On relations between supersymmetric theories in different dimensions

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IAS

Based on work with:

O. Aharony, N. Seiberg, and B. Willett (2013); B. Willett (2014)

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- In the last ~ 25 years we learnt a lot of spectacular facts about supersymmetric QFTs in various dimenions.
- ... and this learning is still going on vigorously.
- These facts include very different ways of encoding the same physics (dualities), exact computations of path integrals, deep relations between theories and computations in various dimensions.
- The questions is whether all this is just a random collection of facts or some of them follow by assuming the others.

In recent years we have wittnessed a lot of progress in answering this question.

Three-legged progress:

- Exact results in QFT.
- Dimensional reductions.
- Spaces of theories.

Example: $\mathcal{N} = 2$ superconformal theories in $4d$.

- Until ~ 2009 we have known a lot of scattered results about these theories.
  - (Self)-duality of $\mathcal{N} = 4$ SYM and $N_f = 2N_c$ SQCD.
  - Existence of some exotic theories (Minahan-Nemeschansky).
  - 2007 - Argyres-Seiberg duality

- In a revolutionary work by Gaiotto in 2008 it was understood that all these scattered facts fall into a big and beautiful structure if one thinks of $\mathcal{N} = 2$ theories as arising upon different dimensional reductions of a unique $6d$ theory. $6 \to 6-2$

- The dualities are equivalent ways to perform the reduction.

- The "exotic theories" are in fact extremely abundant and rather they are the common ones with the theories with known Lagrangian being exotic. Filling gaps in the space of theories made their existance absolutely needed.

- The same time the new technology of computing supersymmetric partition functions $\mathbb{S}^4$ ($\mathbb{S}^3 \times \mathbb{S}^1$, etc) allowed to exploit the structure of the theory space to make other extraordinary connection: *e.g.* AGT.

- Thinking of a particularly vast set of $\mathcal{N}=2$ superconformal theories in $4d$ as arising from compactifications of $6d$ model seems to be the "right" and prolific way to go.

- This allows us to understand quite untuitively and computationally a lot of seemingly mysterious properties of these theories.

- This is a generic lesson which can be applied to situations in other dimensions and other amounts of supersymmetry.

- In this talk we will discuss some of the relations between dualities in D and $D-1$ dimensions. ($D=4,\,3$)
- We will ask the question what a duality in D dimensions implies about relations of theories dimensionally reduced to $D-1$ dimensions.
- Or conversly, how a duality in $D-1$ dimensions is encoded in the D dimensional physics.
- The tool that we will use to make progress will be the different supersymmetric partition functions in various dimensions.

Outline

- Part 一: What an IR duality in D dimensions implies about IR dualities in $D-1$ dimensions?
  - Take-home lesson: orders of limits matter
- Part 二: How is $\mathcal{N}=4$ mirror symmetry in $3d$ encoded in $4d$ physics.
  - Take-home lesson: dualities might not always lift to dualities

Part 一

IR dualities

- IR dualities: different UV descriptions flowing in the IR to the same fixed point.
- 4d IR (Seiberg) dualities:

$$Sp(2n_c)_{2n_f} \qquad \longleftrightarrow \qquad Sp(2(n_f - n_c - 2))_{2n_f}$$

(n_f integer)

- 3d IR (Aharony, Giveon-Kutasov) dualities:

$$Sp(2n_c)_{2n_f}, \; level\ k \quad \longleftrightarrow \quad Sp(2(n_f - n_c - 1 + k))_{2n_f}, \; level\ -k$$

($n_f + k$ integer)

- 2d IR (Hori) dualities:

$$Sp(2n_c)_{2n_f}, \quad \longleftrightarrow \quad Sp(2(n_f - n_c - \frac{1}{2}))_{2n_f}$$

(n_f half-integer)

- Are these dualities in different space-time dimensions related?
- If yes, what is the precise relation?

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General comments about reducing dualities

- To reduce a theory we put it on a circle $\mathbb{S}^1_{\widetilde{r}}$ with finite radius $\widetilde{r}$.
- To take the limit to $D-1$ dimensions we have now to send the radius to zero.
- However, if we are interested in reducing a pair of theories which are dual only in the IR there might, and there is, an issue with order of limits: we first should flow to the IR and only then try to take the limit of small radius.
- A proper way to reduce a duality is thus to consider an effective theory on a circle at energy scales below the compactification radius $\widetilde{r}$. Such a theory looks $D-1$ dimensional.

General comments about reducing dualities

- However, it might be slightly different from the theory we started with: the effective theory might contain new superpotential terms.

- For example: when going from four to three dimensions some symmetries might be anomalous in four dimensions. There are no continuous anomalies in three dimensions and thus these symmetries have to be broken by an explicit superpotential term.

- We then should look for a $D-1$ dimensional UV completion of this effective theories.

- If such a UV completion is found then we obtain a duality in $D-1$ dimensions.

The $4d$ supersymmetric index

- Let us start with reducing $4d$ dualities. We will perform the discussion at the level of the partition functions.

- In $4d$ we have a useful partition function which one can use to study such reductions $\mathbb{S}^3/\mathbb{Z}_r \times \mathbb{S}^1_{\tilde{r}}$.

- For the sake of this talk we will consider only the case of $r = 1$, the supersymmetric index.

$$\mathcal{I} = Tr(-1)^F\, p^{j_1+j_2-\frac{r}{2}}\, q^{j_1-j_2-\frac{r}{2}} \prod_{i=1}^{F} u_i^{F_i}\,.$$

The $4d$ supersymmetric index for a gauge theory

- The index of any gauge theory is easy to compute.

- One performs the computations in a free theory by counting with weights for non-anomalous symmetries all the relevant protected operators and throws away gauge non-singlets

$$\mathcal{I}_{Sp(2n_c)_{2n_f}} = \oint \prod_{i=1}^{n_c} \frac{dz_i}{2\pi i z_i} \frac{\prod_{i=1}^{2n_f} \prod_{j=1}^{n_c} \Gamma_e((pq)^{\frac{R}{2}} z_j^{\pm 1} u_i; p, q)}{\prod_{i<j}^{n_c} \Gamma_e(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{i=1}^{n_c} \Gamma_e(z_i^{\pm 2}; p, q)}$$

- Here R is the non-anomalous R-charge, $R = \frac{n_f - n_c - 1}{n_f}$.

Duality = Identity

- These partition functions are invariant under the RG flow and thus probe the IR physics.
- Duality implies identity on the matrix model integrals,

$$\oint \prod_{i=1}^{n_c} \frac{dz_i}{2\pi i z_i} \frac{\prod_{i=1}^{2n_f}\prod_{j=1}^{n_c} \Gamma_e((pq)^{\frac{R}{2}} z_j^{\pm 1} u_i; p, q)}{\prod_{i<j}^{n_c} \Gamma_e(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{i=1}^{n_c} \Gamma_e(z_i^{\pm 2}; p, q)} =$$

$$\prod_{i<j} \Gamma_e((pq)^R u_i u_j; p, q) \oint \prod_{i=1}^{n_f-n_c-2} \frac{dz_i}{2\pi i z_i} \times$$

$$\frac{\prod_{i=1}^{2n_f}\prod_{j=1}^{n_f-n_c-2} \Gamma_e((pq)^{\frac{1-R}{2}} z_j^{\pm 1} u_i^{-1}; p, q)}{\prod_{i<j}^{n_f-n_c-2} \Gamma_e(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{i=1}^{n_f-n_c-2} \Gamma_e(z_i^{\pm 2}; p, q)}$$

Dimensional reduction at the level of the index

- The index is a function of fugacities, holonomies around the $\mathbb{S}^1$,

$$u_i = e^{2\pi i m_i \widetilde{r}}, \qquad \mathcal{I} = \mathcal{I}(\{u_i\}, p, q)$$

- Here $\widetilde{r}$ is the radius of $\mathbb{S}^1$ and m is the chemical potential.

- To obtain the partition function of the dimensionally reduced theory you consider the limit of the index as $\widetilde{r} \to 0$.

Free chiral example

- Let us give an example of a free chiral field.

- Reducing on $\mathbb{S}^1$ one can write thus the $4d$ index as a product over $\mathbb{S}^3$ partition functions of the KK modes,

$$\mathcal{I}_{(4d)}(u,\, p,\, q) \propto \prod_{n=-\infty}^{\infty} \mathcal{Z}_{(3d)}(\omega_1,\, \omega_2;\, m + \frac{n}{\tilde{r}})$$

- This product should be properly regularized and the 4d index appropriately normalized so that the above becomes an exact equality. *E.g.* for a chiral field one gets

$$e^{\mathcal{I}_0}\ \Gamma_e(e^{2\pi i m\widetilde{r}};\, e^{2\pi i\omega_1\widetilde{r}},\, e^{2\pi i\omega_2\widetilde{r}}) =$$
$$e^{-\Delta}\ \prod_{n=-\infty}^{\infty} e^{-\mathrm{sign}(n)\,\frac{\pi i}{2\omega_1\omega_2}\left((m+\frac{n}{\tilde{r}}-\omega)^2-\frac{\omega_1^2+\omega_2^2}{12}\right)}\Gamma_h(m+\frac{n}{\tilde{r}};\omega_1,\omega_2)\,.$$

  (This equality is mathematically precisely the $SL(3, Z)$ property of elliptic Gamma functions.)

- Sending the radius $\widetilde{r}$ to zero only the zero mass KK mode survives and we get that the 4d index of a chiral reduces to the 3d $\mathbb{S}^3$ partition function.

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Reduction of a gauge theory

- We can now reduce the index of a gauge theory to $\mathbb{S}^3$ partition function.
- The index is given by a contour integral over the gauge fugacities z_i.
- Can we commute the limits of integration and the limit or $\widetilde{r} \to 0$?
- In $4d$ to $3d$ reduction the answer turns out to be yes, as long as we parametrize the integration variables as $z = e^{2\pi i \widetilde{r} \sigma}$.

Duality with superpotential

- Reducing the identity for Seiberg duality we obtain (schematically)

$$\int_{-\infty}^{\infty}\prod_{i=1}^{n_c} d\sigma_i \frac{\prod_{i=1}^{n_c}\prod_{j=1}^{2n_f}\Gamma_h(\pm\sigma_i+m_j)}{\prod_{i<j}\Gamma_h(\pm\sigma_i\pm\sigma_j)\prod_{i=1}^{n_c}\Gamma_h(\pm 2\sigma_i)} = \prod_{j<k}\Gamma_h(m_j+m_k)$$

$$\int_{-\infty}^{\infty}\prod_{i=1}^{n_f-n_c-2} d\sigma_i \frac{\prod_{i=1}^{n_f-n_c-2}\prod_{j=1}^{2n_f}\Gamma_h(\pm\sigma_i+m_j)}{\prod_{i<j}\Gamma_h(\pm\sigma_i\pm\sigma_j)\prod_{i=1}^{n_f-n_c-2}\Gamma_h(\pm 2\sigma_i)}$$

- This expression looks like equality of $\mathbb{S}^3$ partition functions of two $3d$ theories with the same matter content as the $4d$ theory.

- There is one caveat: in $3d$ the axial symmetry is not anomalous and we should be able to refine the partition functions with it. If we do so the equality holds no more.

3d Games

- Thus the equality on the previous slide is an indication that there is a $3d$ duality with a superpotential which explicitly breaks the axial symmetry.

- Once we are in $3d$ we can turn on relevant deformations on both sides of the duality to obtain new dualities.

- In particular we can get rid of some of the superpotential terms and generate CS terms by turning on appropriate real mass deformations.

- Real mass deformations at the level of the $\mathbb{S}^3$ partition functions correspond to sending some of the parameters m_i to infinity.

- These limits in general do not commute with the integrals and great care has to be exercized in taking them!!

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$Sp(N)$ example: after real masses

- Start with $2n_f+2$ flavors and consider the following limit.

$$m_{2n_f+1}\to s+\alpha,\qquad m_{2n_f+2}\to -s+\alpha,\qquad s\to\infty$$

- Taking now the limit of large s under the integrals we obtain the following equality

$$\int_{-\infty}^{\infty}\prod_{i=1}^{n_c}d\sigma_i\frac{\prod_{i=1}^{n_c}\prod_{j=1}^{2n_f}\Gamma_h(\pm\sigma_i+m_j)}{\prod_{i<j}\Gamma_h(\pm\sigma_i\pm\sigma_j)\prod_{i=1}^{n_c}\Gamma_h(\pm 2\sigma_i)}=\Gamma_h(2\sum_{i=1}^{2n_f}m_i-2\omega(n_f-n_c-1))$$

$$\prod_{j<k}\Gamma_h(m_j+m_k)\int_{-\infty}^{\infty}\prod_{i=1}^{n_f-n_c-1}d\sigma_i\frac{\prod_{i=1}^{n_f-n_c-1}\prod_{j=1}^{2n_f}\Gamma_h(\pm\sigma_i+m_j)}{\prod_{i<j}\Gamma_h(\pm\sigma_i\pm\sigma_j)\prod_{i=1}^{n_f-n_c-1}\Gamma_h(\pm 2\sigma_i)}$$

- We do not have the constraint $\sum m_i=0$ anymore.
- This is Aharony duality in $3d$.

The $3d$ index

- Let us next comment on the further reduction of the dualities to two dimensions.

- Again the tool we will use is the supersymmetric index,

$$\mathcal{I} = \text{Tr}(-1)^{2J} q^{\frac{\Delta+J}{2}} \prod_i u_i^{F_i}$$

- For gauge theory the index is again easy to compute. Very schematically it is given by the following

$$\mathcal{I} = \sum_{m_i} \oint \prod_i \frac{dz_i}{2\pi i z_i} \Delta_V(z,m)\; z_i^{k m_i} \prod_{\ell \in matter} \mathcal{I}_\ell(z,m)$$

Reducing the index to $\mathbb{S}^2$ partition function

- The reduction of the index to $\mathbb{S}^2$ partition function proceeds as before.
- The index is a function of fugacities, holonomies around the $\mathbb{S}^1$,

$$u_i = e^{2\pi i \zeta_i \tau}, \qquad \mathcal{I} = \mathcal{I}(\{u_i\}, p, q)$$

- τ is the radius of $\mathbb{S}^1$ and ζ is the chemical potential.
- To obtain the partition function of the dimensionally reduced theory you consider the limit of the index as $\tau \to 0$.
- Example of free chiral,

$$\mathcal{I}(e^{i\zeta}, m) = \frac{e^{-\frac{2}{\tau}\left(Li_2\left(e^{-i\tau\left(\zeta+\frac{\Delta-1}{2}i\right)}\right)-\frac{\pi^2}{6}\right)}}{i^m e^{\pi\left(\zeta+\frac{\Delta-1}{2}i\right)-\frac{1}{2}\tau\left(\zeta+\frac{\Delta-1}{2}i\right)^2}} \times$$

$$\prod_{\ell=0}^{\infty} i^m e^{\Delta-1-2i\zeta-\frac{4i\pi l}{\tau}} \left(\frac{\Delta-1}{2}i+\zeta+\frac{2\pi\ell}{\tau}\right)^{1-\Delta+2i\zeta+\frac{4i\pi\ell}{\tau}} Z(\zeta+\frac{2\pi\,\ell}{\tau}, m)$$

$$\prod_{\ell=1}^{\infty} (-i)^m e^{\Delta-1-2i\zeta+\frac{4\pi i\ell}{\tau}} \left(-\frac{\Delta-1}{2}i-\zeta+\frac{2\pi\ell}{\tau}\right)^{1-\Delta+2i\zeta-\frac{4\pi i\ell}{\tau}} Z(\zeta-\frac{2\pi\,\ell}{\tau}, m)$$

- Taking the $\tau \to 0$ limit only the zero momentum KK mode survives.

Reducing gauge theory

- Reducing indices of gauge theories from $3d$ to $2d$ turns out to be much tougher task than from $4d$ to $3d$.

- The limits of performing the infinite sums and the integrals almost always do not commute!!

- In particular one has to scale the integrated and summed over parameters in non trivial way to pick up the dominant contribution in the limits.

- In some of these scaling for example all of the KK modes become massive in the limit and produce not trivial potential for background fields on $\mathbb{S}^2$.

Reducing $Sp(2n_c)$ Giveon-Kutasov duality

- It so happens that in the particular case of $Sp(2n_c)_{2n_f}$ theory with CS level $k = \frac{1}{2}$ and $Sp(2(n_f - n_c - \frac{1}{2}))_{2n_f}$ with CS level $k = -\frac{1}{2}$ (n_f half-integer), the limits of small radius and the infinite sums/integrals commute.

- These two theories are dual to each other in $3d$. Upon reduction they give an identity between $\mathbb{S}^2$ partition functions which implies a $2d$ duality with the same matter content and same gauge groups.

- This is Hori's duality.

- If one takes $k \neq \frac{1}{2}$ non trivial limits have to be taken and identities indicating possibly new $2d$ dualities are obtained.

Map of reductions

$$4d: \quad Sp(2n_c)_{2n_f} \leftrightarrow Sp(2(n_f - n_c - 2))_{2n_f},\ M,\ W_0$$

$\downarrow$

$$3d: \quad Sp(2n_c)_{2n_f} + W_1 \leftrightarrow Sp(2(n_f - n_c - 2))_{2n_f},\ M,\ W_0 + \widetilde{W}_1$$

$$Sp(2n_c)_{2n_f} \leftrightarrow Sp(2(n_f - n_c - 1))_{2n_f},\ M,\ W_0 + \widetilde{W}'_1$$

$$Sp(2n_c)_{2n_f}^{k=\frac{1}{2}} \leftrightarrow Sp(2(n_f - n_c - \tfrac{1}{2}))_{2n_f}^{k=-\frac{1}{2}},\ M,\ W_0$$

$\rightarrow$ other k

$\downarrow$

$$2d: \quad Sp(2n_c)_{2n_f} \leftrightarrow Sp(2(n_f - n_c - \tfrac{1}{2}))_{2n_f},\ M,\ W_0$$

Summary of part 一

- Given a duality in D dimensions one can always find a proper way to reduce the relevant partition functions indicating dualities in $D-1$ dimensions.

- The proper way might not be trivial: Orders of limits matter!!

- Understanding the physics of the non-commuting limits is an interesting challenge.

Part 二

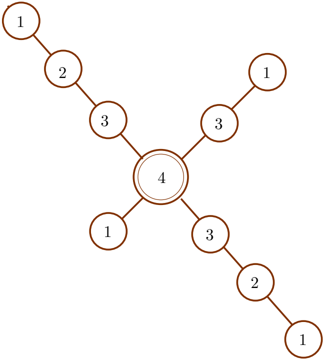
- Interesting question is whether any duality in $D-1$ dimensions descends from a duality in D dimensions.
- What about $\mathcal{N}=4$ mirror symmetries in $3d$?
- Such mirror symmetries have interesting properties, *i.e.* under the duality Coulomb branches are exchanged with Higgs branches.

Some $\mathcal{N} = 4$ mirror symmetries

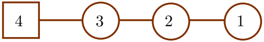
- An interesting set of $\mathcal{N} = 4$ mirror dualities is given by dimensionally reducing $4d$ $\mathcal{N} = 2$ theories of class $\mathcal{S}$.
- In $3d$ these theories are mirror dual to *star shaped quivers*.
- Most class $\mathcal{S}$ theories do not have known description in terms of a Lagrangian but the mirror duals are always given in terms of one.

An example

- Class $\mathcal{S}$ theories are labeled by punctured Riemann surfaces with some discrete information (Young tableaux) associated to the punctures.
- A mirror dual of a theory corresponding to a sphere with 4 punctures of certain type is given in the picture here.



$T_{(1,1,1,1)}[su(4)]$:



$T_{(3,1)}[su(4)]$:

- A lift to $4d$ of side A of the duality is known: class $\mathcal{S}$ theories.

- The mirror side does not have an obvious lift. For example, it contains many $U(1)$ factors with non-trivial dynamics. The coulomb branch is very non trivial.

- What does the $3d$ mirror symmetry mean for the $4d$ lift of side A of the duality.

$4d$ partition functions of class $\mathcal{S}$ theories

- The partition functions on $\mathcal{M}_3 \times \mathbb{S}^1$ (say $\mathcal{M}_3 = \mathbb{S}^3/\mathbb{Z}_r$) of class $\mathcal{S}$ theory corresponding to Riemann surface with genus g and s punctures has the following form

$$Z_{g,\,s} = \sum_{\lambda} C_{\lambda}^{2g-2} \prod_{i=1}^{s} \hat{\psi}_{\lambda}(\Lambda_i[a_i])\,.$$

- There is a roundabout way to derive this expression by using conformal dualities and RG flows of the class $\mathcal{S}$ theories as well as the analytical properties of the partition functions.

What are $\hat{\psi}_\lambda(a)$?

- $\hat{\psi}_\lambda(a)$ are eigenfunctions of certain difference operators,

$$\mathfrak{S}_{z^*}(a) \cdot \hat{\psi}_\lambda(a) = \mathcal{E}_\lambda^{z^*} \, \hat{\psi}_\lambda(a)$$

- The eigenvalues are also residues of $\hat{\psi}_\lambda(a)$,

$$Res_{a \to z^*} \hat{\psi}_\lambda(a) = \mathcal{E}_\lambda^{z^*} \, .$$

- No clear physical $(4d)^*$ meaning for $\hat{\psi}_\lambda(a)$ as of yet.

Reduction to $3d$

- Carefully reducing the expression for the index to $3d$ one obtains

$$\hat{\psi}_\lambda(e^{2\pi i\alpha}) \quad \rightarrow \quad \hat{\psi}(\beta|\alpha)$$

- $\hat{\psi}(\beta|\alpha)$ ($\beta \sim r\lambda$) is the partition function on $\mathcal{M}_3$ of $T[SU(N)]$ theory!!

- The full index (for genus zero theories) becomes

$$Z_{g,s} = \sum_\lambda C_\lambda^{-2} \prod_{i=1}^{s} \hat{\psi}_\lambda(\Lambda_i[a_i]) \quad \rightarrow \quad \int d\beta \Delta_V(\beta) \prod_{i=1}^{s} \hat{\psi}(\beta|\alpha_i) \,.$$

- This is the index of the mirror star-shaped quiver.

- The "mysterious" form of the partition functions in $4d$ reflects the fact that the $3d$ reduction has a mirror dual.

- In fact the eigenvalue equation defining th functions $\hat{\psi}_\lambda(a)$ has a natural $3d$ meaning.

- The reduction of $\mathfrak{S}_{z^*}(a)$ becomes a difference operator introducing a line defect into the partition function on $\mathcal{M}_3$.

- The reduction of the eigenvalue $\mathcal{E}^{z^*}(b)$ is the expectation value of a Wilson line.

- The eigenvalue equation

$$\mathfrak{S}_{z^*}(a)\hat{\psi}(b|a) = \mathcal{E}^{z^*}(b)\hat{\psi}(b|a)$$

is the fact that Wilson line for one flavor symmetry is the same as the defect line operator for the other.

Summary of 二

- The $3d$ duality here does not seem to lift to a duality in $4d$
- Although it is unlikely that there is a dual in $4d$ which directly reduces to the star-shaped quiver description, the existance of such a mirror dual in $3d$ is encoded in $4d$ in the particular form the partition functions take.

Summary

- Many of the dualities in $D-1$ dimensions can be obtained by assuming dualities in D dimensions.
- Some of the dualities in $D-1$ dimensions do not lift in a simple manner to D dimensions. Though the existance of the $D-1$ dimensional duality might be encoded in a non-trivial way in D dimensions.
- There are many interesting questions: *e.g.* understanding thoroughly the physics of the reductions and uplifts; reducing dualities from D dimensions to $D-2$; spaces of $\mathcal{N}=1$ theories in $4d$; etc

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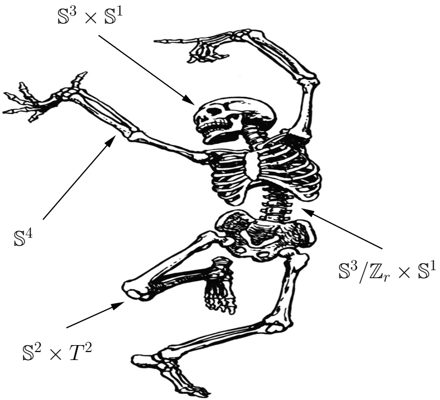
Biological Analogy

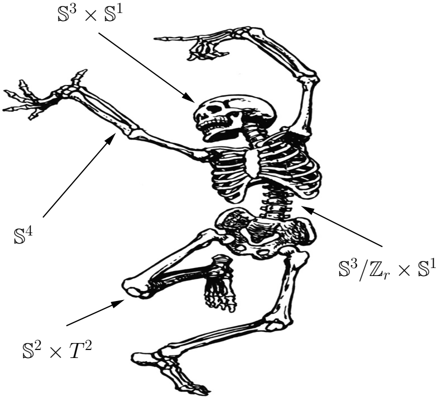
- Theories $\leftrightarrow$ Organisms

- Rhobust features: Exact Partition Functions $\leftrightarrow$ Parts of the Skeleton



- More (super)symmetry $\leftrightarrow$ more bones

- Rhobust features: Exact Partition Functions ↔ Parts of the Skeleton

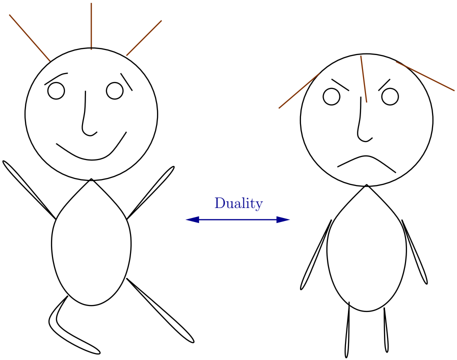


- More (super)symmetry ↔ more bones

- Dimensional reductions and RG flows $\leftrightarrow$ "Evolution"

- “Non-Lagrangian” theories $\leftrightarrow$ Exotic/extinct animals, “missing links”

- Duality $\leftrightarrow$ Different appearances of the same organism



- When we are interested in some beautiful and complicated creature (the theory of Nature) it might be usefull to consider all the other beasts in the animal kingdom.

ありがとうございます – Thank You!!