# ODE/IM correspondence and modified affine Toda field equations


**Katsushi Ito and Christopher Locke**

Tokyo Institute of Technology



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**Abstract**

We study the two-dimensional affine Toda field equations for affine Lie algebra $\hat{\mathfrak{g}}$ modified by a conformal transformation and the associated linear equations. In the conformal limit, the associated linear problem reduces to a (pseudo-)differential equation. For classical affine Lie algebra $\hat{\mathfrak{g}}$, we obtain a (pseudo-)differential equation corresponding to the Bethe equations for the Langlands dual of the Lie algebra $\mathfrak{g}$, which were found by Dorey et al. [1] in study of the ODE/IM correspondence.


## Lie algebra preliminaries

Each affine Toda field equation is associated with an affine Lie algebra $\hat{\mathfrak{g}}$. The generators of a simple Lie algebra $\mathfrak{g}$ of rank $r$ are $\{E_\alpha, H^i\}$ ($\alpha \in \Delta$, $i = 1, \cdots r$), satisfying the commutation relations

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \quad \text{for } \alpha+\beta \in \Delta, \tag{1}$$

$$[E_\alpha, E_{-\alpha}] = \frac{2\alpha \cdot H}{\alpha^2}, \tag{2}$$

$$[H, E_\alpha] = \alpha\, E_\alpha. \tag{3}$$

We normalize the long root of $\mathfrak{g}$ to have length squared of 2.

The affine Lie algebra $\hat{\mathfrak{g}}$ is obtained by adding the root $\alpha_0 = -\theta$ where $\theta$ is the highest root. The integers $n_i$ ($n_i^\vee$), called the (dual) Coxeter labels, are defined to satisfy $0 = \sum_{i=0}^r n_i \alpha_i = \sum_{i=0}^r n_i^\vee \alpha_i^\vee$, with $n_0^\vee = 1$. The (dual) Coxeter number denoted as $h$ ($h^\vee$) is defined as the sum of $n_i$ ($n_i^\vee$). We define the (co)Weyl vector $\rho$ ($\rho^\vee$) as the sum of the (co)fundamental weights.

## Modified affine Toda equation

The equation of motion of the 2d affine Toda field theory associated to $\hat{\mathfrak{g}}$ is

$$\beta \partial_+ \partial_- \phi - m^2 \sum_{i=0}^r n_i \alpha_i \exp(\beta \alpha_i \phi) = 0\,. \tag{4}$$

where $\phi$ is an $r$-component scalar field, $m$ is a mass parameter, and $\beta$ a dimensionless coupling parameter. We will use complexified coordinates $(z^+, z^-)$ given by $z^\pm = \frac{1}{2}(x^0 \pm i x^1)$ and $z^+ = \rho\, e^{i\theta}$.

Using a conformal transformation along with a field redefinition, the affine Toda field equation can be put into a modified form which has been used in recent papers [2, 3],

$$z^\pm \to \tilde{z}^\pm = f^\pm(z^\pm)\,, \quad \phi \to \tilde{\phi} = \phi - \frac{1}{\beta}\rho^\vee \log(\partial_+ f^+ \partial_- f^-)\,. \tag{5}$$

Using the above conformal transformation, we get the modified affine Toda field equation

$$\beta \partial_+ \partial_- \phi - m^2 \left[\sum_{i=1}^r n_i \alpha_i \exp(\beta \alpha_i \phi) + p^+(z^+) p^-(z^-) n_0 \alpha_0 \exp(\beta \alpha_0 \phi)\right] = 0, \tag{6}$$

where the conformal factors have been absorbed into the definitions of $p^\pm(z^\pm)$, where $M > \frac{1}{h-1}$,

$$p^\pm(z) = (\partial_\pm f)^h = (z^\pm)^{hM} - (s^\pm)^{hM}. \tag{7}$$

The equation of motion can be written as a zero curvature condition $F = \mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A} = 0$ for the one-form $\mathbf{A} = A_+\, \mathrm{d}z^+ + A_-\, \mathrm{d}z^-$. Our choice that gives this modified affine Toda equation (6) is

$$A_\pm = \pm \frac{\beta}{2} \partial_\pm \phi \cdot H + m e^{\pm\lambda} \left[\sum_{i=1}^r \sqrt{n_i^\vee} E_{\pm\alpha_i} \exp\left(\frac{\beta}{2}\alpha_i \phi\right) + p^\pm(z^\pm) \sqrt{n_0^\vee} E_{\pm\alpha_0} \exp\left(\frac{\beta}{2}\alpha_0 \phi\right)\right] \tag{8}$$

Here a spectral parameter $\lambda$ has been introduced.

## Asymptotic behavior

The large and small $\rho$ asymptotic solutions of the equation of motion (6) are

$$\phi(z, \bar{z}) = \frac{M\hat{\rho}}{\beta} \log(z\bar{z}) + \mathcal{O}(1) \quad \text{as } \rho \to \infty, \tag{9}$$

$$\phi(z, \bar{z}) = g \log(z\bar{z}) + \mathcal{O}(1) \quad \text{as } \rho \to 0, \tag{10}$$

where $g$ is a vector that controls the asymptotic behavior near $0$.

Applying the gauge transformation

$$\tilde{A}_\pm = U A_\pm U^{-1} + U \partial_\pm U^{-1}, \quad \tilde{\Psi} = U\Psi, \tag{11}$$

$$U = e^{\beta\phi \cdot H/2}, \tag{12}$$

gives a simple form of the connection with no diagonal part in the large $\rho$ limit,

$$A_\pm = m e^{\pm\lambda} \rho^{2M} \Lambda_\pm\,, \quad \text{where } \Lambda_\pm = \sum_{i=0}^r \sqrt{n_i^\vee} E_{\pm\alpha_i}\,. \tag{13}$$

We take the vector $\Psi$ to be in the *vector representation* of $\mathfrak{g}$. The large $\rho$ behavior of the fastest decaying solution to the linear problem along the positive real axis can be written in terms of the eigenvalue of $\Lambda_+$ with the largest real part, $\mu$. The top component of this fastest decaying solution is

$$\Xi \sim \exp\left(-2\mu \frac{\rho^{M+1}}{M+1} m \cosh(\lambda + i\theta(M+1))\right)\,. \tag{14}$$

Solutions to the linear problem defined for $\rho \to 0$ can be chosen so that the components satisfy

$$(\xi_i)_j = \delta_{ij} e^{-(\lambda+i\theta)\beta g \cdot h_i} + \mathcal{O}(\rho), \quad \chi_i \equiv (\xi_i)_1\,. \tag{15}$$

## $Q$-functions and Bethe ansatz equations

The linear problem and equation of motion are invariant under the transformation $\hat{\Omega}_k$ for integer $k$,

$$\hat{\Omega}_k : \begin{cases} z^\pm \to z^\pm\, e^{\pm 2\pi k i/hM} \\ s^\pm \to s^\pm\, e^{\pm 2\pi k i/hM} \\ \lambda \to \lambda - \frac{2\pi k i}{hM} \end{cases}. \tag{16}$$

This motivates the definition of a $k$-Symanzik rotated function to be

$$f_k(\rho, \theta; \lambda) = \hat{\Omega}_k f(\rho, \theta; \lambda)\,. \tag{17}$$

For the $A_r^{(1)}$ case, the following set of functional relations can be shown to hold [1] for a set of auxiliary functions $\psi^{(a)}$,

$$W^{(2)}[\psi^{(a)}_{-1/2}, \psi^{(a)}_{1/2}] = \psi^{(a-1)}\psi^{(a+1)}, \quad \psi^{(a)} = W^{(a)}[\psi^{(1)}_{\frac{1-a}{2}}, \ldots, \psi^{(1)}_{\frac{a-1}{2}}], \tag{18}$$

$$\psi^{(1)} = \Xi, \quad \psi^{(0)} = \psi^{(r+1)} = 1, \tag{19}$$

where $W^{(a)}[f_1, \ldots, f_a]$ is the Wronskian. Since (15) forms a basis of solutions to the linear problem, we can expand the decaying solution (14) as

$$\Xi(z^+, z^-; \lambda, g) = \sum_{i=1}^r Q_i(\lambda, g)\, \chi_i(z^+, z^-; \lambda, g)\,, \tag{20}$$

$$\psi^{(a)}(z^+, z^-; \lambda, g) = \sum_{\mathbf{i}} Q^{(a)}_{[i_1,\ldots,i_a]}(\lambda, g)\, W^{(a)}[\chi_{i_1}, \ldots, \chi_{i_a}]\,. \tag{21}$$

Substituting the expansion (21) into the functional relations (18) and picking off the coefficient of the most divergent term near $0$ gives the Bethe ansatz equations where .

$$\left.\frac{Q^{(a-1)}_{-1/2} Q^{(a)}_{1} Q^{(a+1)}_{-1/2}}{Q^{(a-1)}_{1/2} Q^{(a)}_{-1} Q^{(a+1)}_{1/2}}\right|_{\lambda_i^{(a)}} = -e^{\frac{2\pi i}{h(M+1)}(1+\beta\alpha_a \cdot g)}\,, \quad \text{where } Q^{(a)}(\lambda_i^{(a)}, g) = 0\,. \tag{22}$$

From the above equations, it has been shown [2, 3] that these $Q$-functions found from analysis of the spectral determinant of classical modified Toda field theories *corresponds* to some quantum IM for $A_1^{(1)}$ and $A_2^{(2)}$ cases. It is postulated to hold also for the whole $A_r^{(1)}$ chain as well.

## Compatibility (pseudo-)ODEs

Applying gauge transformation with $U_\pm = e^{\mp\beta\phi \cdot H/2}$ gives another useful form of the connection,

$$A_\pm = \pm\beta\partial_\pm\phi \cdot H + m e^{\pm\lambda} \left\{\sum_{i=1}^r \sqrt{n_i^\vee} E_{\pm\alpha_i} + p^\pm(z^\pm)\sqrt{n_0^\vee} E_{\pm\alpha_0}\right\}. \tag{23}$$

For the simplest case $A_r^{(1)}$, writing out the first order matrix equation $(\partial_\pm + A_\pm)\tilde{\Psi} = 0$ gives a system of equations. This system of equations can be solved in terms of the top/bottom component of $\tilde{\Psi}$,

$$\mathbf{h} = (h_1, \ldots, h_{r+1})\,, \quad \mathbf{h}^\dagger = (-h_{r+1}, \ldots, h_1)\,, \tag{24}$$

$$D_\pm(h) \equiv \partial_\pm + \beta\partial\phi \cdot h\,, \quad D_\pm(\mathbf{h}) \equiv D_\pm(h_{r+1}) \cdots D_\pm(h_1)\,, \tag{25}$$

$$D_+(\mathbf{h})\psi^+ = (-m e^\lambda)^h\, p^+(z^+)\psi^+\,, \tag{26}$$

$$D_-(\mathbf{h}^\dagger)\psi^- = (-m e^\lambda)^h\, p^-(z^-)\psi^-\,. \tag{27}$$

This analysis can be carried out for other algebras as well, and the compatibility equations are listed below in table 1. These are the ODE part of the ODE/IM correspondence. They determine the behavior of $\Psi$, which is related to a corresponding quantum IM.

| | | |
|---|---|---|
| $A_r^{(1)}$ | $D_+(\mathbf{h})\psi = (-m e^\lambda)^h p^+(z^+)\psi$ | $\mathbf{h} = (h_1, \ldots h_{r+1})$ |
| $D_r^{(1)}$ | $D_+(\mathbf{h}^\dagger)\partial_+^{-1} D_+(\mathbf{h})\psi = 2^{r-1}(m e^\lambda)^h \sqrt{p^+(z^+)}\partial_+\sqrt{p^+(z^+)}\psi$ | $\mathbf{h} = (h_1, \ldots h_r)$ |
| $B_r^{(1)}$ | $D_+(\mathbf{h}^\dagger)\partial_+ D_+(\mathbf{h})\psi = 2^r(m e^\lambda)^h \sqrt{p^+(z^+)}\partial_+\sqrt{p^+(z^+)}\psi$ | $\mathbf{h} = (h_1, \ldots h_r)$ |
| $A_{2r-1}^{(2)}$ | $D_+(\mathbf{h}^\dagger) D_+(\mathbf{h})\psi = -2^{r-1}(m e^\lambda)^h \sqrt{p^+(z^+)}\partial_+\sqrt{p^+(z^+)}\psi$ | $\mathbf{h} = (h_1, \ldots h_r)$ |
| $C_r^{(1)}$ | $D_+(\mathbf{h}^\dagger) D_+(\mathbf{h})\psi = (m e^\lambda)^h p^+(z^+)\psi$ | $\mathbf{h} = (h_1, \ldots h_r)$ |
| $D_{r+1}^{(2)}$ | $D_+(\mathbf{h}^\dagger)\partial_+ D_+(\mathbf{h})\psi = 2^{r+1}(m e^\lambda)^{2h} p^+(z^+)\partial_+^{-1} p^+(z^+)\psi$ | $\mathbf{h} = (h_1, \ldots h_r)$ |
| $A_{2r}^{(2)}$ | $D_+(\mathbf{h}^\dagger)\partial_+ D_+(\mathbf{h})\psi = -2^r\sqrt{2}(m e^\lambda)^h p^+(z^+)\psi$ | $\mathbf{h} = (h_1, \ldots h_r)$ |
| $G_2^{(1)}$ | $D_+(\mathbf{h}^\dagger)\partial_+ D_+(\mathbf{h})\psi = 8(m e^\lambda)^h \sqrt{p^+(z^+)}\partial_+\sqrt{p^+(z^+)}\psi$ | $\mathbf{h} = (h_1, h_2, h_3)$ |
| $D_4^{(3)}$ | $\big[D_+(\mathbf{h}^\dagger)\partial_+ D_+(\mathbf{h}) + (\zeta+1)2\sqrt{3}(m e^\lambda)^4 D_+(\mathbf{h}^\dagger) p^+(z^+)$ $-\zeta 4\sqrt{3}(m e^\lambda)^4 D_+(-h_1)(\partial_+ p^+(z^+) + p^+(z^+)\partial_+) D_+(h_1)$ $-(\zeta+1)2\sqrt{3}(m e^\lambda)^4 p^+(z^+) D_+(\mathbf{h})$ $+(\zeta-1)^2 12(m e^\lambda)^8 p^+(z^+)\partial_+^{-1} p^+(z^+)\big]\psi_1 = 0$ | $\mathbf{h} = (h_1, h_2, h_3)$ |

**Table 1:** Compatibility equations for various algebras with weight vectors $h_i$

The above (pseudo-)ODEs agree with that of [1] if we make the following identification:

| Dorey et al. [1] | K. Ito, CL [4] |
|---|---|
| $\mathfrak{g} \longleftarrow$ | $\longrightarrow \hat{\mathfrak{g}}^\vee$ |

## Conclusions

- Obtained (pseudo-)ODEs corresponding to type $A$, $B$, $C$, $D$, and $G$ affine Toda equations.
- The equations were found to correspond to that of [1] for the Langlands dual of $\hat{\mathfrak{g}}$.
- We also have verified that Lie algebra isomorphisms hold at the level of the linear problem.

## Future directions

- What is the meaning of the new equations that are not in [1], for instance of $B_r^{(1)}$, $C_r^{(1)}$, and $A_{2r}^{(2)}$?
- Can we find Bethe ansatz equations corresponding to these new cases?
- What about the other exceptional cases of $E$ and $F$ type?

## References


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