Sine-Square Deformation (SSD) and its Relevance to String Theory

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Based on work with N. Ishibashi
and [arXiv:1404.6343]

Conformal Field Theory in 2 dim.

(Holomorphic part of)

$$\mathcal{H}_{\mathrm{amiltonian}} \sim L_0 = \frac{1}{2\pi i}\oint dz\ z\,T(z)$$

Let us consider a simple (almost trivial) modification to the Hamiltonian

$$Add\ L_1\ and\ L_{-1}$$

L_{-1}: $\frac{1}{2\pi i}\oint dz\ T(z)$

L_1: $\frac{1}{2\pi i}\oint dz\ z^2 T(z)$

L_0, L_1, L_{-1} form $SL(2, \mathbb{R})$ subalgebra of Virasoro algebra

Global Conformal Transformation on the Riemann surface

Introduce

$$L_+ \equiv \frac{L_1 + L_{-1}}{2} \qquad L_- \equiv \frac{L_1 - L_{-1}}{2i}$$

Casimir Operator $(L_0)^2 - (L_+)^2 - (L_-)^2$

Now the modification

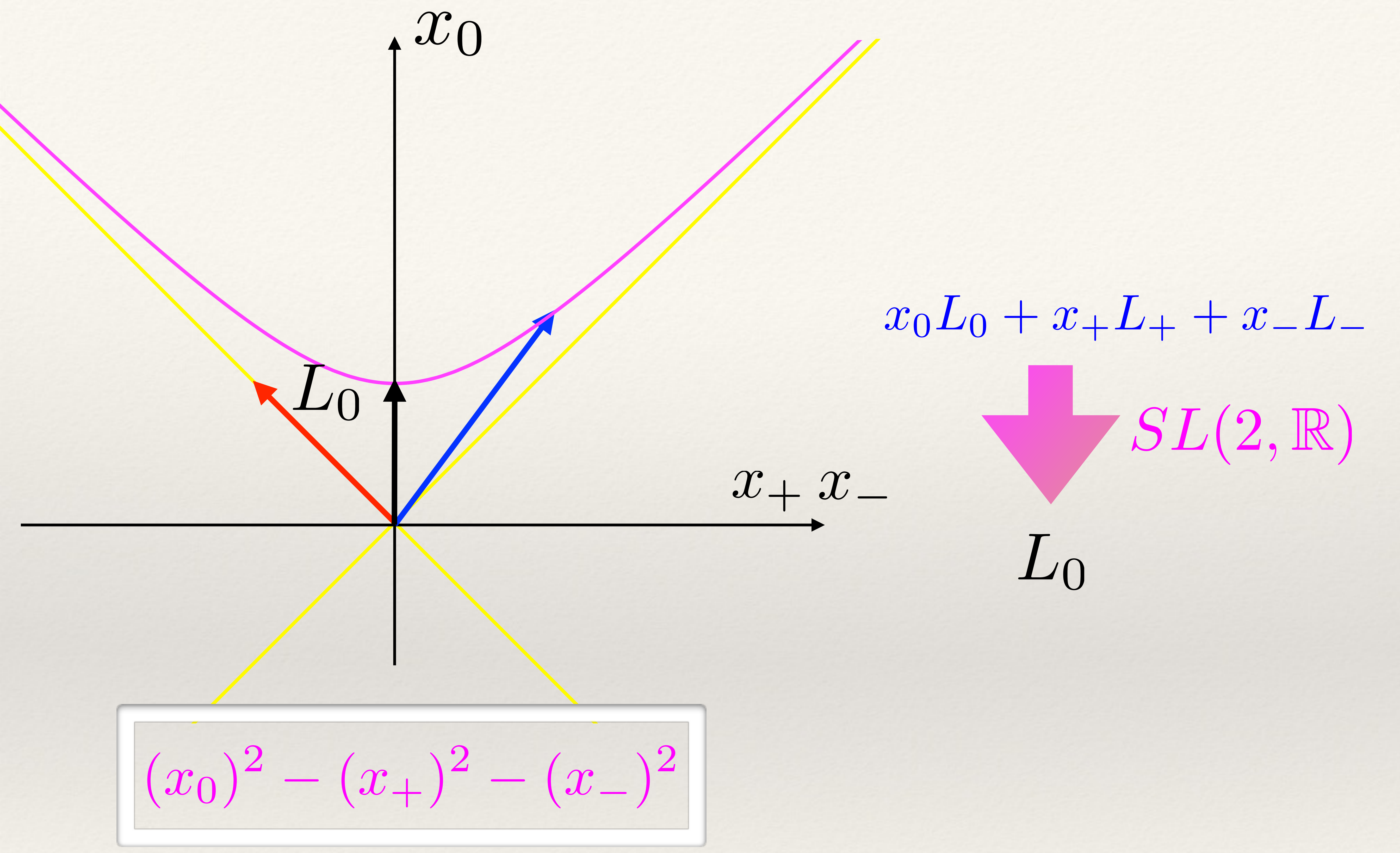
$$e^{-it_0 L_0 - it_+ L_+ - it_- L_-}\Big(x_0 L_0 + x_+ L_+ + x_- L_-\Big)e^{it_0 L_0 + it_+ L_+ + it_- L_-}$$

$SL(2,\mathbb{R})$

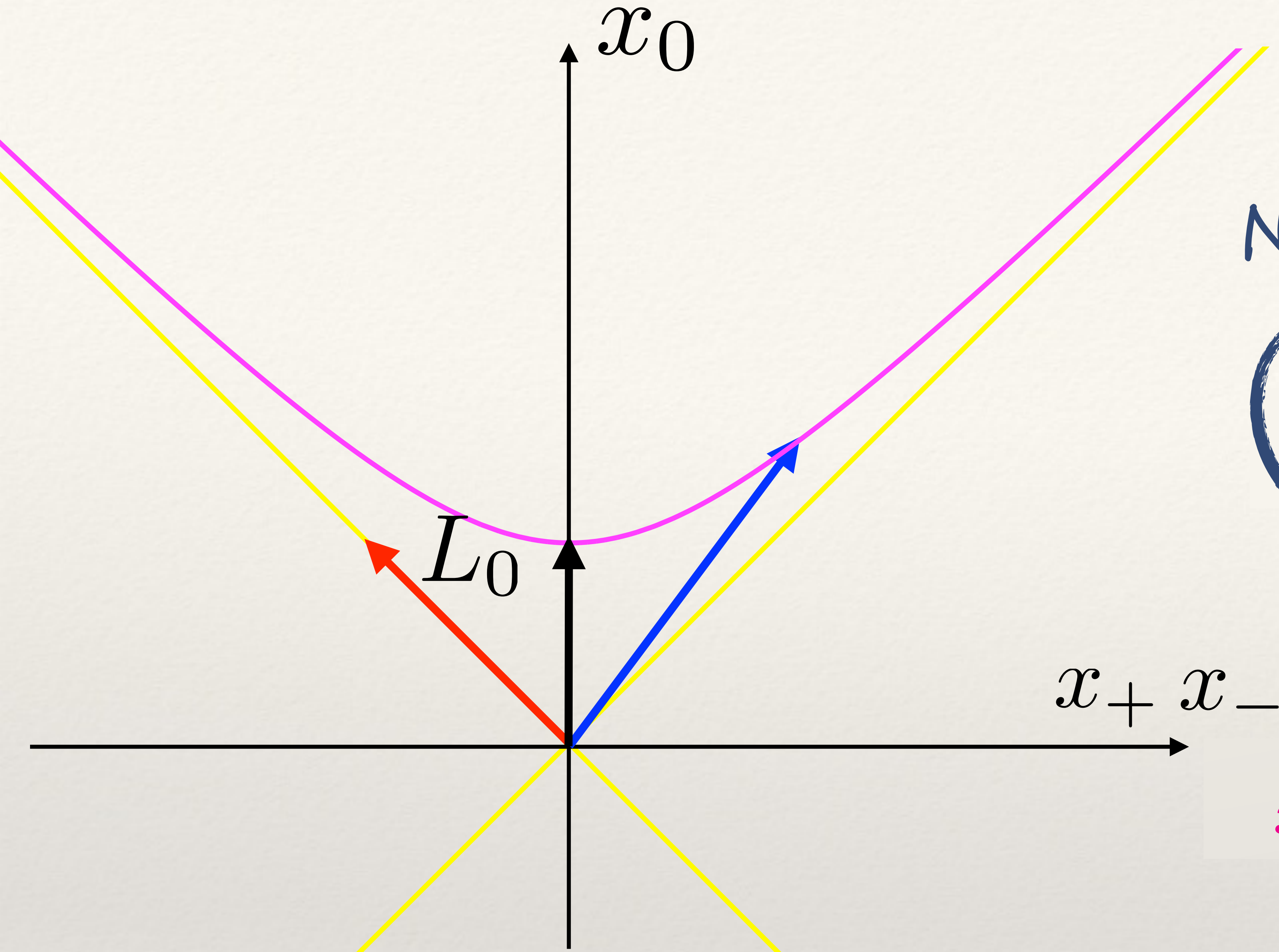
$$= x'_0 L_0 + x'_1 L_+ + x'_2 L_-$$

$$= L_0 \qquad x'_0 = 1, x'_1 = x'_2 = 0$$

$$(x_0)^2 - (x_1)^2 - (x_2)^2 = (x'_0)^2 - (x'_1)^2 - (x'_2)^2$$



x_0
L_0
$x_+ \; x_-$
$x_0 L_0 + x_+ L_+ + x_- L_-$
$SL(2,\mathbb{R})$
L_0
$(x_0)^2 - (x_+)^2 - (x_-)^2$



Non-trival modification

$$L_0 \; - \; L_+$$

$SL(2,\mathbb{R})$

$$x_0 L_0 + x_+ L_+ + x_- L_-$$

with $(x_0)^2 - (x_+)^2 - (x_-)^2 \; = 0$

$$(x_0)^2 - (x_+)^2 - (x_-)^2$$

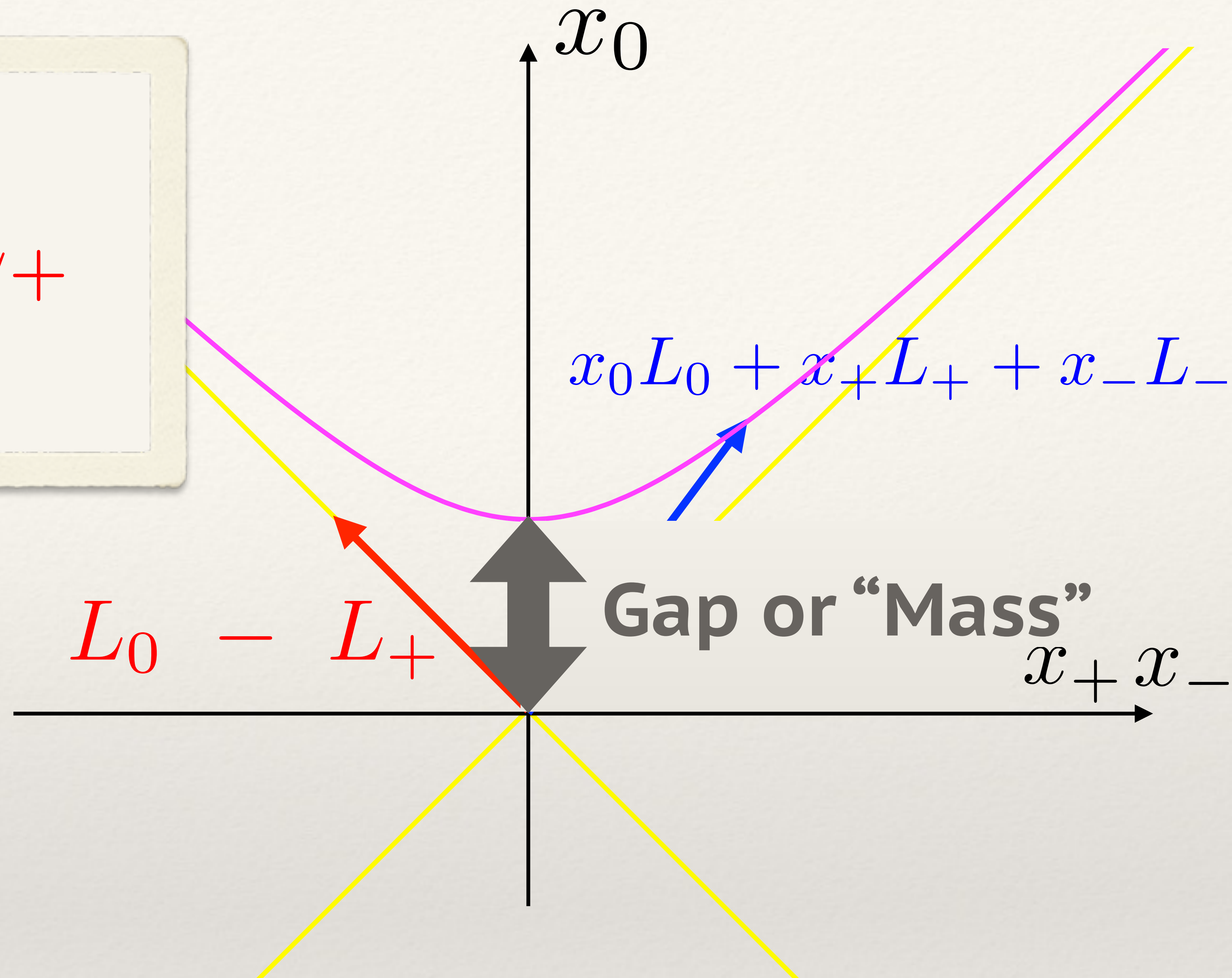
No way to realize

$$x_0 = 1, \;\; x_+ = x_- = 0$$

What does

$\mathcal{H} \sim L_0 - L_+$

suggest?



"Continuous Spectrum"

c.f. "Level" structure of excited states in CFT

$$L_0 \; - \; L_+$$

To motivate further, let me introduce an interesting work by A. Gendiar, R. Krcmar and T. Nishino

Prog. Theor. Phys. 122 (2009) 953; ibid. 123 (2010) 393.

They Started With

Gendiar, Krcmar, Nishino (2009)

1d systems w/ nearest neighbor coupling

$$\mathcal{H} = -\sum J_{n,n+1}\left(\sigma_n \cdot \sigma_{n+1}\right)$$

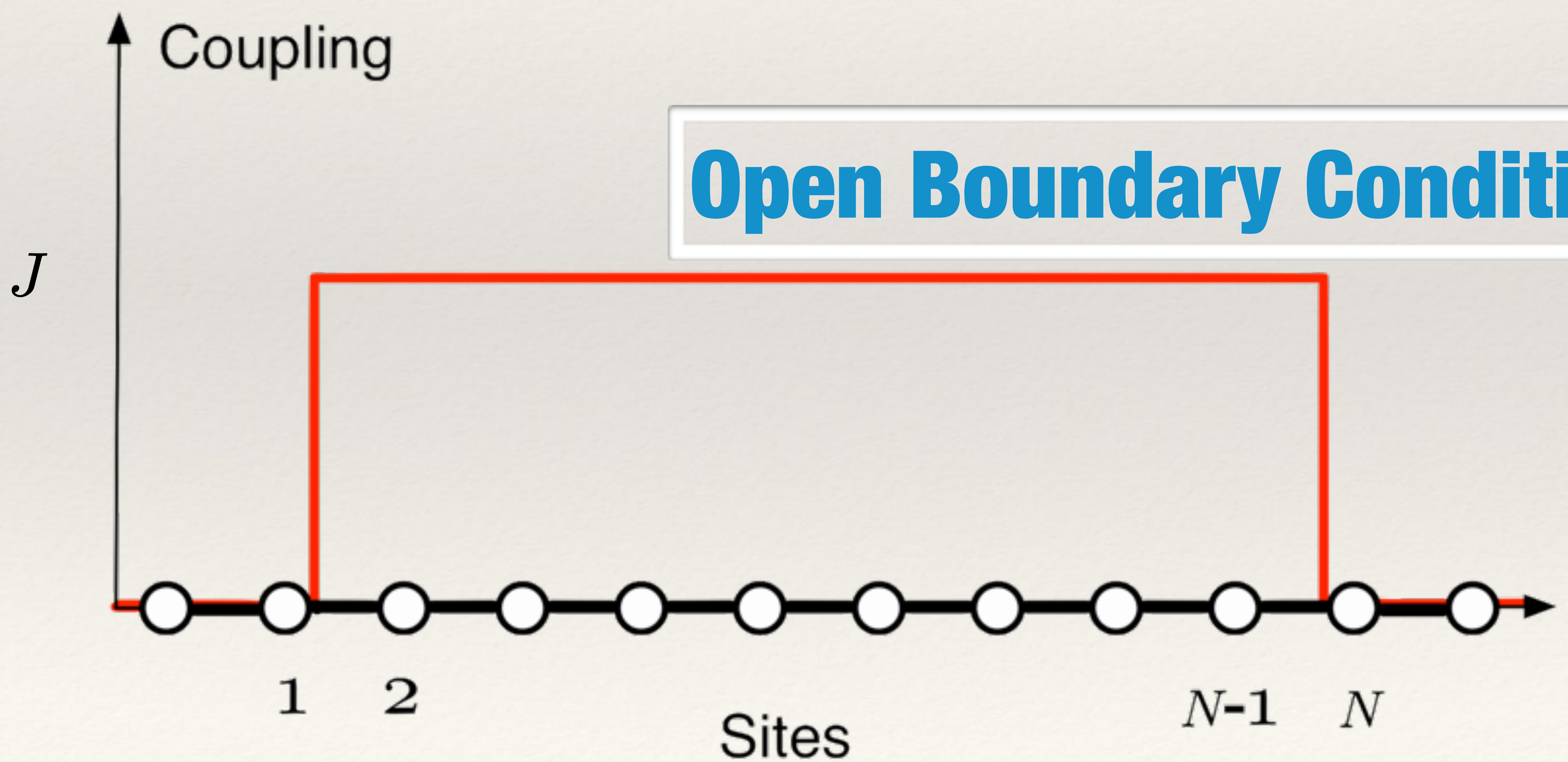
and

Open Boundary Condition

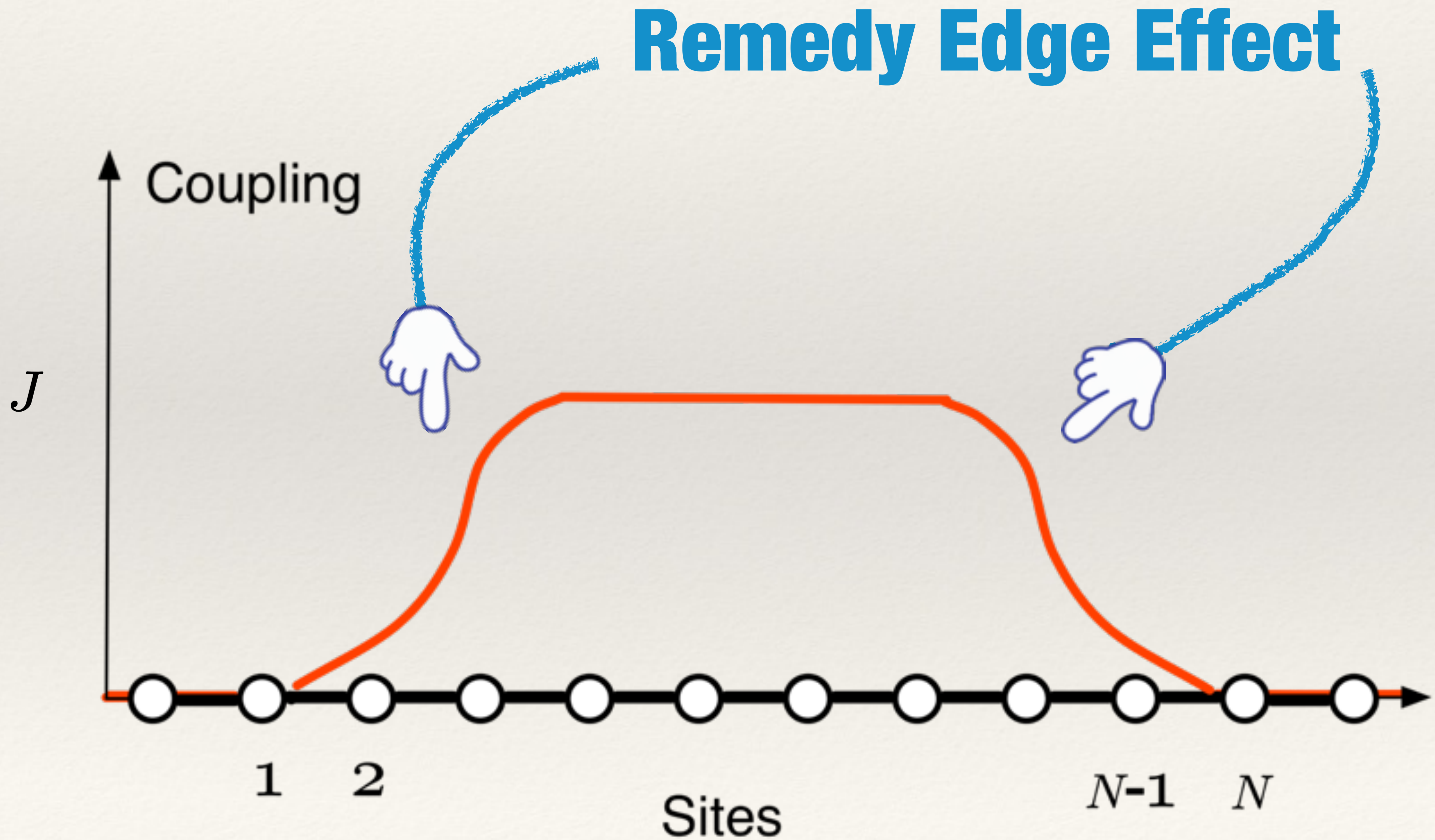
$$\mathcal{H} = -\sum J_{n,n+1} \left(\sigma_n \cdot \sigma_{n+1}\right)$$

$$J_{1,2} = J_{2,3} = \cdots = J_{N-1,N} \equiv J$$

$$J_{0,1} = J_{N,N+1} = 0$$



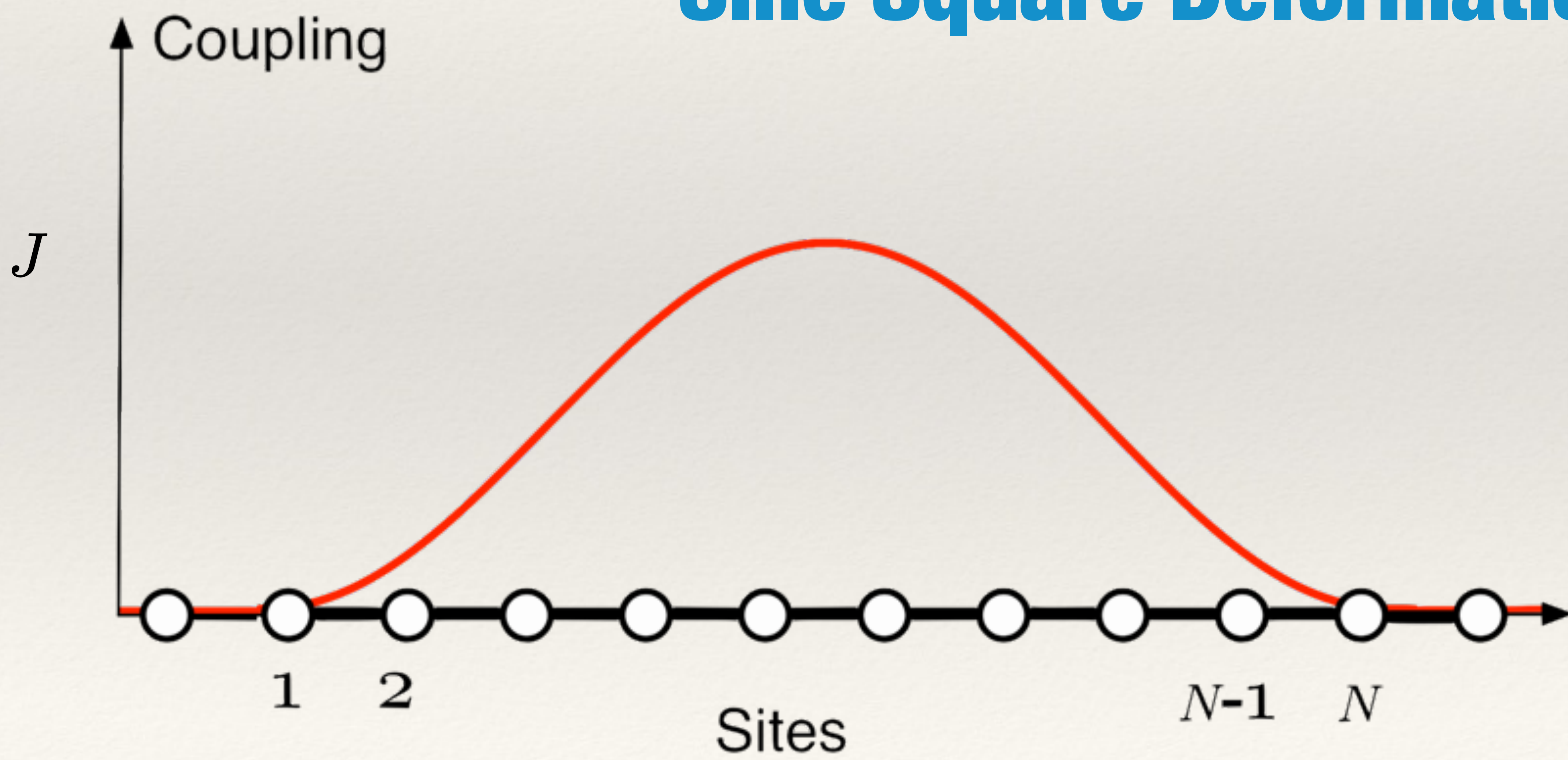
$$\mathcal{H} = -\sum J_{n,n+1} (\sigma_n \cdot \sigma_{n+1})$$

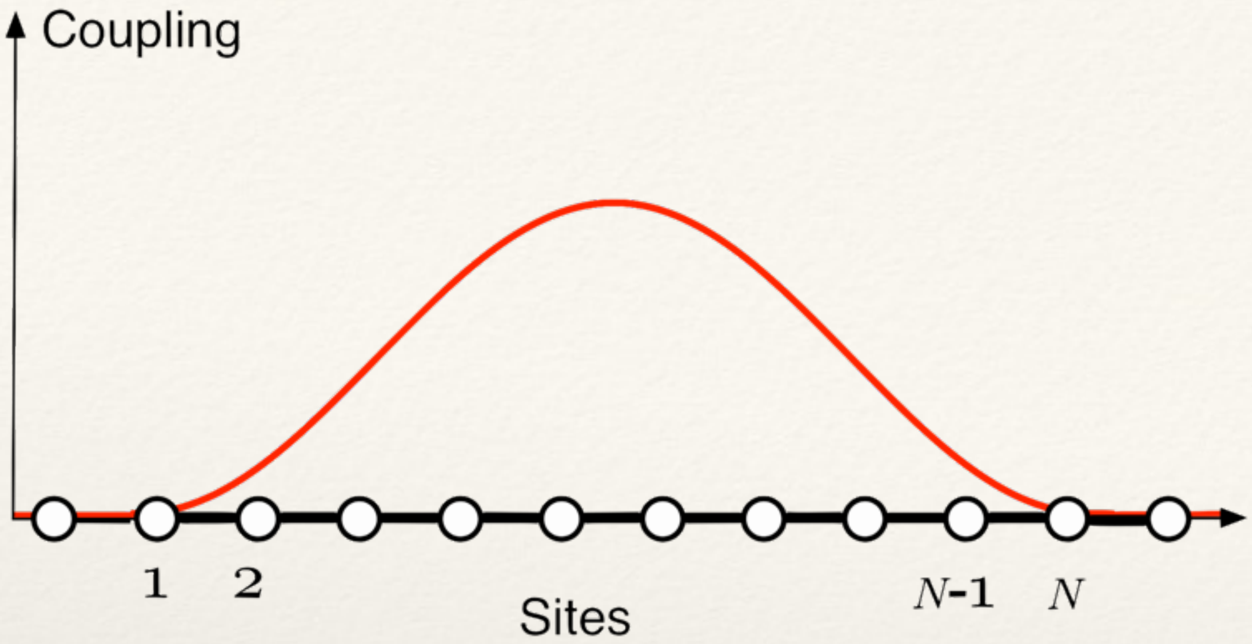


$$\mathcal{H} = -\sum J_{n,n+1} \left(\sigma_n \cdot \sigma_{n+1} \right)$$

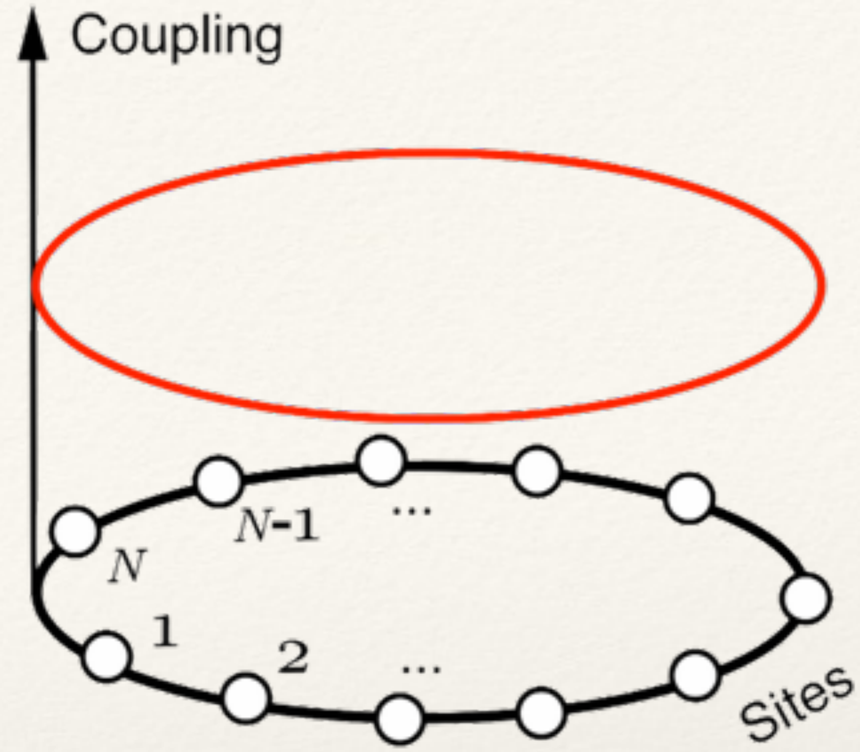
$$J_{i,i+1} \equiv J \sin^2 \left(\frac{n}{N} \pi \right)$$

Sine Square Deformation





Sine Square Deformation



Closed

Same Ground State

A. Gendiar, R. Krcmar and T. Nishino

Prog. Theor. Phys. 122 (2009) 953; ibid. 123 (2010) 393.

The mechanism behind this deformation was clarified by H. Katsura and his collaborators.

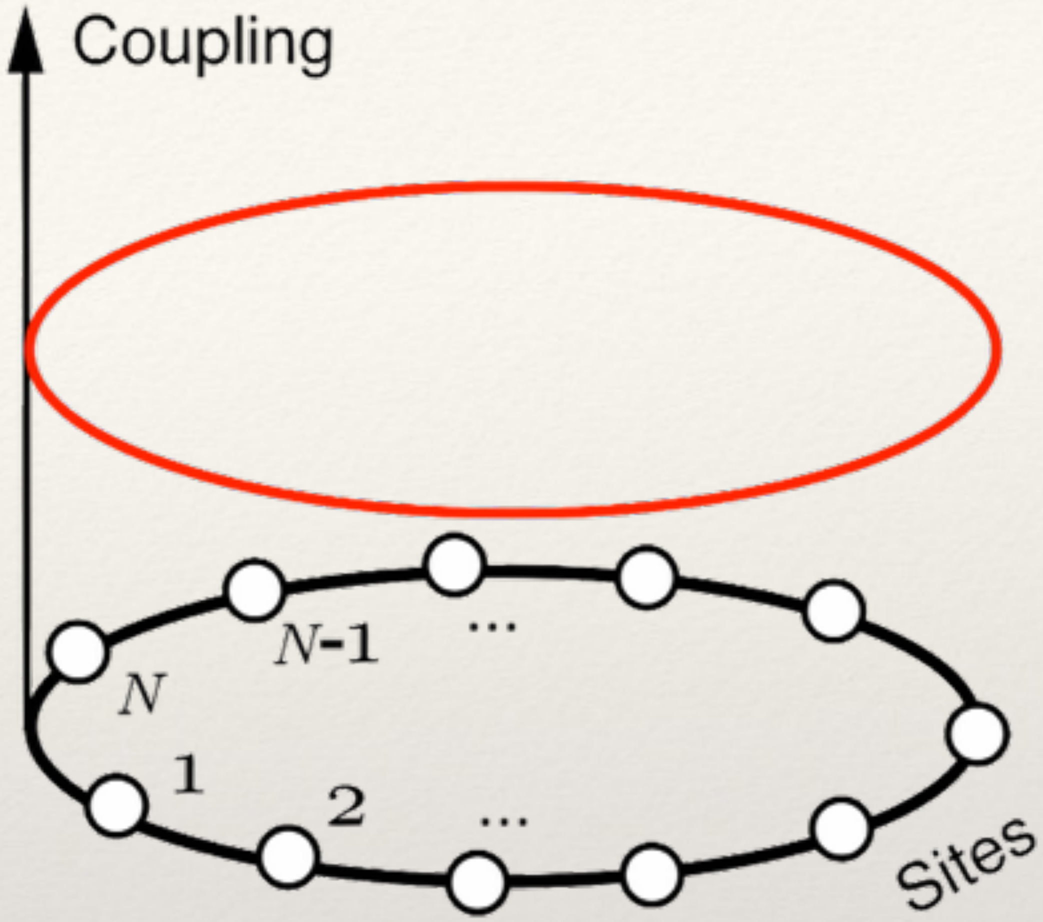
H. Katsura, J. Phys. A:Math.Theor. 44 (2011) 252001

I. Maruyama, H. Katsura and T. Hikihara, Phys.Rev.B84(2011)165132

Closed Hamlitonian

$$\mathcal{H}_c = \sum_{n=1}^{N} h_{n,n+1}$$

$h_{N,1} \neq 0$



$$\mathcal{H}_{\pm 1} = \sum_{n=1}^{N} e^{\pm 2\pi i \frac{n}{N}} h_{n,n+1}$$

$$\mathcal{H}_c = \sum_{n=1}^{N} h_{n,n+1}$$

$$\mathcal{H}_{\pm 1} = \sum_{n=1}^{N} e^{\pm 2\pi i \frac{n}{N}} h_{n,n+1}$$

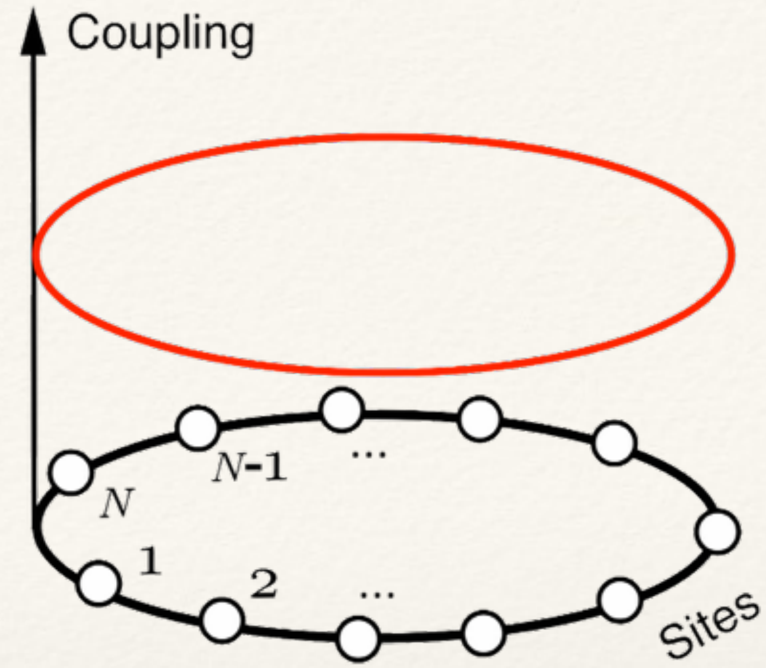
$$\mathcal{H}_{SSD} \equiv \frac{1}{2}\mathcal{H}_c - \frac{1}{4}(\mathcal{H}_{+1} + \mathcal{H}_{-1})$$

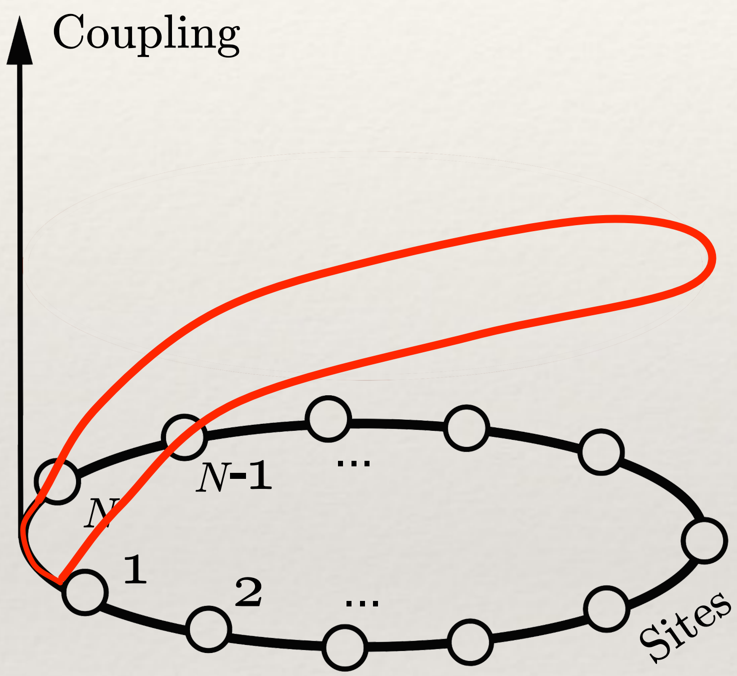
$$\frac{1}{2} - \frac{1}{4}\left(e^{2\pi i \frac{n}{N}} + e^{-2\pi i \frac{n}{N}}\right) = \frac{1}{2}\left(1 - \cos 2\pi \frac{n}{N}\right)$$

$$= \sin^2 \pi \frac{n}{N}$$

$$\Rightarrow \quad \mathcal{H}_{SSD} = \sum_{n=1}^{N} \sin^2(\pi \frac{n}{N}) h_{n,n+1}$$

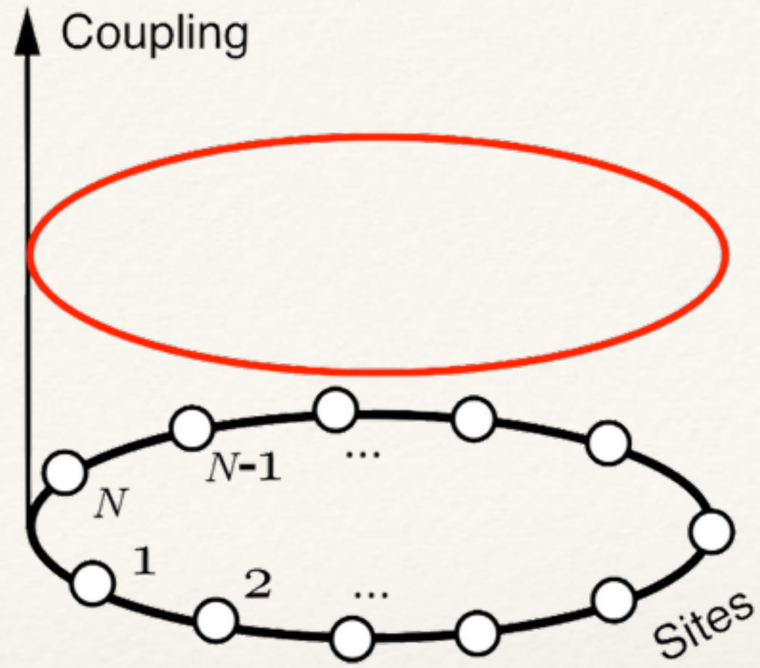
$$\mathcal{H}_c = \sum_{n=1}^{N} h_{n,n+1}$$

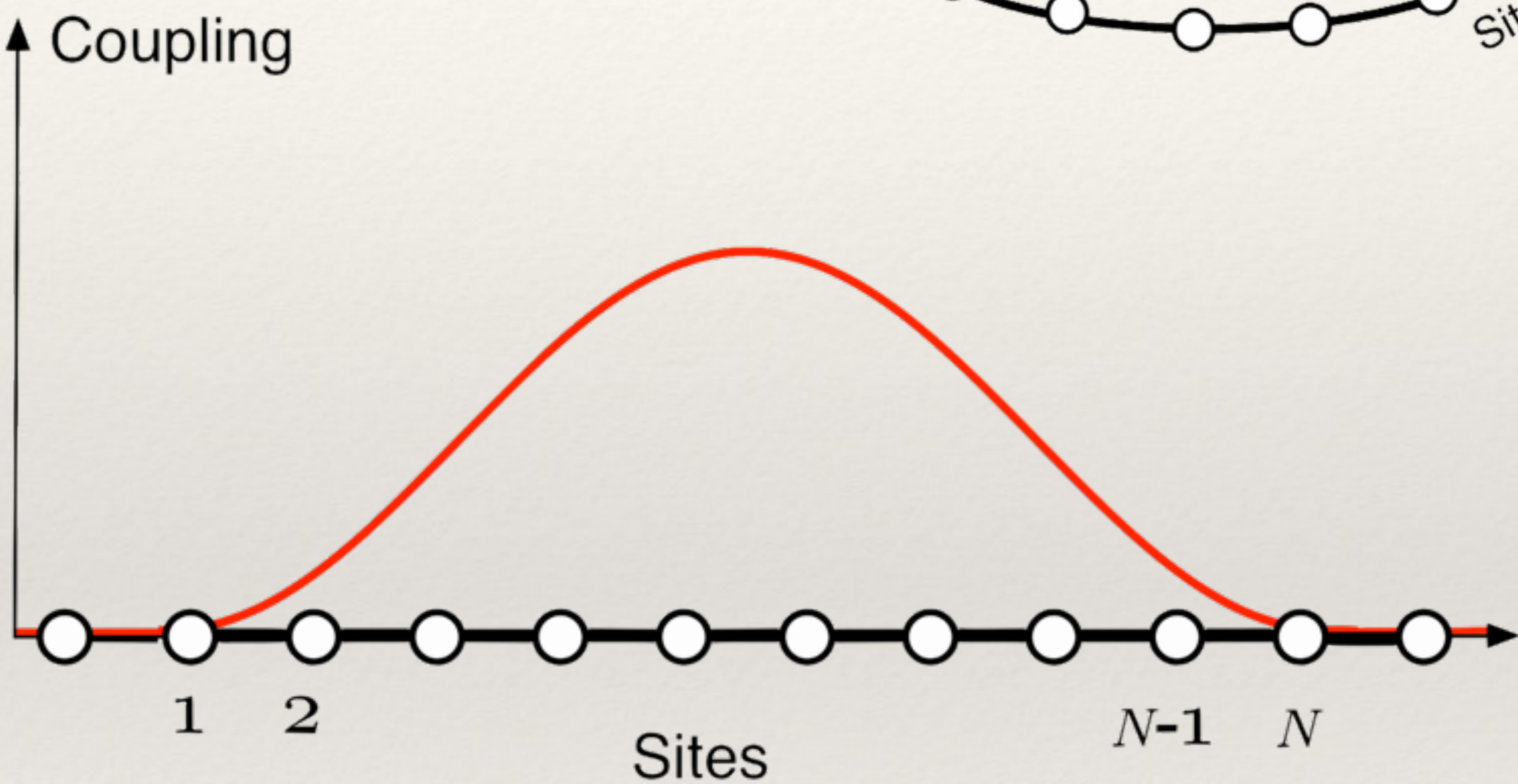




$$\mathcal{H}_{SSD} = \sum_{n=1}^{N} \sin^2(\pi \frac{n}{N}) h_{n,n+1}$$

$$\mathcal{H}_c = \sum_{n=1}^{N} h_{n,n+1}$$





$$\mathcal{H}_{SSD} = \sum_{n=1}^{N} \sin^2(\pi \frac{n}{N}) h_{n,n+1}$$

Katsura (2011), Maruyama, Katsura, Hikihara (2011)

Provided

- ☑ $\mathcal{H}_{\pm 1}$ annihilates $\mathcal{H}_c$'s vacuum $|\mathrm{vac}\rangle$

  $$\mathcal{H}_{\pm 1}|\mathrm{vac}\rangle = 0$$

- ☑ Either $\mathcal{H}_{\mathrm{SSD}}$'s vacuum is unique or $\mathcal{H}_{\mathrm{SSD}}$ *is bounded below*

$|\mathrm{vac}\rangle$ *is also* $\mathcal{H}_{\mathrm{SSD}}$'s vacuum

$$\mathcal{H}_{\pm 1}|\text{vac}\rangle = 0$$

$$\mathcal{H}_{SSD} \equiv \frac{1}{2}\mathcal{H}_c - \frac{1}{4}(\mathcal{H}_{+1} + \mathcal{H}_{-1})$$

$$\mathcal{H}_c|\text{vac}\rangle = E_0|\text{vac}\rangle$$

$$\mathcal{H}_{\text{SSD}}|\text{vac}\rangle = \frac{E_0}{2}|\text{vac}\rangle$$

2D Cft On A Cylinder

$$\mathcal{H}_c = \frac{2\pi}{\ell}\left(L_0 + \bar{L}_0\right) - \frac{\pi c}{6\ell}$$

$$\mathcal{H}_{\pm 1} = \frac{2\pi}{\ell}\left(L_{\pm 1} + \bar{L}_{\mp 1}\right)$$

$L_0|0\rangle = \bar{L}_0|0\rangle = 0$ ⇔ $\mathcal{H}_c$'s vacuum $|0\rangle$

sl(2,c)invariance $\quad L_{\pm 1}|0\rangle = \bar{L}_{\pm 1}|0\rangle = 0$

⇒ $\mathcal{H}_{\pm 1}|\text{vac}\rangle = 0$

$$\mathcal{H}_c = \frac{2\pi}{\ell}\left(L_0 + \bar{L}_0\right) - \frac{\pi c}{6\ell} \qquad \mathcal{H}_{\pm 1} = \frac{2\pi}{\ell}\left(L_{\pm 1} + \bar{L}_{\mp 1}\right)$$

$$\mathcal{H}_{SSD} = \frac{1}{2}\mathcal{H}_c - \frac{1}{4}(\mathcal{H}_{+1} + \mathcal{H}_{-1})$$

$$\sim \frac{1}{2}\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) + \text{(anti-holomorphic)}$$

$$\mathcal{H}_{\mathrm{SSD}}|0\rangle = \frac{E_0}{2}|0\rangle \quad \Longleftrightarrow \quad \mathcal{H}_c|0\rangle = E_0|0\rangle$$

$$E_0 = -\frac{\pi c}{6l}$$

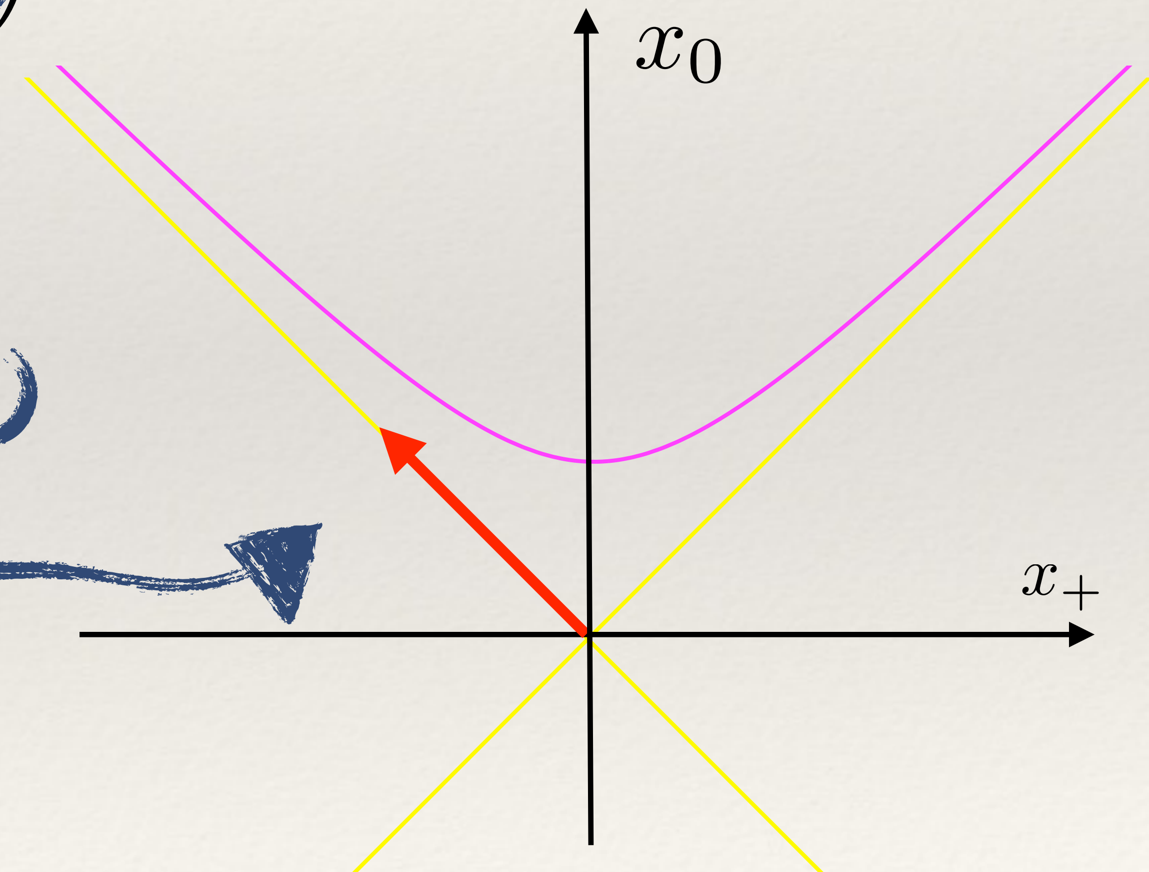
H. Katsura, J. Phys. A: Math. Theor. 45 (2012) 115003.

$$\mathcal{H}_c = \frac{2\pi}{\ell}\left(L_0 + \bar{L}_0\right) - \frac{\pi c}{6\ell} \qquad \mathcal{H}_{\pm 1} = \frac{2\pi}{\ell}\left(L_{\pm 1} + \bar{L}_{\mp 1}\right)$$

$$\mathcal{H}_{SSD} = \frac{1}{2}\mathcal{H}_c - \frac{1}{4}\left(\mathcal{H}_{+1} + \mathcal{H}_{-1}\right)$$

$$\sim \frac{1}{2}\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) + \text{(anti-holomorphic)}$$

$$L_0 \; - \; L_+$$



$$\mathcal{H}_{\mathrm{SSD}}|0\rangle = \frac{E_0}{2}|0\rangle \quad \Longleftrightarrow \quad \mathcal{H}_c|0\rangle = E_0|0\rangle$$

$$E_0 = -\frac{\pi c}{6l}$$

Non-trival modification

H. Katsura, J. Phys. A: Math. Theor. 45 (2012) 115003.

Implication For String Theory?

Non-Trivial Modification (Deformation)

Affects Boundary Condition

World Sheet Dynamics Of

D-Brane

Open/Closed Duality

?

Implication For String Theory?

Non-Trivial Modification (Deformation)

Modification Of World Sheet Metric

Affects B

World Sheet Dynamics Of

D-Brane

Open/Closed Duality

Worth Further Exploration

Let Me Elaborate

- Boundary condition — set by hand
  - Compartmentalize characteristic physics
  - Useful to concentrate each idiosyncrasy
- Often non-perturbative effects involve different boundary conditions
  - D-brane, open closed duality
- Understanding Non-perturbative dynamics in terms of the world sheet gravity

Lagrangean

$$\mathcal{L}_\alpha = \frac{1}{2}\int_0^\ell dx\,\{(\partial_t\varphi)\,F(x)\,(\partial_t\varphi) - (\partial_x\varphi)\,G(x)\,(\partial_x\varphi)\}$$

$$F(x) = N\sum_{k\in\mathbb{Z}} r^{|k|} e^{2\pi i k x/\ell}$$

$$G(x) = 1 - \alpha\cos\frac{2\pi x}{\ell}$$

$$= \frac{g\ell}{2}\sum_{n,k}\dot{\phi}_n\dot{\phi}_{-n-k} N r^{|k|}$$

$$-\frac{2\pi^2 g}{\ell}\left\{n^2\phi_n\phi_{-n} - \frac{\alpha}{2}\left(n\,(n+1)\,\phi_n\phi_{-n-1} + n\,(n-1)\,\phi_n\phi_{-n+1}\right)\right\}$$

$$\mathcal{L}_\alpha = \frac{g\ell}{2}\sum_{n,k}\dot{\phi}_n\dot{\phi}_{-n-k}Nr^{|k|}$$

$$-\frac{2\pi^2 g}{\ell}\left\{n^2\phi_n\phi_{-n} - \frac{\alpha}{2}\left(n\left(n+1\right)\phi_n\phi_{-n-1} + n\left(n-1\right)\phi_n\phi_{-n+1}\right)\right\}$$

Now conjugate momenta are

$$\pi_n = g\ell\sum_k Nr^{|k|}\dot{\phi}_{-n-k}$$

$$\mathcal{H}_\alpha = \sum_n \pi_n\dot{\phi}_n - \mathcal{L}_\alpha$$

Provided $r = \frac{1-\sqrt{1-\alpha^2}}{\alpha}\,, \qquad N = \frac{1}{\sqrt{1-\alpha^2}}$

$$= \frac{1}{2g\ell}\Big[\pi_n\pi_{-n} - \frac{\alpha}{2}\pi_n\pi_{-n+1} - \frac{\alpha}{2}\pi_n\pi_{-n-1}$$

$$+\left(2\pi g\right)^2 n^2\phi_n\phi_{-n} - \frac{\alpha}{2}\left(2\pi g\right)^2 n\left(n+1\right)\phi_n\phi_{-n-1}$$

$$-\frac{\alpha}{2}\left(2\pi g\right)^2 n\left(n-1\right)\phi_n\phi_{-n+1}\Big]$$

$$\mathcal{H}_\alpha = \frac{1}{2g\ell}\Big[\pi_n\pi_{-n} - \frac{\alpha}{2}\pi_n\pi_{-n+1} - \frac{\alpha}{2}\pi_n\pi_{-n-1}$$

$$+ (2\pi g)^2 n^2 \phi_n\phi_{-n} - \frac{\alpha}{2}(2\pi g)^2 n(n+1)\phi_n\phi_{-n-1}$$

$$- \frac{\alpha}{2}(2\pi g)^2 n(n-1)\phi_n\phi_{-n+1}\Big]$$

$$= \frac{2\pi}{\ell}\left(L_0 + \bar{L}_0 - \frac{\alpha}{2}\left(L_1 + \bar{L}_1 + L_{-1} + \bar{L}_{-1}\right)\right)$$

$$\mathcal{H}_{+1} + \mathcal{H}_{-1}$$

$$= \frac{2\pi}{\ell}\left(L_1 + \bar{L}_1 + L_{-1} + \bar{L}_{-1}\right) = \frac{1}{2g\ell}\sum_{n\in\mathbb{Z}}\Big\{\pi_n\pi_{-(n+1)} + \pi_n\pi_{-(n-1)}$$

$$+ \quad (2\pi g)^2 n(n+1)\phi_n\phi_{-(n+1)} + (2\pi g)^2 n(n-1)\phi_n\phi_{-(n-1)}\Big\}$$

$$\mathcal{L}_\alpha = \frac{1}{2}\int_0^\ell dx\,\{(\partial_t\varphi)\,F(x)\,(\partial_t\varphi) - (\partial_x\varphi)\,G(x)\,(\partial_x\varphi)\}$$

$$F(x) = N\sum_{k\in\mathbb{Z}} r^{|k|}e^{2\pi i k x/\ell} \qquad = N\delta(x)$$

$$\mathcal{H}_\alpha = \frac{2\pi}{\ell}\left(L_0 + \bar{L}_0 - \frac{\alpha}{2}\left(L_1 + \bar{L}_1 + L_{-1} + \bar{L}_{-1}\right)\right)$$

$$G(x) = 1 - \alpha\cos\frac{2\pi x}{\ell}$$

$$= 2\sin^2\frac{\pi x}{\ell}$$

$$N \equiv \frac{1}{\sqrt{1-\alpha^2}}.$$

$$\alpha = 1$$

$$r \equiv \frac{1-\sqrt{1-\alpha^2}}{\alpha}$$

$$\mathcal{H}_{SSD} = \frac{1}{2}\mathcal{H}_c - \frac{1}{4}(\mathcal{H}_{+1} + \mathcal{H}_{-1})$$

$$= \frac{\pi}{\ell}\left(L_0 + \bar{L}_0 - \frac{L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}}{2}\right) - \frac{\pi c}{12\ell}$$

Worldsheet Metric g^{ab}

g^{00} $\updownarrow$ g^{11} $\updownarrow$

$$\mathcal{L}_{\rm SSD} = \frac{1}{2}\int_0^{\ell} dx \left\{ (\partial_t \varphi)\, N\delta(x)\, (\partial_t \varphi) - (\partial_x \varphi)\, 2\sin^2\frac{\pi x}{\ell}\, (\partial_x \varphi) \right\}$$

$N \to \infty$

$$\mathcal{H}_{\rm SSD} = \frac{\pi}{\ell}\left(L_0 + \bar{L}_0 - \frac{L_1 + L_{-1} + \bar{L}_1 + \bar{L}_{-1}}{2} \right) - \frac{\pi c}{12\ell}$$

Non-Trivial Divergence Confirmed

Difficult To Tackle Directly

Explore States Other Than $|0\rangle$

Other Than $|0\rangle$

- “Excited” states - work in progress

A candidate for the implied “continuous” states

$$\sum_{r=2}^{\infty}\sum_{k=3}^{r+1}\frac{\mu^k}{k!}\frac{(r-2)!}{(k-3)!(r-k+1)!}L_{-r}|0\rangle$$

μ : continuous parameter

- Exotic states

Other Than $|0\rangle$

- Exotic states

by H. Katsura

$$\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) \sum_{r=2}^{\infty} L_{-r}|0\rangle = 0$$

The lowest energy state for $\mathcal{H}_{\text{SSD}}$

Other Than $|0\rangle$

- Exotic states

by H. Katsura

$$\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) \qquad = 0$$

The lowest energy state for $\mathcal{H}_{\mathrm{SSD}}$

$$\left|\sum_{r=2}^{\infty} L_{-r}|0\rangle\right| \longrightarrow \infty$$

Other Than $|0\rangle$

- Exotic states

by H. Katsura

$$\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) \quad \sum_{r=2}^{\infty}\sum_{k=3}^{r+1} \frac{\mu^k}{k!} \frac{(r-2)!}{(k-3)!(r-k+1)!} L_{-r}|0\rangle$$

te

$$\left|\sum_{r=2}^{\infty} L_{-r}|0\rangle\right| \longrightarrow \infty$$

So as the previously mentioned candidate states

Other Than $|0\rangle$

- Exotic states

$$L_0|h\rangle = h|h\rangle$$

$$L_n|h\rangle = 0$$

$$(n > 0)$$

$$\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) e^{L_{-1}}|h\rangle = 0$$

The lowest energy state for $\mathcal{H}_{\rm SSD}$

Other Than $|0\rangle$

- Exotic states

$$L_0|h\rangle = h|h\rangle$$

$$L_n|h\rangle = 0$$

$$(n > 0)$$

$$\left(L_0 - \frac{L_1 + L_{-1}}{2}\right) = 0$$

The lowest energy state for $\mathcal{H}_{\text{SSD}}$

$$\left|e^{L_{-1}}|h\rangle\right| \longrightarrow \infty$$

Need More Work To Understand The Whole Structure

Summary

Sine Square Deformation

String Theory

Duality

Divergence In Worldsheet

Dynamics

Condensation of world sheet metric

Thank You For Your Attention