Brane solutions of Hopf soliton in seven dimensions

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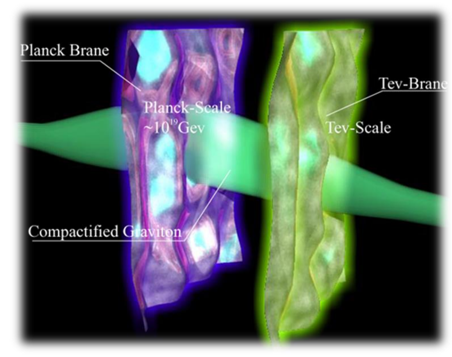
We consider a seven-dimensional brane world scenario model. It is a high dimensional theory which solves the hierarchy problem and cosmological constant problem. The standard model (SM) particles and forces are confined to a 3-brane. We assume the seven dimensional space-time world using toroidal coordinates and that the brane is described by a localized solution to the extended Skyrme-Faddeev model embedding in the extra dimensions. The solutions are axially symmetric knotted solitons with non-zero Hopf charge Q_H and are numerically obtained the coupled system of the Einstein and the matter field equations by Newton-Raphson method in the case of $Q_H = 2$ ($m = 1, n = 2$). We have obtained the several solutions with both of signs of constant and also with / without of the effect of the inflation. We discuss the localizing property of the gravity. In terms of the inherent chiral character of the solutions, we are able to discuss the property of the localized chiral fermions in our branes.



I. The matter fields - Static Hopfions in the extended Skyrme-Faddeev model

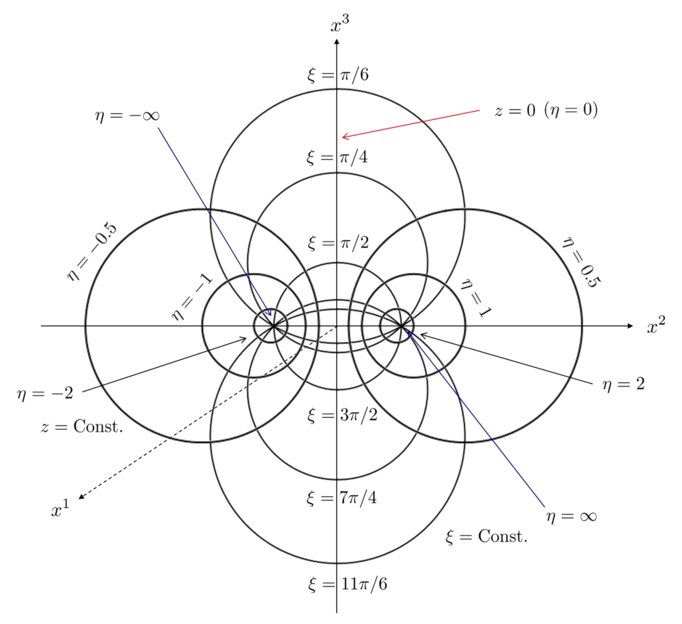
L. A. Ferreira, Nobuyuki Sawado, Kouichi Toda, *Static Hopfions in the extended Skyrme-Faddeev model*, J. High Energy Phys. JHEP 11 (2009) 124

Lagrangian density

$$\mathcal{L} = A^2\partial_\mu\vec{n}\cdot\partial^\mu\vec{n} - \frac{1}{e^2}(\partial_\mu\vec{n}\times\partial^\nu\vec{n})^2 + \frac{b}{2}(\partial_\mu\vec{n}\cdot\partial^\mu\vec{n})^2$$
$$A>0,\quad e^2<0,\quad b<0,\quad b\,e^2>1$$

Toroidal coordinates

$$x^1 = \frac{r_0}{p}\sqrt{z}\cos\varphi,\quad x^2 = \frac{r_0}{p}\sqrt{z}\sin\varphi,\quad x^3 = \frac{r_0}{p}\sqrt{1-z}\sin\xi$$
$$p \equiv 1-\cos\xi\sqrt{1-z} \qquad z = \tanh^2\eta$$
$$-\infty<\eta<\infty \;\Leftrightarrow\; 0\le z\le 1,\ -\pi\le\theta\le\pi,\quad 0\le\varphi\le 2\pi$$

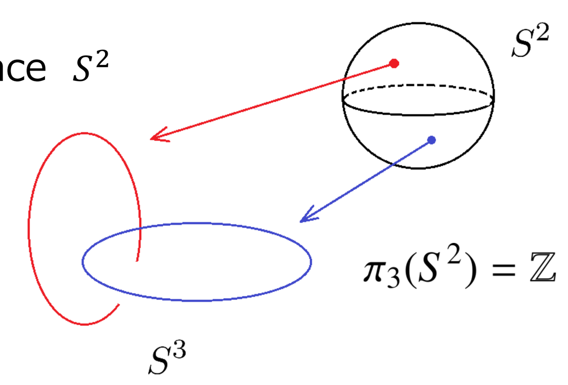


Hopf fibration and Hopf toplogical charge

maps from the three dimensional space $\mathbb{R}^3 \sim S^3$ to the target space S^2

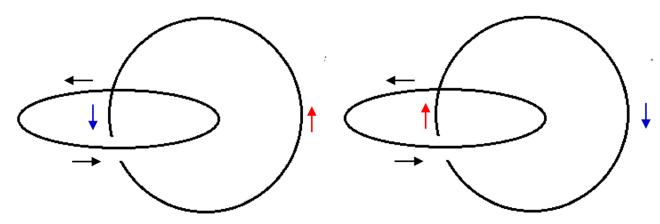
The linking number of closed loops on S^3
$\Leftrightarrow$ The Hopf topological charge

Under the axially symmetric ansatz $Q_H = mn$

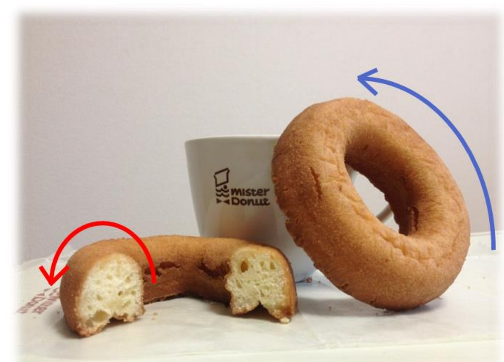


The chiral structure

The Hopf topological charge corresponds to the linking number of Hopfions, and can be either positive or negative. This reflects the chiral structure of knotted solitons.

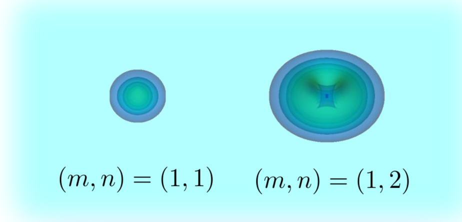


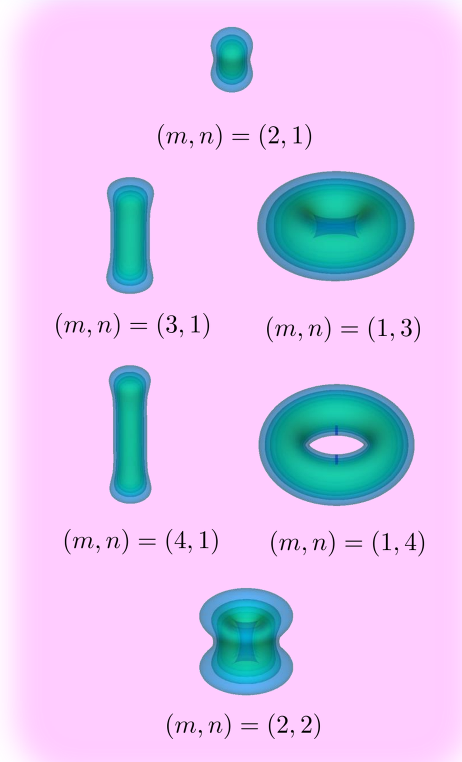
Linking number +2 Linking number -2



The axially symmetric solutions

Very probably, The solutions with Hopf charge 3 and 4 correspond to excited states. On the other hand, The solutions Hopf charge 1 or 2 may correspond to the minimum of energy. That is in fact what happens in the Skyrme-Faddeev model.





II. The brane solutions - A coupled system of the Hopfions and gravity

The actions

$$S = S_{\rm matter} + S_{\rm gravity} \qquad S_{\rm matter} = \int d^7x\sqrt{g}\,\mathcal{L} \qquad S_{\rm gravity} = \int d^7x\sqrt{g}\left(\frac{1}{2\chi_{(7)}}R - \Lambda_{(7)}\right)$$

The metric in seven dimensions

$$g_{MN} = M^2(z,\xi)g^{(4)}_{\mu\nu}dx^\mu dx^\nu - \left(\frac{r_0}{p}\right)^2\left[\frac{L^2(z,\xi)}{M^2(z,\xi)}\left\{\frac{dz^2}{4z(1-z)} + (1-z)d\xi^2\right\} + \frac{K^2(z,\xi)}{M^2(z,\xi)}zd\varphi^2\right]$$

$$ds^2_{(4)} = g^{(4)}_{\mu\nu}\,dx^\mu dx^\nu = dt^2 - \delta_{ij}e^{2H(t)}dx^i dx^j$$

The ansatz for the matter fields $F = F(z,\xi)$ and $\Theta = \Theta(z,\xi)$

$$\vec{n} = (\sin F\cos[\Theta + n\varphi],\ \sin F\sin[\Theta+n\varphi],\ \cos F),\quad \vec{n}\cdot\vec{n} = 1$$

Boundary conditions

$$F(z=0,\xi)=0,\ F(z=1,\xi)=\pi,\quad [\partial_z\Theta(z,\xi)]_{z=0} = [\partial_z\Theta(z,\xi)]_{z=1} = 0 \qquad \text{for}\quad 0\le z\le 1$$
$$\Theta(z,\xi=0)=0,\ \Theta(z,\xi=\pi)=m\pi,\ [\partial_\xi F(z,\xi)]_{\xi=0} = [\partial_\xi F(z,\xi)]_{\xi=\pi}=0 \qquad \text{for}\quad 0\le\xi\le\pi$$
$$L(z=1,\xi)=1,\ M(z=1,\xi)=1,\ K(z=1,\xi)=1 \qquad \text{for}\quad 0\le\xi\le\pi$$

We are thinking that the Neumann conditions for the warped functions might NOT BE NEEDED.

Einstein equations

$$\frac{p^2}{a^2(1-z)}\frac{M^2}{L^2}\Bigg[u\frac{\partial_z^2L}{L} + \frac{\partial_\xi^2L}{L} + u\frac{\partial_z^2M}{M} + \frac{\partial_\xi^2M}{M} + u\frac{\partial_z^2K}{K} + \frac{\partial_\xi^2K}{K} - 2u\frac{\partial_z^2p}{p} - 2\frac{\partial_\xi^2p}{p} - u\left(\frac{\partial_zL}{L}\right)^2 - 3\left(\frac{\partial_\xi L}{L}\right)^2 + 3u\left(\frac{\partial_zM}{M}\right)^2$$
$$+3\left(\frac{\partial_\xi M}{M}\right)^2 + 3u\left(\frac{\partial_zp}{p}\right)^2 + 3\left(\frac{\partial_\xi p}{p}\right)^2 + u\frac{\partial_zK}{K}\frac{\partial_zM}{M} + \frac{\partial_\xi K}{K}\frac{\partial_\xi M}{M} - 2u\frac{\partial_zK}{K}\frac{\partial_zp}{p} - 2\frac{\partial_\xi K}{K}\frac{\partial_\xi p}{p}$$
$$+2(1-z)\left\{(1-3z)\frac{\partial_zL}{L} + 2(1-2z)\frac{\partial_zM}{M} + (3-5z)\frac{\partial_zK}{K} - 4(1-2z)\frac{\partial_zp}{p}\right\} - 3(1-z)\Bigg] - \frac{3\gamma}{M^2} = \alpha(\tau_0 - \beta)$$

$$\frac{p^2}{a^2(1-z)}\frac{M^2}{L^2}\Bigg[3u\frac{\partial_z^2M}{M} + 3\frac{\partial_\xi^2M}{M} + u\frac{\partial_z^2K}{K} + \frac{\partial_\xi^2K}{K} - u\frac{\partial_z^2p}{p} - \frac{\partial_\xi^2p}{p} + 6u\left(\frac{\partial_zM}{M}\right)^2 + 6\left(\frac{\partial_\xi M}{M}\right)^2 + 6u\frac{\partial_zK}{K}\frac{\partial_zM}{M} + 6\frac{\partial_\xi K}{K}\frac{\partial_\xi M}{M}$$
$$-2u\frac{\partial_zK}{K}\frac{\partial_zp}{p} - 2\frac{\partial_\xi K}{K}\frac{\partial_\xi p}{p} - 6u\frac{\partial_zM}{M}\frac{\partial_zp}{p} - 6\frac{\partial_\xi M}{M}\frac{\partial_\xi p}{p}$$
$$+2(1-z)(3-5z)\left\{-3\frac{\partial_zM}{M} + \frac{\partial_zK}{K} - \frac{\partial_zp}{p}\right\} - 2(1-z)\Bigg] - \frac{12\gamma}{M^2} = \alpha(\tau_z + \tau_\xi - 2\beta)$$

$$\frac{p^2}{a^2(1-z)}\frac{M^2}{L^2}\Bigg[3u\frac{\partial_z^2M}{M} + 3\frac{\partial_\xi^2M}{M} + u\frac{\partial_z^2L}{L} + \frac{\partial_\xi^2L}{L} - u\frac{\partial_z^2p}{p} - \frac{\partial_\xi^2p}{p} + 7u\left(\frac{\partial_zM}{M}\right)^2 + 7\left(\frac{\partial_\xi M}{M}\right)^2 + u\left(\frac{\partial_zp}{p}\right)^2 + \left(\frac{\partial_\xi p}{p}\right)^2 + u\frac{\partial_zK}{K}\frac{\partial_zp}{p}$$
$$+\frac{\partial_\xi K}{K}\frac{\partial_\xi p}{p} - u\left(\frac{\partial_zL}{L}\right)^2 - \left(\frac{\partial_\xi L}{L}\right)^2 + 2(1-z)(1-3z)\left\{3\frac{\partial_zM}{M} + \frac{\partial_zL}{L} - \frac{\partial_zp}{p}\right\} - (1-z)\Bigg] - \frac{6\gamma}{M^2} = \alpha(\tau_\varphi - \beta)$$

The localizing property of the gravity on the brane

The Planck mass in 3+1 dimensions

$$M_{\rm pl}^2 = M_{(7)}^5\int_0^{2\pi}\int_{-\pi}^{\pi}\int_1^0 \frac{L(z,\xi)^2K(z,\xi)}{M(z,\xi)}\frac{r_0^3}{2p^3}\,dz\,d\xi\,d\varphi$$

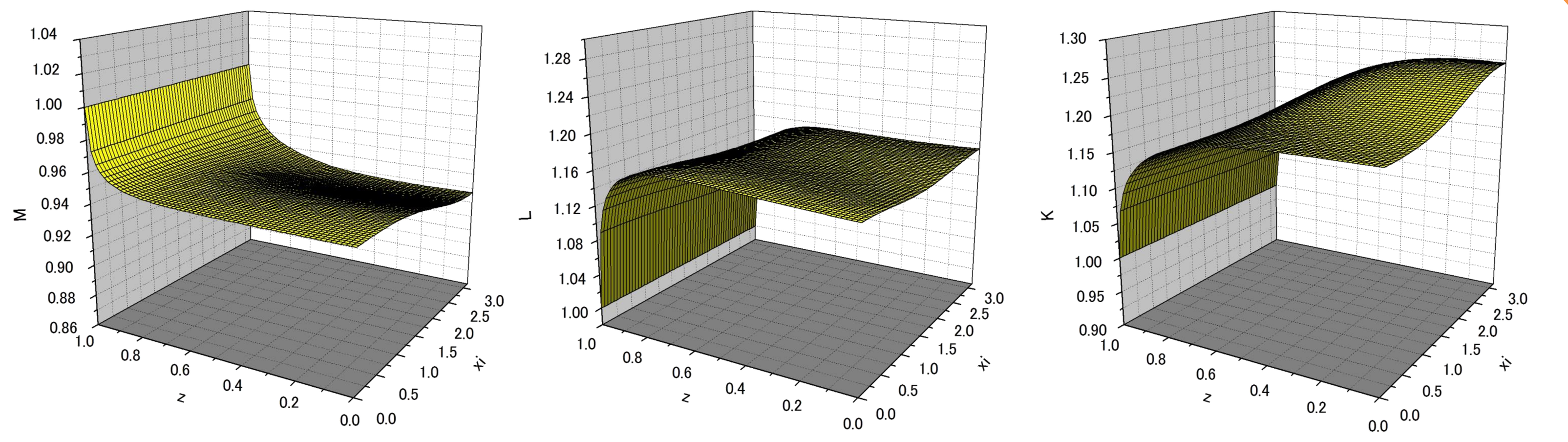
convergence → Gravity localization

divergence → Gravity delocalization

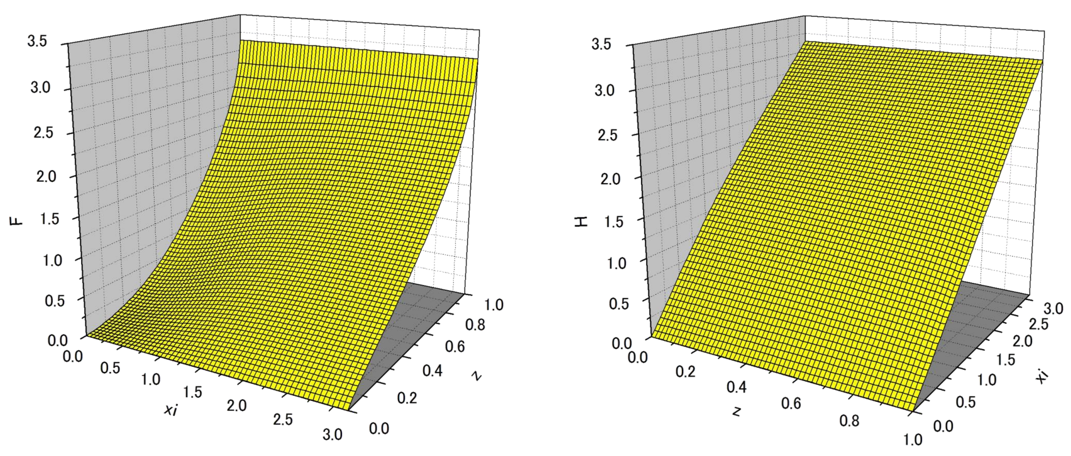
Zero cosmological constant	Positive cosmological constant	Negative cosmological constant	Inflation
×	×	*	*

$$G_{(4)} \propto \frac{1}{M_{\rm pl}^2}$$

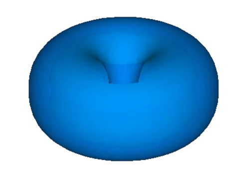
In these results we have been able to find no explicit solutions that the gravity localized on the brane. However, we are thinking that it might be possible to localize the gravity in the case of the negative cosmological constant or the effect of the inflation. After that, we will investigate other solutions in these cases.



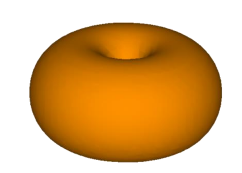
(a) The warped function $M(z,\xi)$ (b) The warped function $L(z,\xi)$ (c) The warped function $K(z,\xi)$



(d) The matter field $F(z,\xi)$ (e) The matter field $\Theta(z,\xi)$

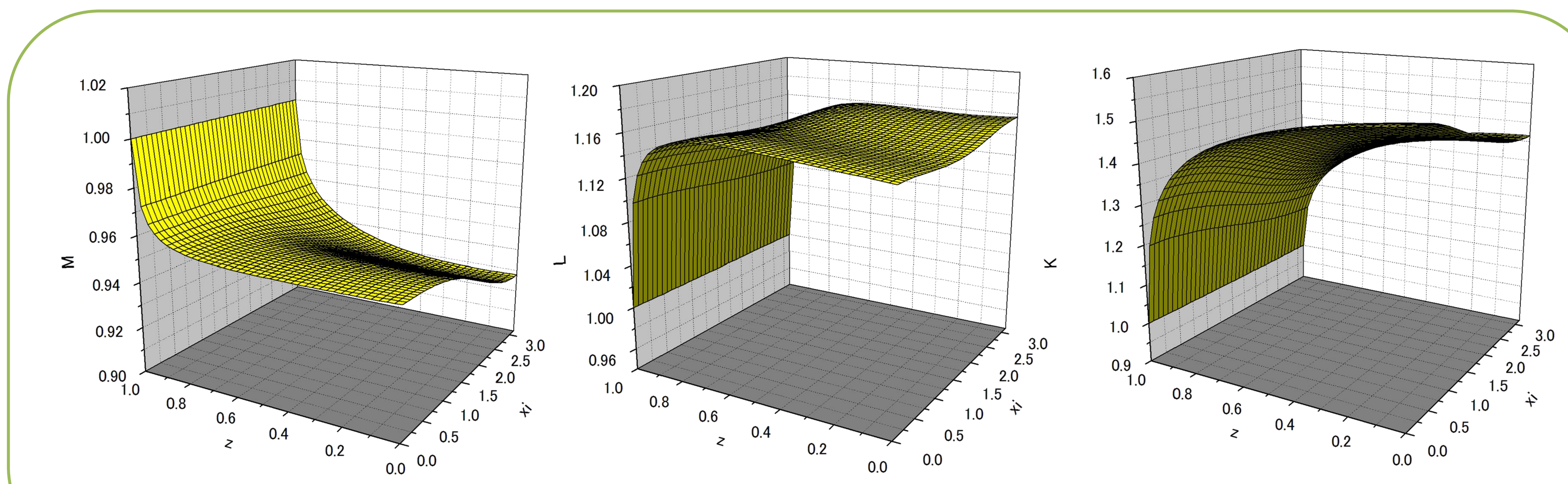


(f) The isosurface of the energy density



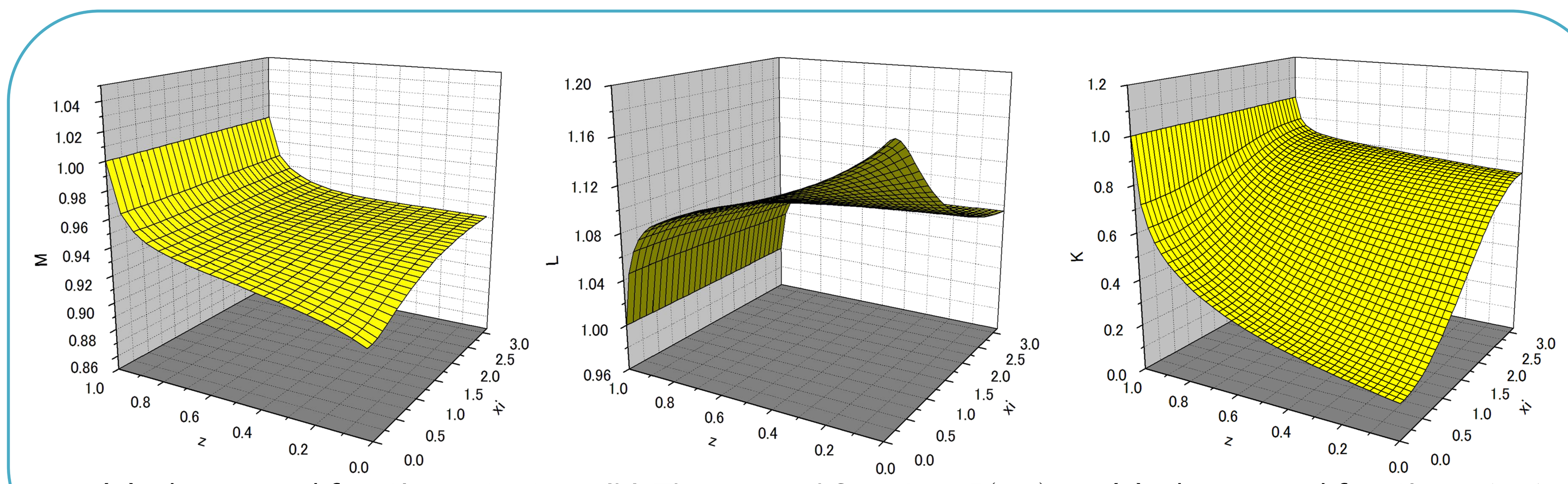
(g) The isosurface of the scalar curvature

The numerical solutions for $\alpha = 1.0\times10^{-2}$, $\beta = 0.0$, and $\gamma = 0.0$



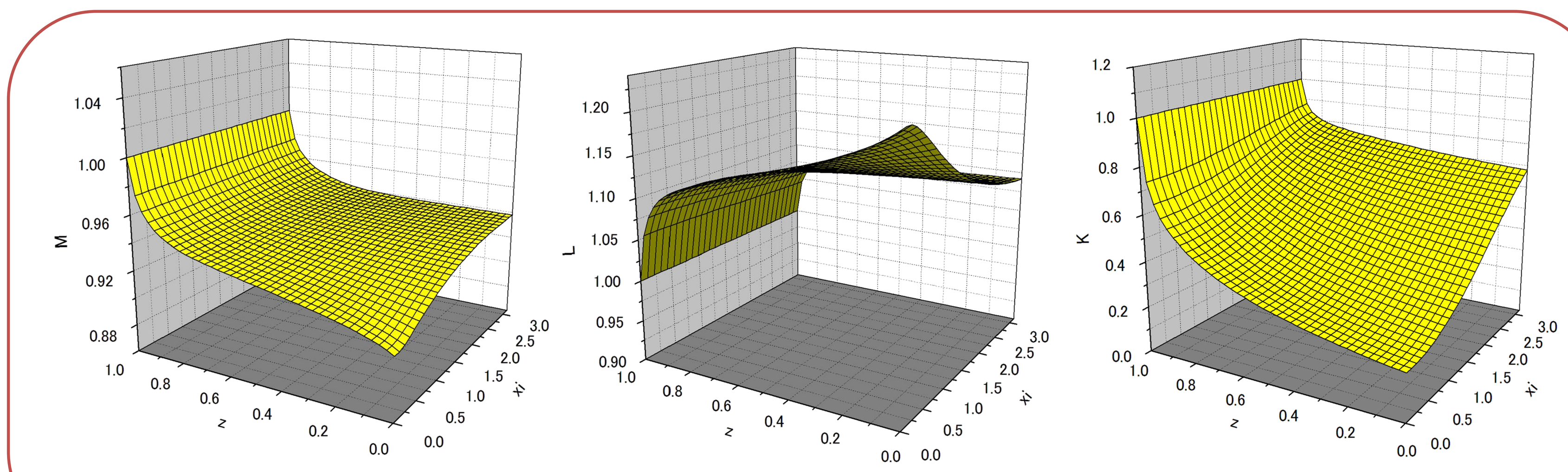
(a) The warped function $M(z,\xi)$ (b) The warped function $L(z,\xi)$ (c) The warped function $K(z,\xi)$

The numerical solutions for $\alpha = 1.0\times10^{-2}$, $\beta = 1.0\times10^{-4}$, and $\gamma = 0.0$



(a) The warped function $M(z,\xi)$ (b) The warped function $L(z,\xi)$ (c) The warped function $K(z,\xi)$

The numerical solutions for $\alpha = 1.0\times10^{-2}$, $\beta = -1.0\times10^{-3}$, and $\gamma = 0.0$



(a) The warped function $M(z,\xi)$ (b) The warped function $L(z,\xi)$ (c) The warped function $K(z,\xi)$

The numerical solutions for $\alpha = 1.0\times10^{-2}$, $\beta = 0.0$, and $\gamma = 1.0\times10^{-5}$