X^{i_1}_{j_1}X^{i_2}_{j_2} - X^{i_2}_{j_1}X^{i_1}_{j_2}\right) \Rightarrow \frac{1}{2}\left(trXtrX - tr(XX)\right) = O_{[1.1]}$$

Multi-trace　　Single-trace

Representation basis

The trace structure can be conveniently encoded in the Young diagram.

- Finite N constraint (⇒cut-off for angular momentum)

$$c_1(R) \leq N$$

$$tr\left(X^3\right) = \frac{3}{2} trXtr\left(X^2\right) - \frac{1}{2}\left(trX\right)^3 \qquad (N=2)$$

⇔

$$tr_3\left(p_{[1,1,1]} X^{\otimes 3}\right) = 0$$

- Orthogonal (complete) at classical level

Representation basis seems to be a useful basis to organise gauge inv. operators *when N is not large enough*.

In this talk, I will study the mixing problem using a representation basis.

Construction of a representation basis for the *X*-*Y* sector

$$tr_{m.n}\left(P^{\gamma} X^{\otimes m} \otimes Y^{T\otimes n}\right)$$

Projector associated with an irreducible rep. γ of *GL(N)*

$$X^{i_1}_{j_1} X^{i_2}_{j_2} \cdots X^{i_m}_{j_m} Y^{T\, k_1}_{l_1} Y^{T\, k_2}_{l_2} \cdots Y^{T\, k_n}_{l_n} \rightarrow \gamma$$

3*Xs* 2*Ys*

$(3,2)$	γ_+	γ_-
$k=0$	$[3]$	$[2]$
	$[2,1]$	$[2]$
	$[1,1,1]$	$[2]$
	$[3]$	$[1,1]$
	$[2,1]$	$[1,1]$
	$[1,1,1]$	$[1,1]$
$k=1$	$[2]$	$[1]$
	$[1,1]$	$[1]$
$k=2$	$[1]$	$\emptyset$

The irreducible representation of *GL(N)*:

$$\gamma = (\gamma_+, \gamma_-, k),$$

$$0 \le k \le \min(m,n), \quad \gamma_+ \mapsto (m-k), \ \gamma_- \mapsto (n-k)$$

Roughly speaking, this *k* represents the mixing between *X* and *Y*.

See the example of the simplest case $N \otimes \bar{N} = \left(N^2 - 1\right) \oplus 1$

$$X^{i_1}_{j_1}\left(Y^T\right)^{k_1}_{l_1} = \underline{X^{i_1}_{j_1}\left(Y^T\right)^{k_1}_{l_1} - \frac{1}{N}\delta^{i_1k_1}X^{m}_{j_1}\left(Y^T\right)^{m}_{l_1}} + \underline{\frac{1}{N}\delta^{i_1k_1}X^{m}_{j_1}\left(Y^T\right)^{m}_{l_1}}$$

$$O^{k=0} = trXtrY - \frac{1}{N}tr\left(XY\right)$$

k=0

$$O^{k=1} = \frac{1}{N}tr\left(XY\right)$$

k=1

$$\left\langle O_{k=0}{}^{\dagger} O_{k=1} \right\rangle_0 = 0$$

In general, the *k=0* have the following structure:

$$O_R\left(X\right)O_S\left(Y\right) + O\left(1/N\right)\cdots$$

$$O_R(X) = tr_m\left(p_R \cdot X^{\otimes m}\right), \quad O_S(Y) = tr_n\left(p_S \cdot Y^{\otimes n}\right)$$

Finite N constraint (stringy exclusion principle)

[YK-Ramgoolam 0709.2158]

$$tr_{m.n}\left(P^{\gamma} X^{\otimes m} \otimes Y^{T\otimes n}\right)$$

$$c_1(\gamma_+)+c_1(\gamma_-)\leq N$$

$$\gamma_+ \mapsto (m-k),\quad \gamma_- \mapsto (n-k)$$

$$tr_{m.n}\left(P^{\gamma(k=0,R,S)} X^{\otimes m} \otimes Y^{T\otimes n}\right) = tr_m\left(p_R X^{\otimes m}\right) tr_n\left(p_S Y^{\otimes m}\right)+\cdots$$

$$c_1(R)+c_1(S)\leq N$$

$$\gamma_+ = R \mapsto m,\quad \gamma_- = S \mapsto n$$

This is stronger than the naively expected one : $c_1(R)\leq N,\ \ c_1(S)\leq N$

$$tr\left(X^2 Y\right) = trXtrXY - \frac{1}{2}(trX)^2 trY + tr\left(X^2\right)trY \quad (N=2)$$

$$tr_{2,1}\left(P_{[1,1][1]} X^{\otimes 2} \otimes Y^T\right) = 0$$

This would give a cut-off for the angular momentum of the composite system.

One-loop analysis

[YK 1002.2424]

The one-loop mixing problem : to look for eigenstates of *H*.

$$\hat{H} \equiv tr\left(\left[X, Y\right]\left[\partial_X, \partial_Y\right]\right)$$

Our goal will be to understand the mixing pattern in terms of Young diagrams.

It is not so easy in general, but we can find some eigenstates easily *based on the new language.*

Easy to find that the *k=0* ops are annihilated by *H*:

$$(\partial_X \partial_Y)_{pq} X^i_j Y^{T\,k}_{\ \ l} = \underline{\delta_{ik}} \delta_{pj} \delta_{ql}$$

$$(\partial_Y \partial_X)_{pq} X^i_j Y^{T\,k}_{\ \ k} = \underline{\delta_{jl}} \delta_{pk} \delta_{qi}$$

$$C X^i_j Y^{T\,k}_{\ \ l} = \underline{\delta_{ik}} X^s_j Y^{T\,s}_{\ \ l}$$

$$X^i_j Y^{T\,k}_{\ \ l} \overleftarrow{C} = \underline{\delta_{jl}} X^i_s Y^{T\,k}_{\ \ s}$$

$$\hat{H} \cdot tr_{m.n}\left(P^{\gamma(k=0)} X^{\otimes m} \otimes Y^{T\otimes n}\right) = 0 \quad \Leftarrow \quad C \cdot P^{\gamma(k=0)} = 0$$

The other eigenstates

$$\hat{H}\cdot tr_{m.n}\left(P^{\gamma}X^{\otimes m}\otimes Y^{T\otimes n}\right)$$

$$=\sum_{r,s} tr_{m,n}\left(\left[P^{\gamma},C_{r,s}\right]X^{\otimes r-1}\otimes[X,Y]\otimes X^{\otimes m-r}\otimes Y^{T\otimes s-1}\otimes 1\otimes Y^{T\otimes n-s}\right)$$

$$=0$$

$$C_{r,s}P^{\gamma}=P^{\gamma}C_{r,s} \qquad \text{(Schur's lemma)}$$

<u>This is valid for any m, n, N.</u>

$$\gamma=(\gamma_+,\gamma_-,k),$$

$$0\le k\le \min(m,n),\quad \gamma_+\mapsto(m-k),\ \gamma_-\mapsto(n-k)$$

3*Xs* 2*Ys*

$(3,2)$	γ_+	γ_-
$k=0$	$[3]$	$[2]$
	$[2,1]$	$[2]$
	$[1,1,1]$	$[2]$
	$[3]$	$[1,1]$
	$[2,1]$	$[1,1]$
	$[1,1,1]$	$[1,1]$
$k=1$	$[2]$	$[1]$
	$[1,1]$	$[1]$
$k=2$	$[1]$	$\emptyset$

On the complete set

[YK Ramgoolam 0709.2158, 0807.3696]

$$tr_{m.n}\left(P^{\gamma} X^{\otimes m} \otimes Y^{T\otimes n}\right)$$

The number of the operators is not enough to provide a complete set. A complete basis is given by

$$O^{\gamma}_{A,ij}(X,Y) \equiv tr_{m,n}\left(Q^{\gamma}_{A,ij} X^{\otimes m} \otimes (Y^T)^{\otimes n}\right)$$

$$P^{\gamma} = \sum_{A,i} Q^{\gamma}_{A,ii}$$

$$\gamma_+ \mapsto (m-k), \quad \gamma_- \mapsto (n-k) \quad R \mapsto m, \quad S \mapsto n$$

$$A = (R,S)$$

$$M_{\gamma\to A} = \sum_{\delta\mapsto k} g(\gamma_+,\delta;R) g(\gamma_-,\delta;S)$$

$$\left\langle O^{\gamma_1}_{A_1,i_1 j_1}(X,Y)^{\dagger} O^{\gamma_2}_{A_2,i_2 j_2}(X,Y)\right\rangle_0 \propto \delta^{\gamma_1\gamma_2}\delta_{A_1A_2}\delta_{i_1i_2}\delta_{j_1j_2}$$

[Orthogonality]

When $k=0$, $i,j=1$ and $\gamma=A=(R,S)$.

$$Q^{\gamma(k=0)}_{A,ij} = P^{\gamma(k=0,\gamma_+,\gamma_-)} = P_{RS}$$

On the mixing pattern

An *X* and a *Y* are always combined after the action of the dilatation operator.

This means the *k=0* operators can not appear as the image of the dilatation operator.

$$\hat{H} \cdot O^{k=0} = 0$$

$$\hat{H} \cdot O^{k\neq 0} = \sum_{k'\neq 0} O^{k'}$$

In this sense, the *k=0* operators are not mixed with the other sectors ($k\neq 0$).

$$\left\langle O^{\gamma(k=0)}(X,Y)^{\dagger} O^{\gamma_2 (k\neq 0)}_{A_2, i_2 j_2}(X,Y) \right\rangle_1 = 0$$

$$\left\langle O^{\gamma_1(k=0)}(X,Y)^{\dagger} O^{\gamma_2 (k=0)}(X,Y) \right\rangle_1 \propto \delta^{\gamma_1 \gamma_2}$$

Summary

- Proposed to use *the representation basis* at finite N

  Young diagrams ⇔ multi-trace structure, flavour structure

  Finite N constraint (stringy exclusion principle)

  Orthogonal at classical level

- Will be useful to solve the mixing problem

  The operator labelled by an irreducilbe rep of *GL(N)* is anihilated by *D*.

$$\hat{H} \cdot tr_{m.n}\left(P^{\gamma} X^{\otimes m} \otimes Y^{T\otimes n}\right) = 0$$

  Looks like we moved to a proper language, with the help of *Brauer algebra*.

Examples of the basis

1 X 1 Y

$$O^{\gamma(k=0)} = trXtrY - \frac{1}{N}tr(XY) \qquad O^{\gamma(k=1)} = \frac{1}{N}tr(XY)$$

$(1,1)$	γ_+	γ_-
$k=0$	$[1]$	$[1]$
$k=1$	$\emptyset$	$\emptyset$

2 Xs 1 Y

$$O^{\gamma(k=0,[2],[1])} = \frac{1}{2}\left(\left(trX\right)^2 + tr\left(X\right)^2\right)trY - \frac{1}{N+1}\left(trXtr\left(XY\right) + tr\left(X^2Y\right)\right)$$

$$O^{\gamma(k=0,[1,1],[1])} = \frac{1}{2}\left(\left(trX\right)^2 - tr\left(X\right)^2\right)trY - \frac{1}{N+1}\left(trXtr\left(XY\right) - tr\left(X^2Y\right)\right)$$

$$O^{\gamma(k=1,[1],[0])} = \frac{2}{N^2-1}\left(NtrXtr\left(XY\right) - tr\left(X^2Y\right)\right)$$

$(2,1)$	γ_+	γ_-
$k=0$	$[2]$	$[1]$
	$[1,1]$	$[1]$
$k=1$	$[1]$	$\emptyset$