Certification of many-body quantum states

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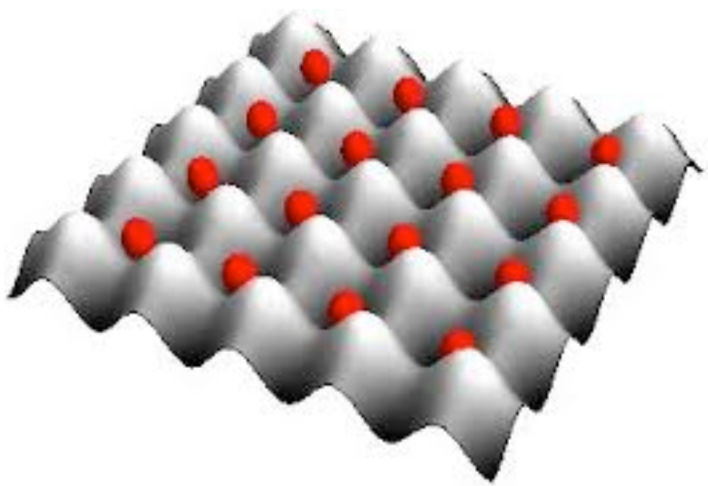
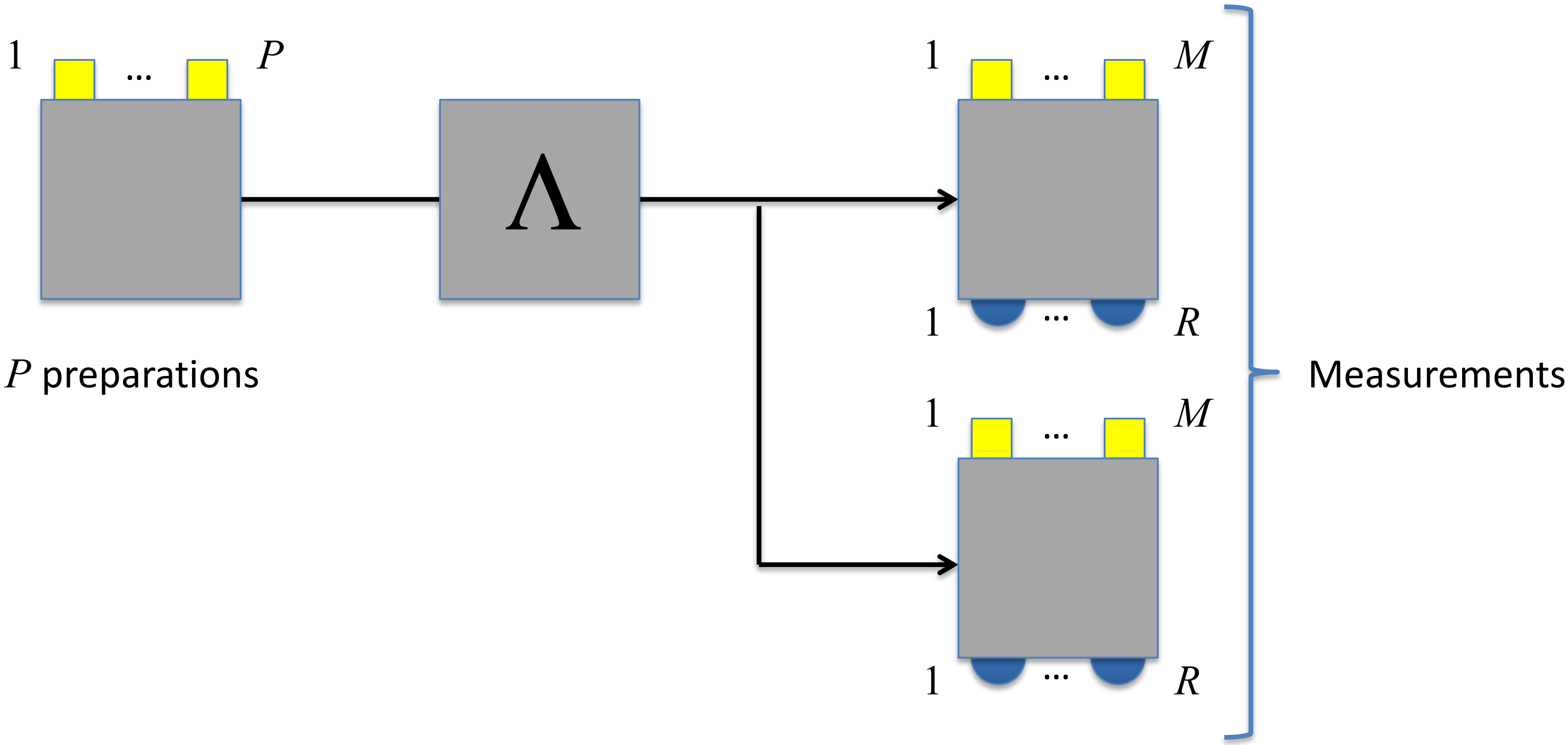
The problem



Does this complex quantum system have a given quantum property? E.g.: is it entangled?

Quantum estimation

Standard scenario: (i) different quantum states are prepared, (ii) a quantum operation is applied to these states and (iii) measurements are performed.



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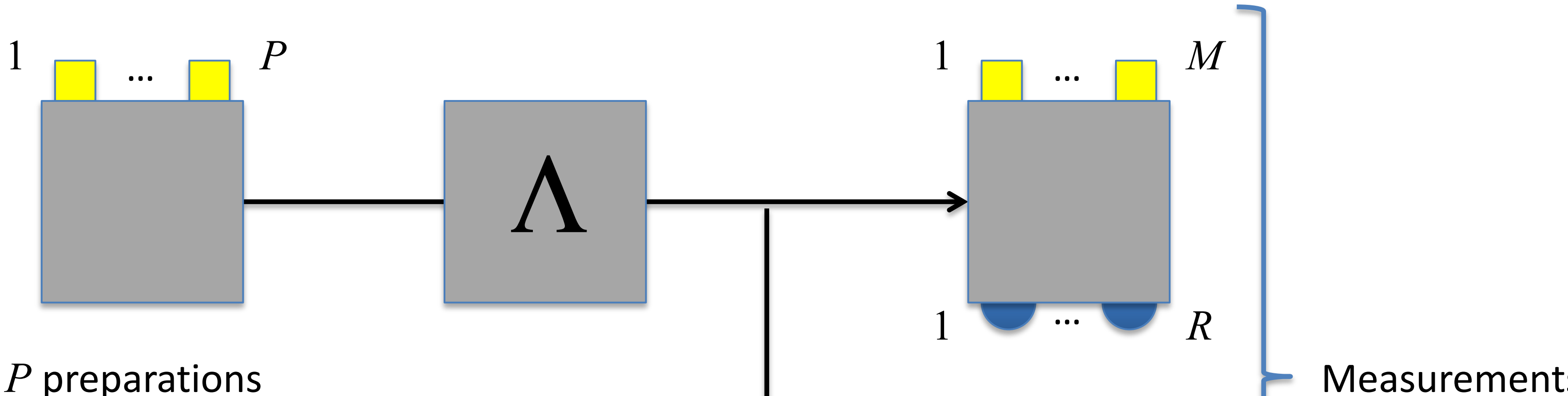

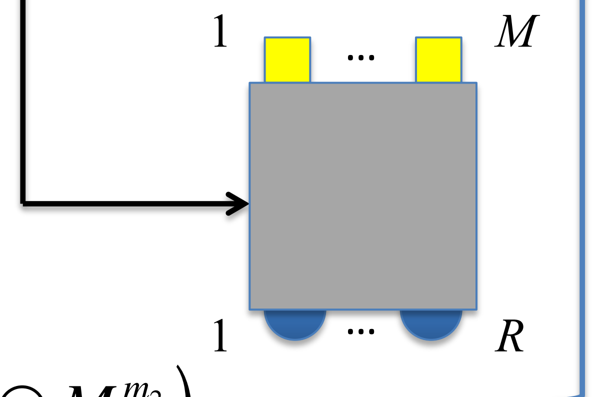

Goal: to extract information about the system from the measurement statistics.

$$P\left(r_1, r_2 \middle| p, m_1, m_2\right) = \mathrm{tr}\left(\Lambda\left(\rho_p\right) M_{r_1}^{m_1} \otimes M_{r_2}^{m_2}\right)$$

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- **State estimation:** no channel, the measurement operators are known, the prepared state(s) are inferred from the observed statistics.

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Assumptions in quantum estimation

Any quantum estimation problem problem relies on crucial assumptions.

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Assuming perfect knowledge of the devices in the experiment may be questionable, in view of the high complexity of experimental setups.

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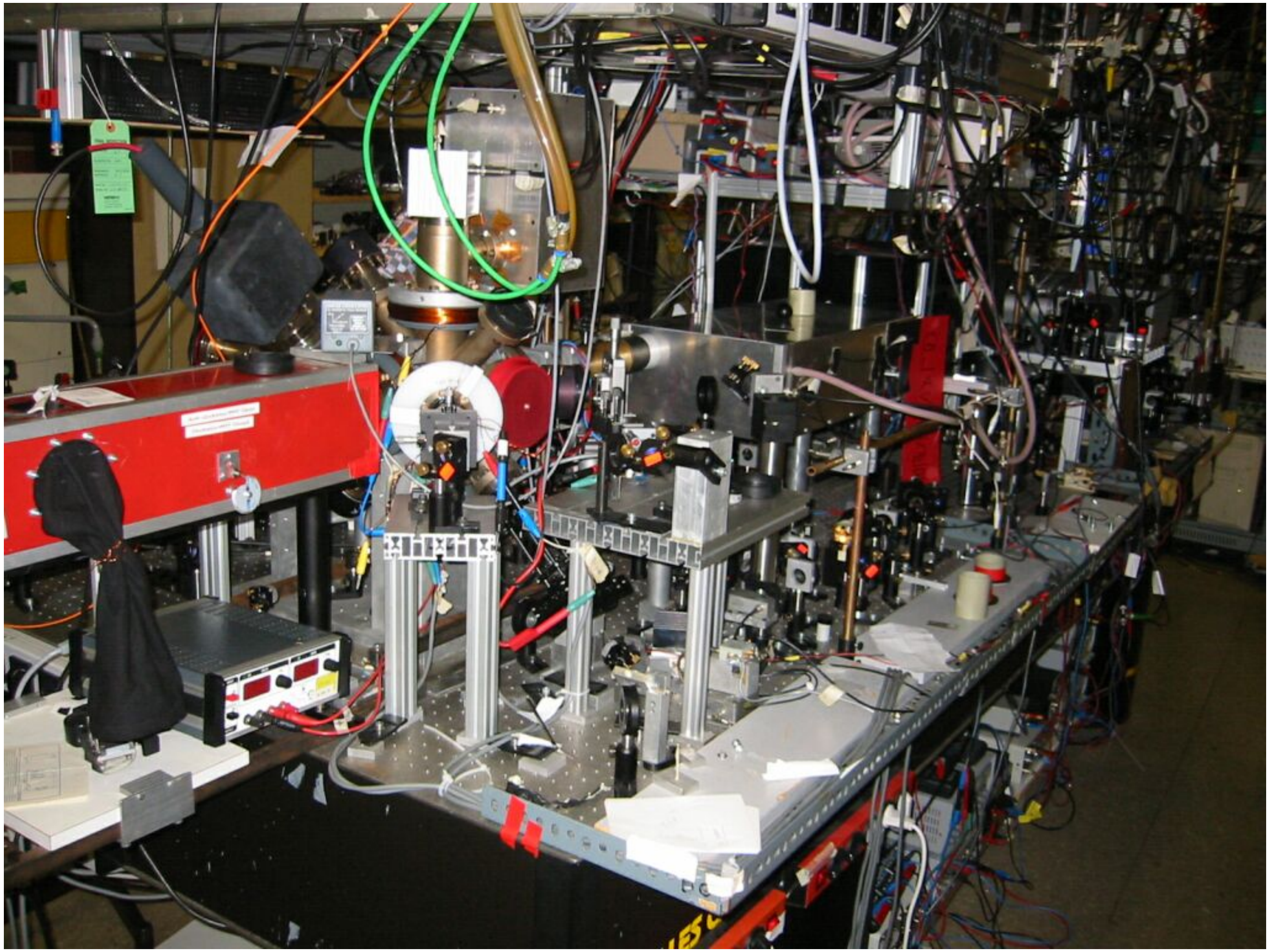

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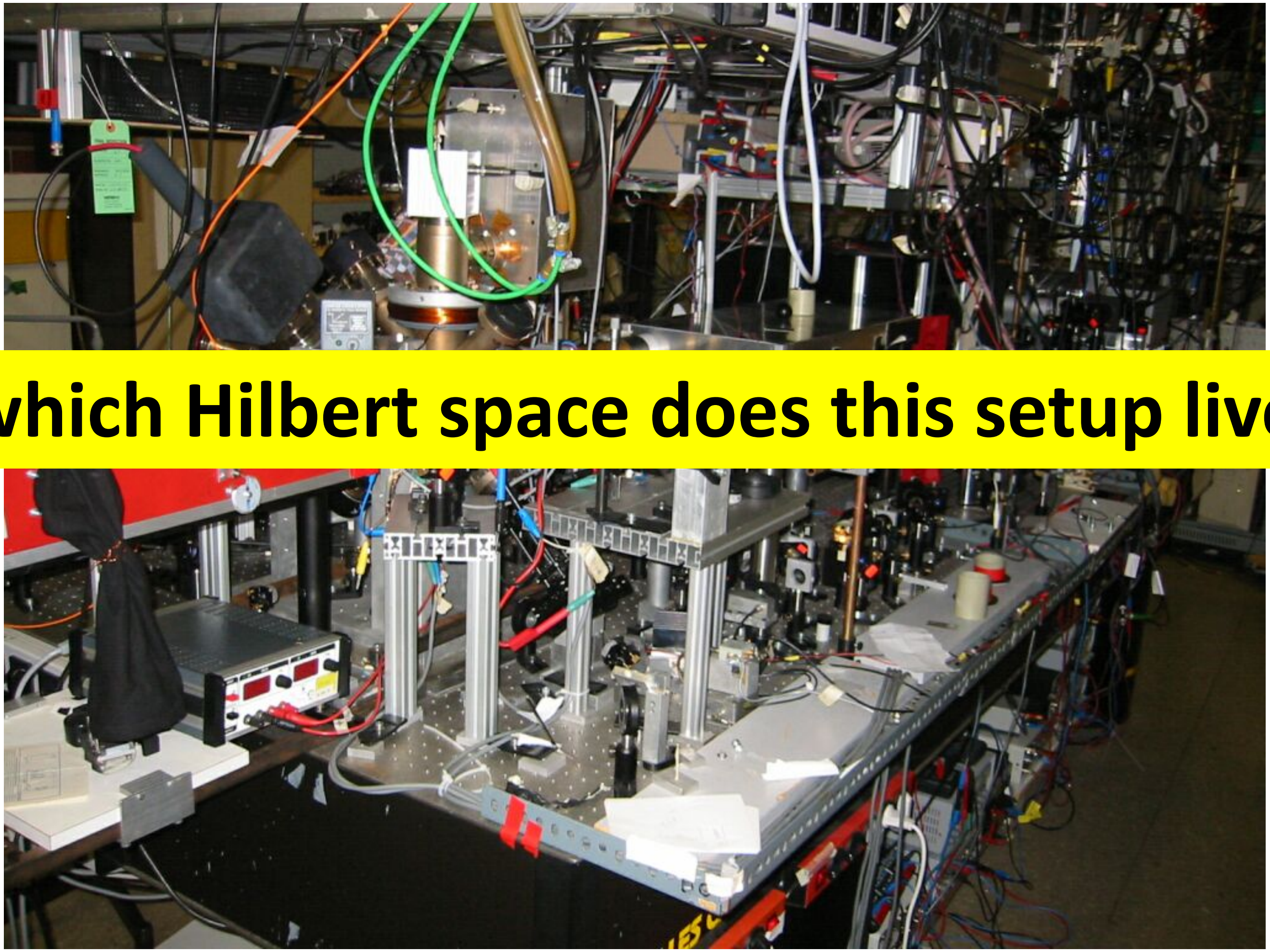
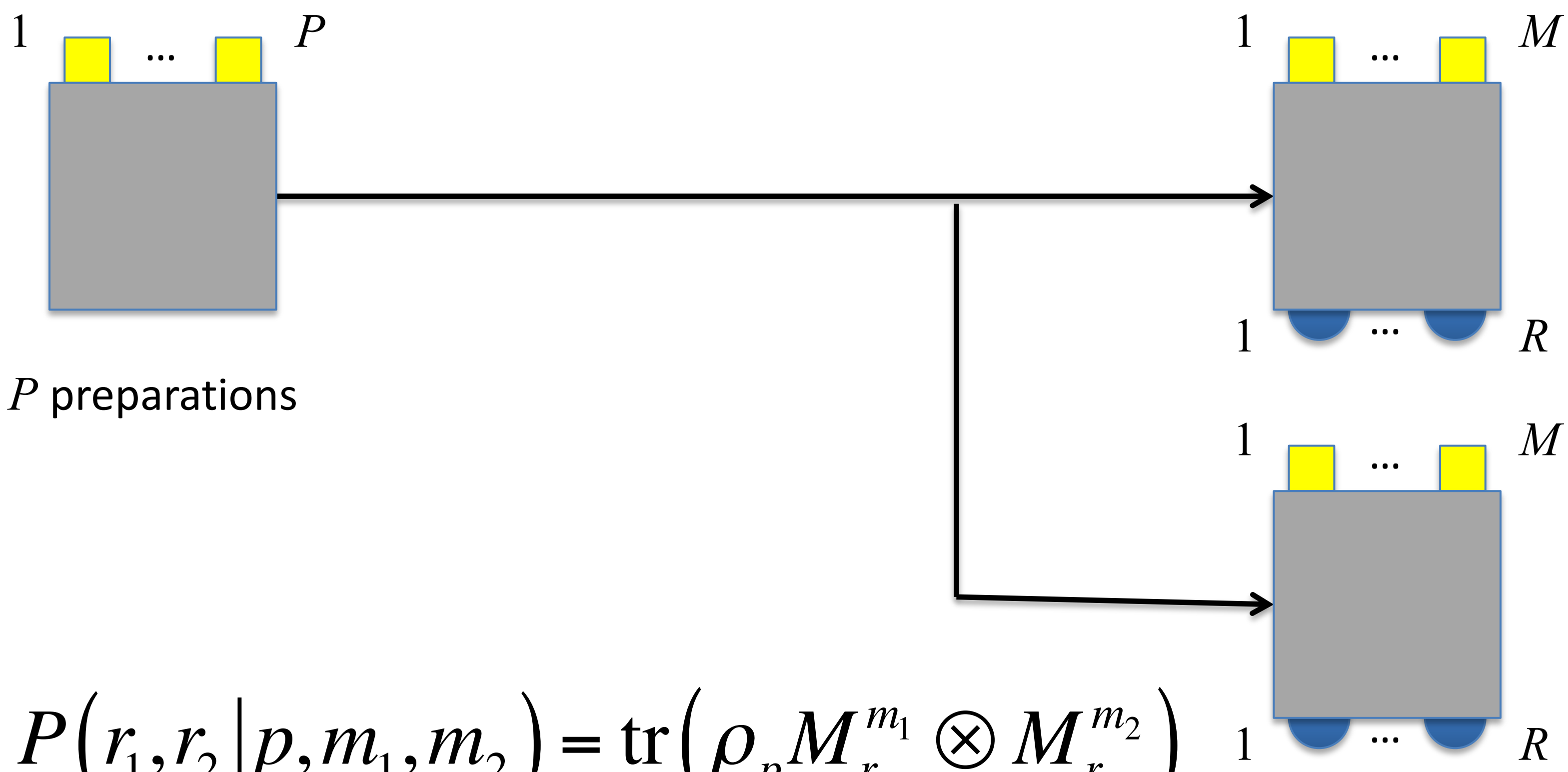


In which Hilbert space does this setup live?!

Device-independent estimation



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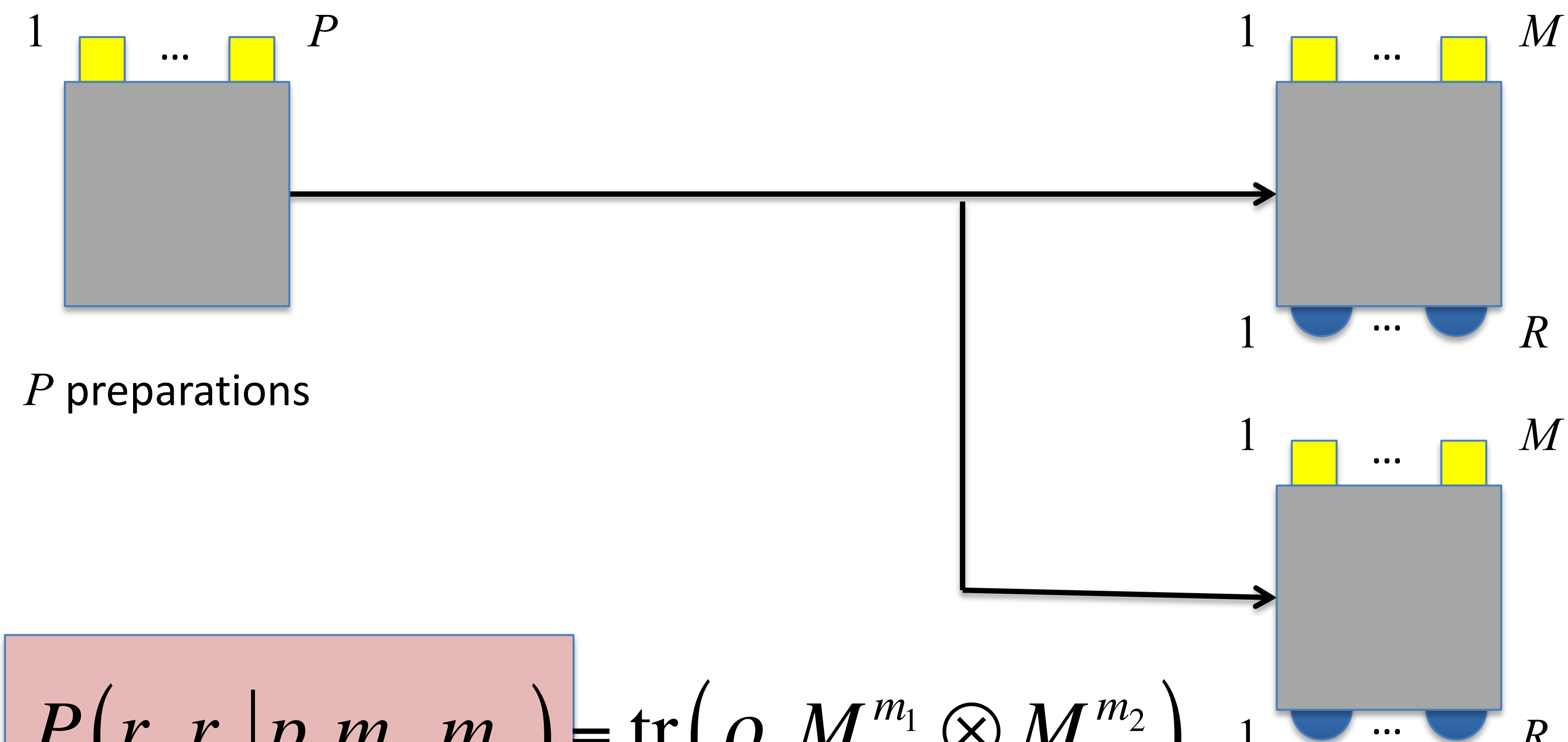


P preparations

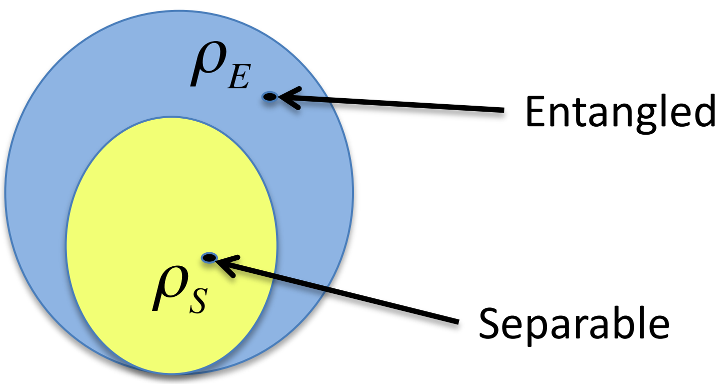
$$\boxed{P\left(r_1, r_2 \middle| p, m_1, m_2\right)} = \mathrm{tr}\left(\rho_p M_{r_1}^{m_1} \otimes M_{r_2}^{m_2}\right)$$

Goal: to extract information about the system **only** from the measurement statistics and without making any modeling or a priori assumption on the devices, which are seen as black boxes.

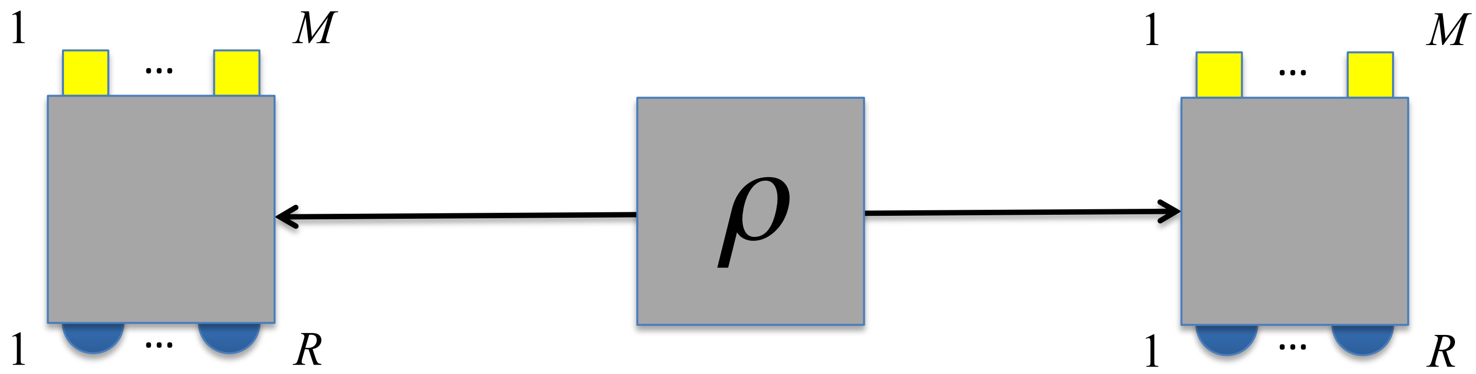
Entanglement detection

An N-particle quantum state is entangled if: $\rho \neq \sum_{i=1}^{N} p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)}$

States that can be written in this form are separable, or non-entangled. They can be prepared using only local quantum operations and classical communication.



Standard entanglement detection



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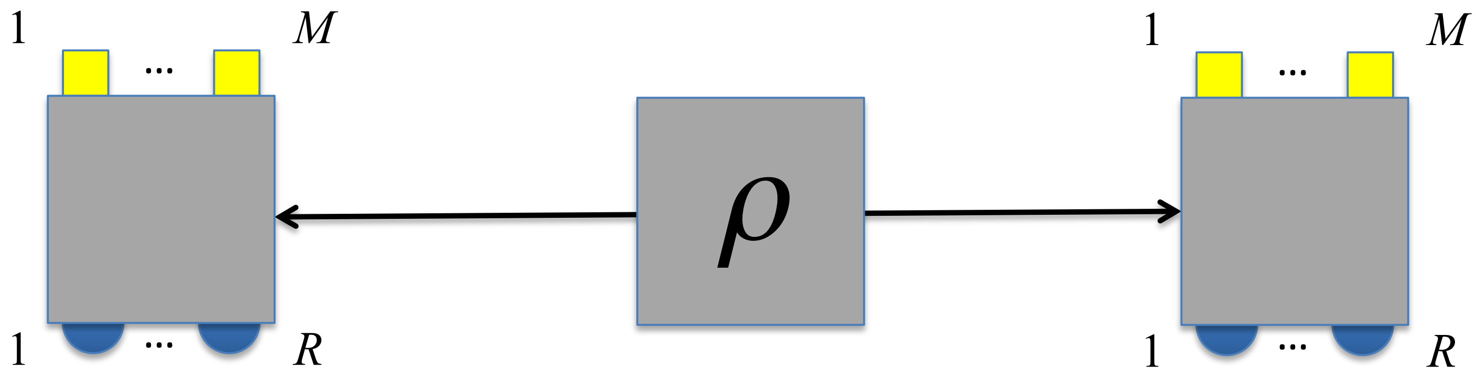

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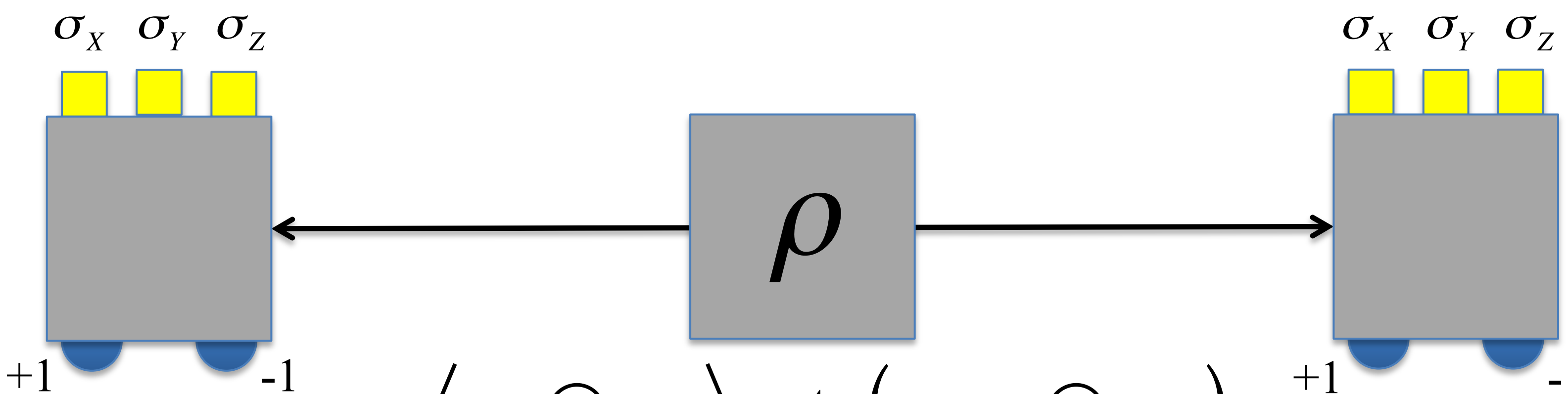
By performing an enough number of measurements, one can compute the value of an entanglement witness or even reconstruct the whole state and check its entanglement properties.

Standard entanglement detection

Example: two qubits. $C^2 \otimes C^2$



$$\langle \sigma_i \otimes \sigma_j \rangle = \mathrm{tr}\left(\rho\, \sigma_i \otimes \sigma_j\right)$$

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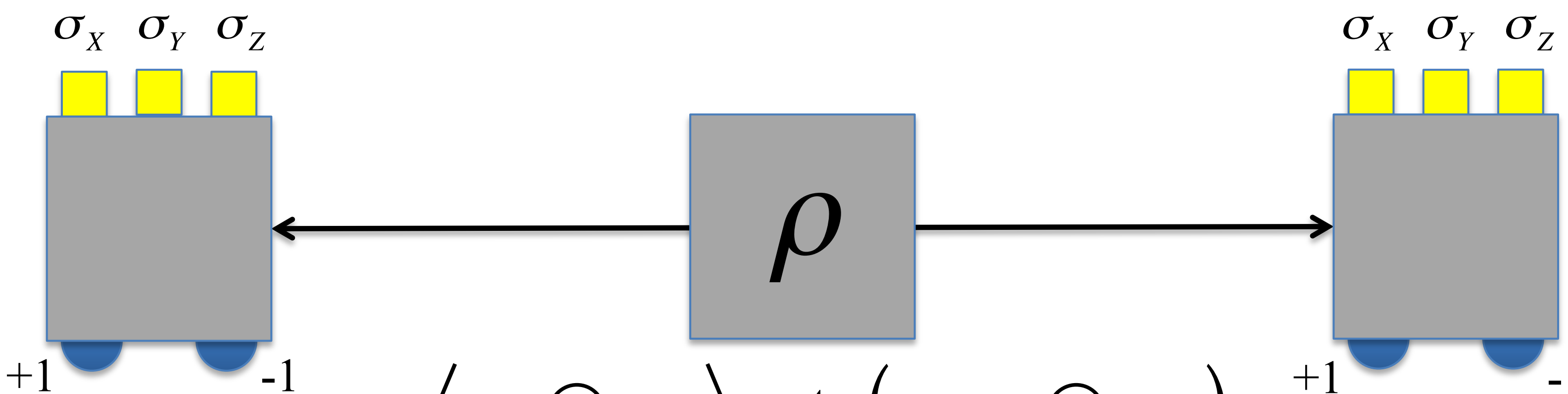


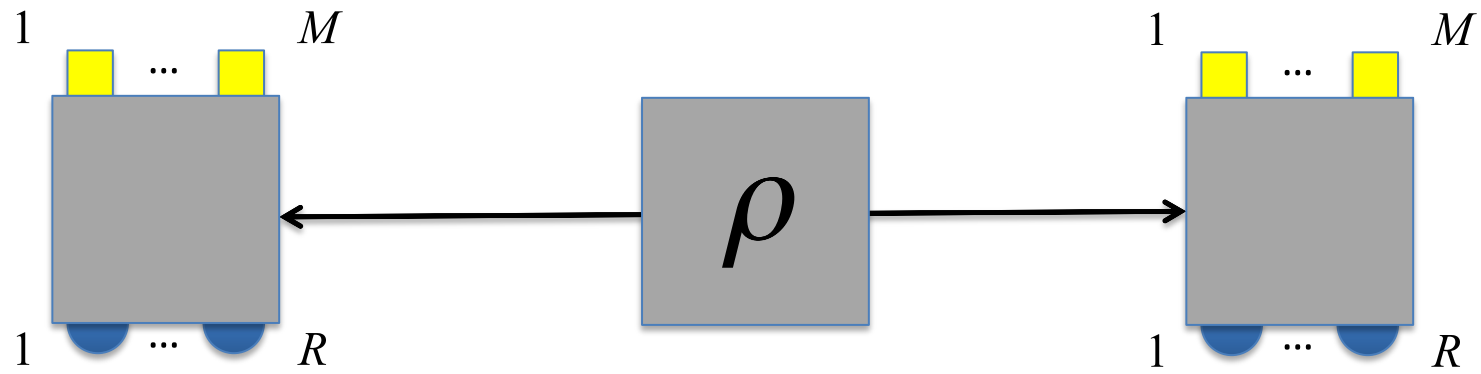
$$\langle \sigma_i \otimes \sigma_j \rangle = \mathrm{tr}(\rho\, \sigma_i \otimes \sigma_j)$$

One can now measure an entanglement witness, e.g.: $\langle \sigma_X \otimes \sigma_X \rangle + \langle \sigma_Z \otimes \sigma_Z \rangle \leq 1$

One can perform full tomography, reconstruct the quantum state and apply an entanglement criterion, such as partial transposition.

$$\rho = \begin{pmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{pmatrix}$$

DI entanglement detection



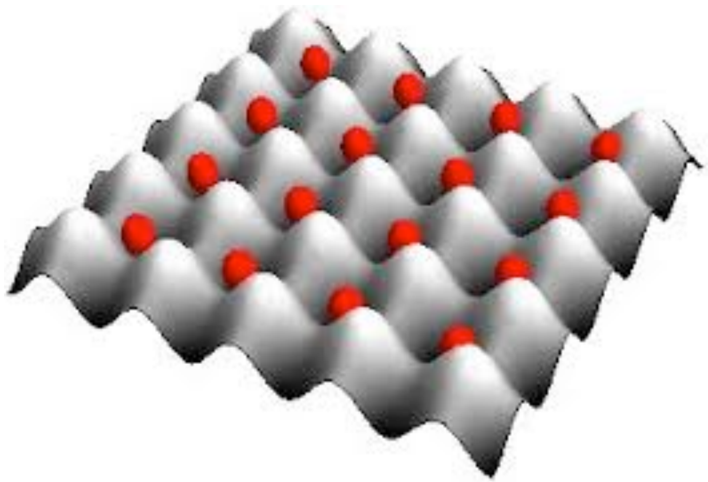
$$P\left(r_1, r_2 \middle| m_1, m_2\right) = \mathrm{tr}\left(\rho M_{r_1}^{m_1} \otimes M_{r_2}^{m_2}\right)$$

Now, no assumption is made on the applied measurements.

Bell inequalities are the only device-independent entanglement witnesses.

Full reconstruction is impossible as the Hibert space is unknown.

The problem



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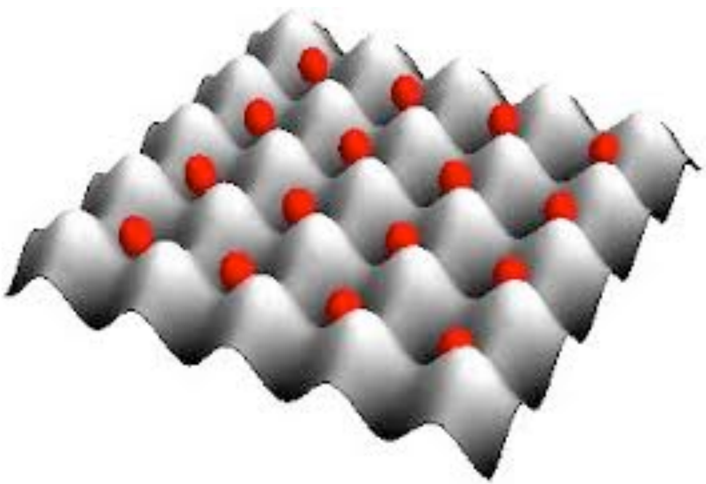

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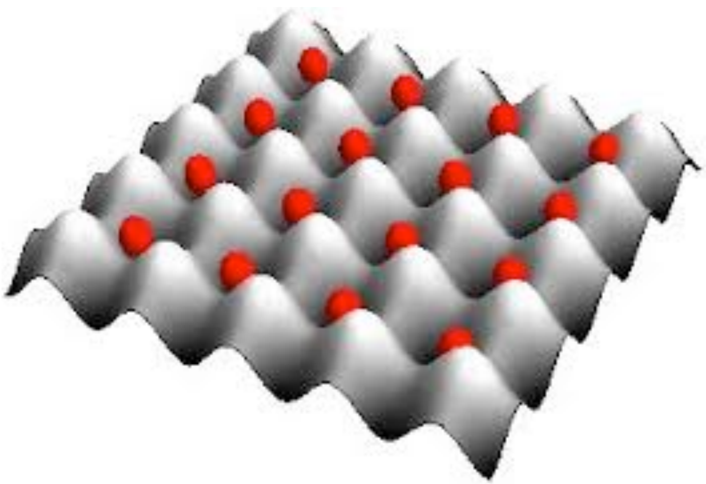

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1) The **full reconstruction** of a quantum state requires measuring a **number of parameters** that grows **exponentially** with the number of particles.

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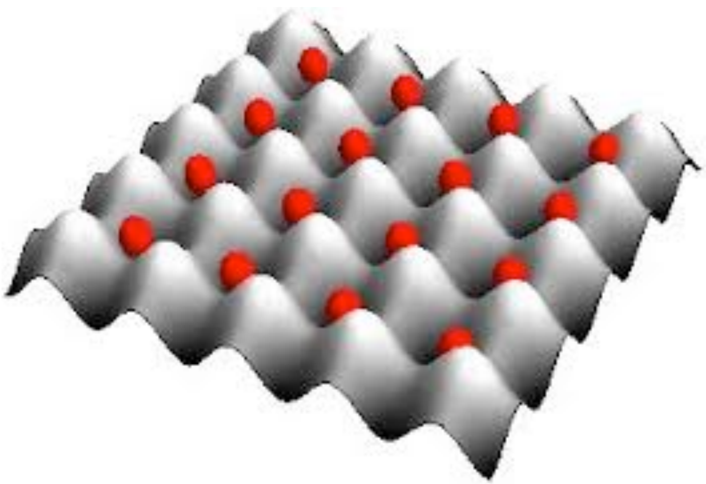

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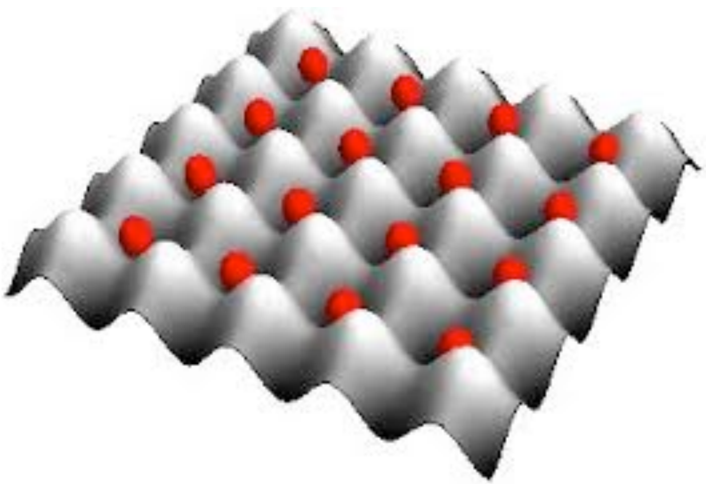

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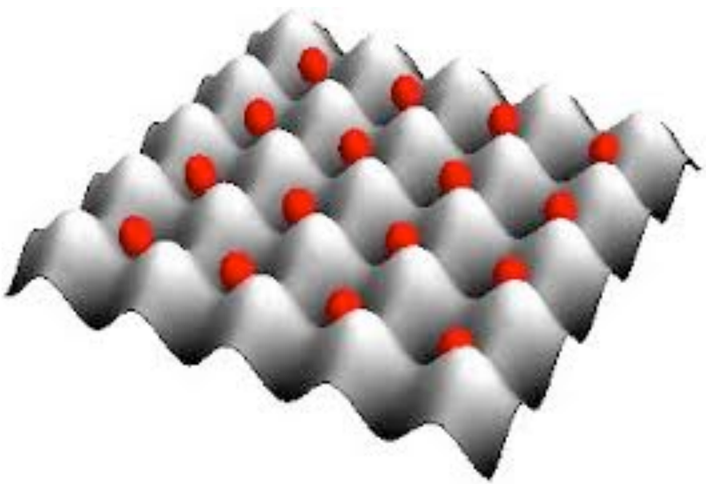

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One needs to efficiently extract relevant information from restricted data.

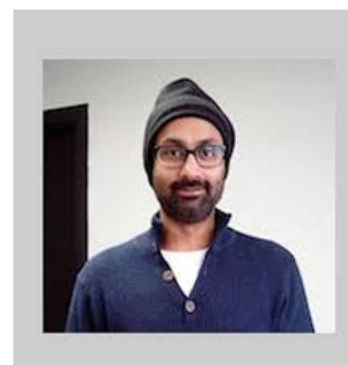
PHYSICAL REVIEW X

Open Access

Connector Tensor Networks: A Renormalization-Type Approach to Quantum Certification

Miguel Navascués, Sukhbinder Singh, and Antonio Acín
Phys. Rev. X **10**, 021064 – Published 19 June 2020

Miguel



Sukhi

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$$\rho_N = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)}$$

Consider a non-positive map $W: B\left(C^{d_i} \otimes C^{d_i}\right) \rightarrow B\left(C^{d_o}\right)$ such that for all states $\rho, \sigma \geq 0$ one has:

$$W(\rho \otimes \sigma) \geq 0$$

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If $d_o < d_i^2$ the map contracts the space. If we apply the map to the initial state, say on the last two particles, one gets:

$$(1_{1\ldots N-2} \otimes W)(\rho_N) = \tilde{\rho}_{N-1}$$

Connector tensor network

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If the initial state is separable, then the resulting operator is also a separable state.

$$\tilde{\rho}_{N-1} = (1_{1\dots N-2} \otimes W)\left(\sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)}\right) = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes W\left(\rho_i^{(N-1)} \otimes \rho_i^{(N)}\right) \geq 0$$

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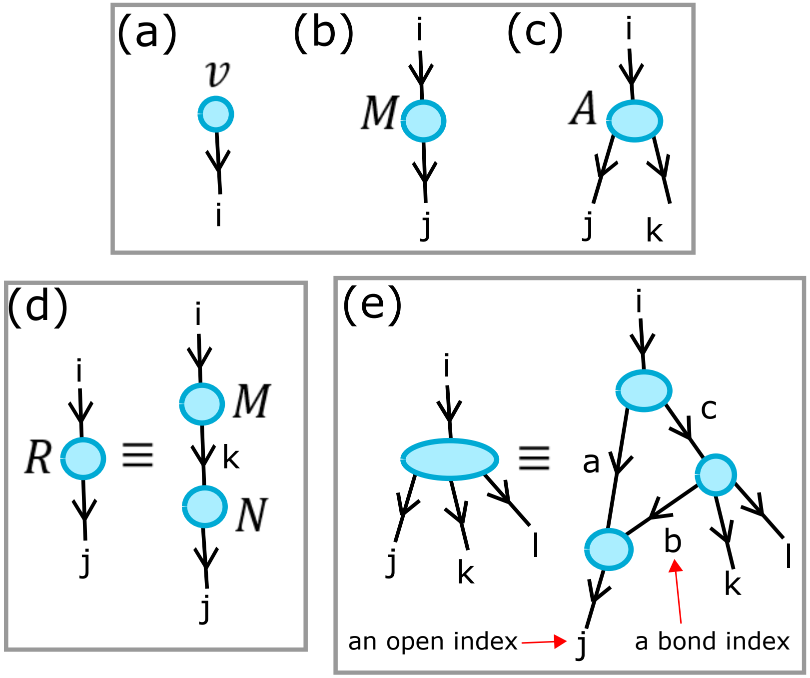
Remark: an entanglement witness is an example of these maps where the final state is a scalar.

$$W(\rho \otimes \sigma) \geq 0$$

Tensor networks

Tensor network notation: a tensor made of the contraction of different tensors. The open indices in the network are the indices of the resulting tensor, while the connected, also known as bond, indices are contracted.



(a) $|v\rangle = \sum_i v_i |i\rangle$

(b) $M = \sum_{ij} M_{ij}\, |j\rangle\langle i|$

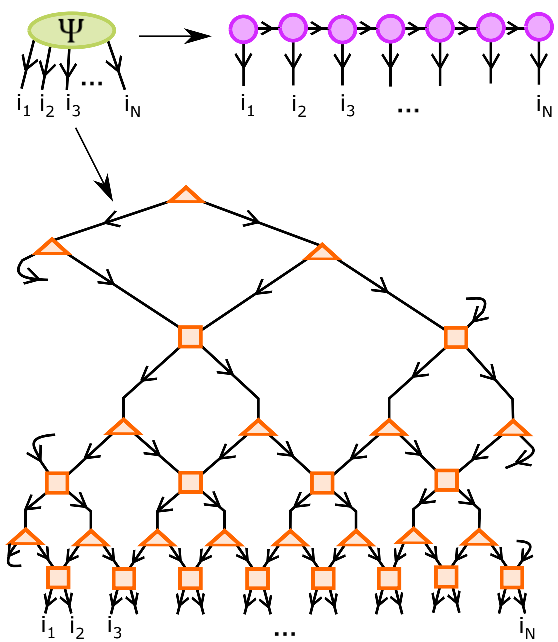
(c) $A = \sum_{ijk} A_{ijk}\, |j\rangle|k\rangle\langle i|$

(d) Matrix multiplication: $R = NM$

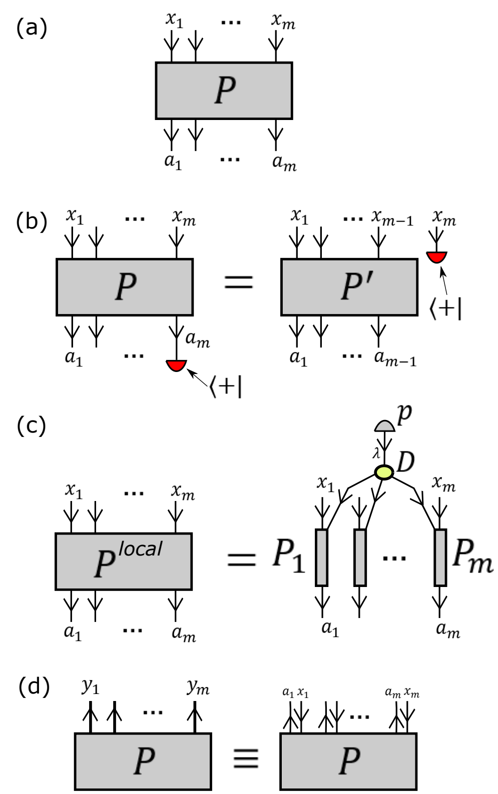
(e) General tensor network

Tensor networks

Usually employed to represent quantum states in a many-body context, but they can apply to any object consisting of indices, for example correlations.

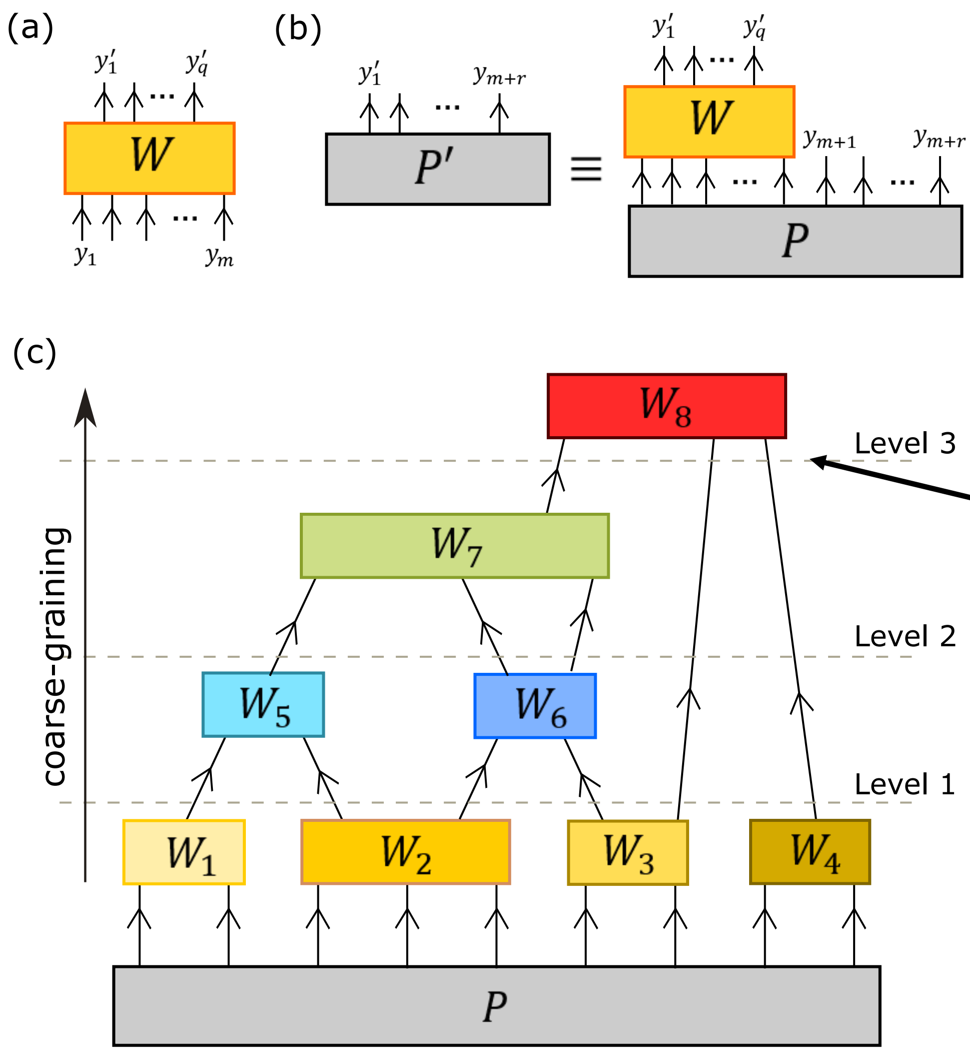


Known forms: MPS and MERA



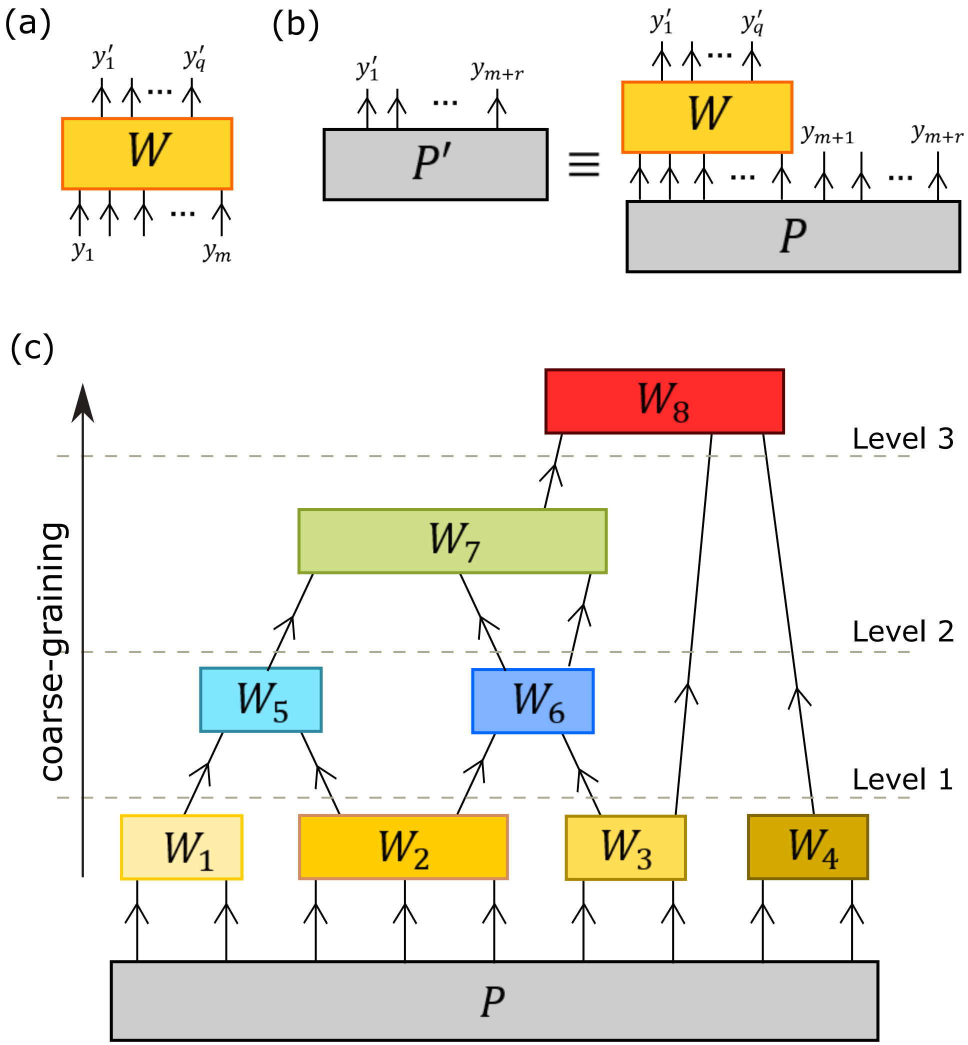
$P(a_1 \ldots a_N | x_1 \ldots x_N)$

Connector tensor network



The object to be detected is coarse grained until a standard criterion, say a witness, is applied to it. This witness can usually also be seen as the last step of the connector network.

Connector tensor network



See-saw: all connectors $W_{j\neq i}$ are fixed but W_i that is optimized. Denote by C the tensor resulting from applying $W_{j\neq i}$ to the initial tensor P. Then we look for:

$$\min W_i(C)$$

such that W_i is a connector. If a negative value is obtained the object P has the desired properties.

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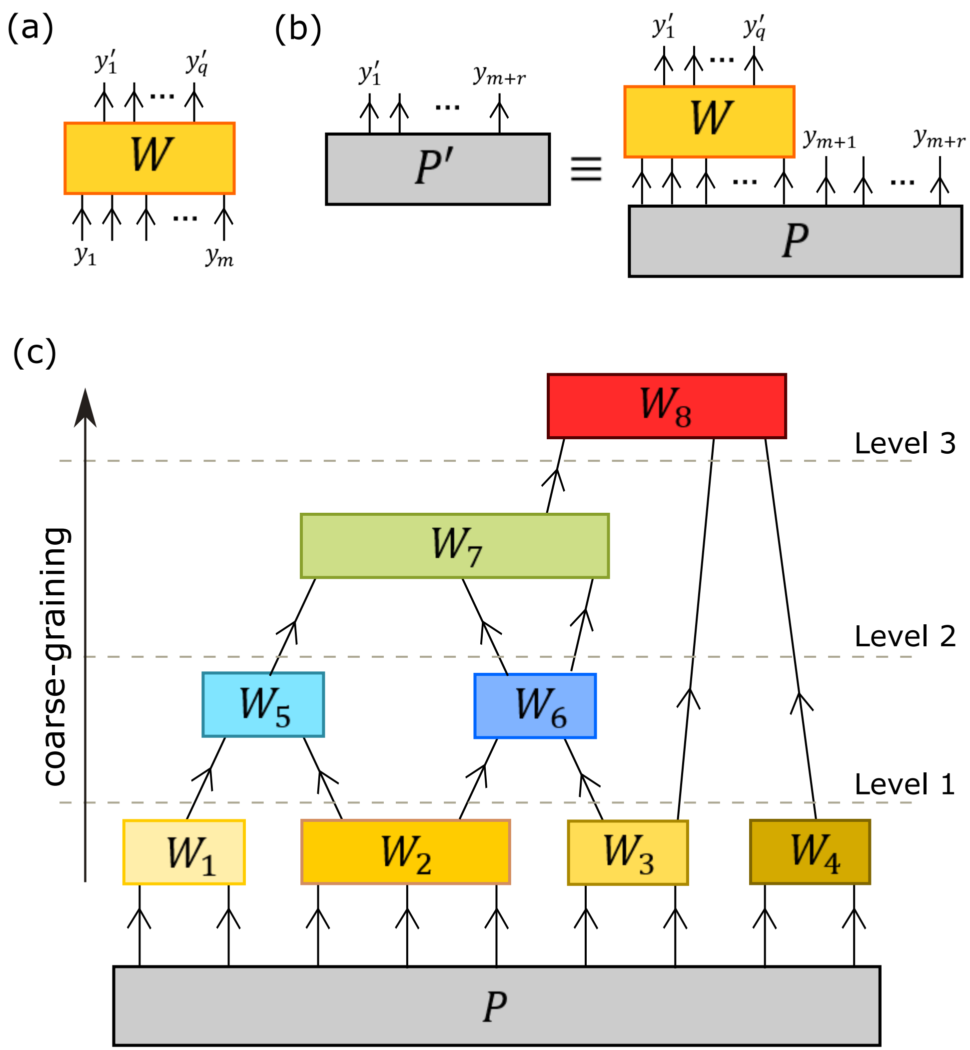

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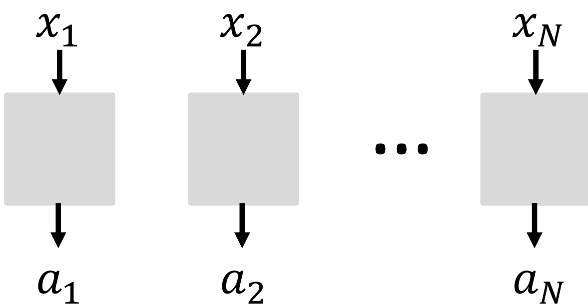
The set of connectors or subsets of it can be characterized by linear or semi-definite programming.

Bell non-locality



$P(a_1 \dots a_N | x_1 \dots x_N)$

The problem is defined by the number of parties N, of measurements m and of results r.

The observed correlations $P(a_1 \dots a_N | x_1 \dots x_N)$ are defined by a vector of size $m^N r^N$.

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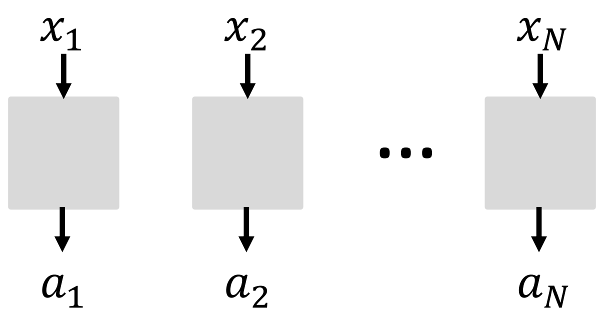

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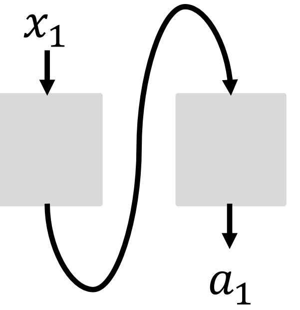
Detecting if a point is local can be solved by linear programming:

$$\exists\ p_i \geq 0 \text{ such that } P(a_1 \ldots a_N | x_1 \ldots x_N) = \sum_{i=1}^{n} p_i\, P_{L,i}^{\text{ext}}$$

The number of extreme points is $n = r^{mN}$. Already in the simplest scenario, $r = m = 2$, the linear program becomes prohibitive for $N \gtrsim 10$.

Bell connectors

- A wiring is a (uninteresting) connector.



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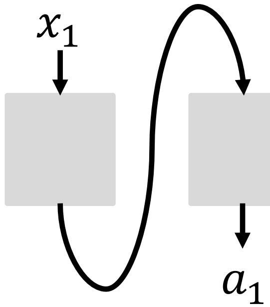
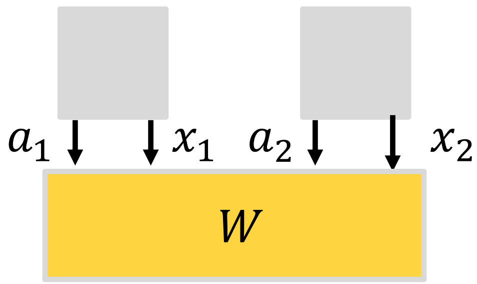


- A (normalized) Bell inequality is a connector:

$$1 \geq \beta = \sum_{a_1 a_2 x_1 x_2} w_{a_1 a_2 x_1 x_2} P_L(a_1 a_2 | x_1 x_2) \geq 0$$



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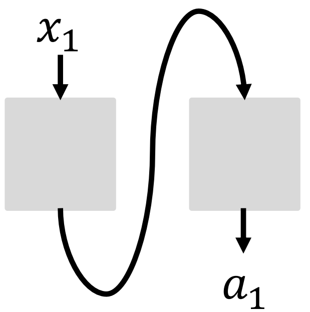

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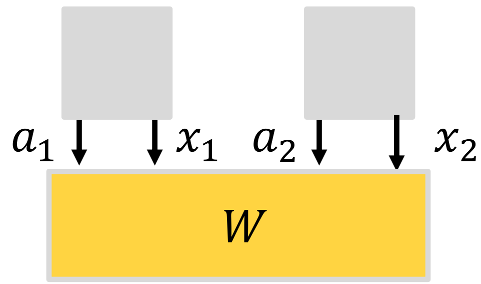
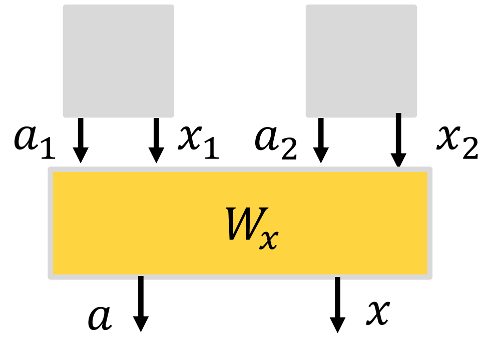


- A family of n Bell inequalities can be used to define a connector, where the resulting conditional probability distribution has n inputs and two outputs, where $P(a = 0|x) = \beta_x$.



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- Connectors can be characterized by linear programming: they should map local correlations in the input scenario (N_i, m_i, r_i) into local correlations of the output scenario (N_o, m_o, r_o).

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- Given a map W, one should prove that, for all extreme local points $P_{L_i,k}^{\text{ext}}$ in (N_i, m_i, r_i), one has:

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- Same considerations apply to optimizations over connectors.

Results

Connectors are especially powerful when the object under study also has a tensor-network structure.

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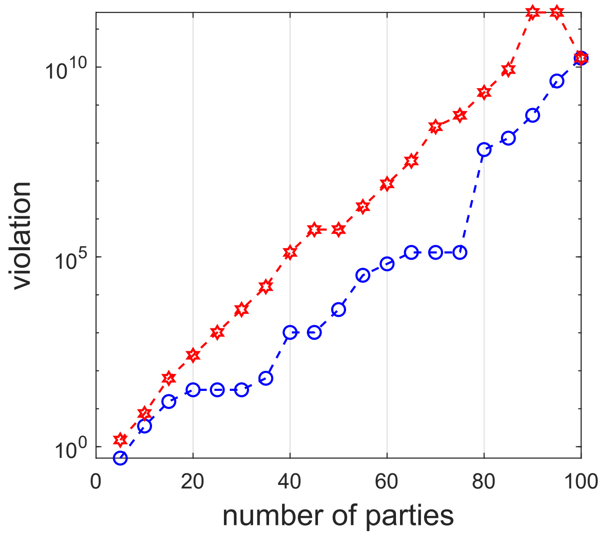
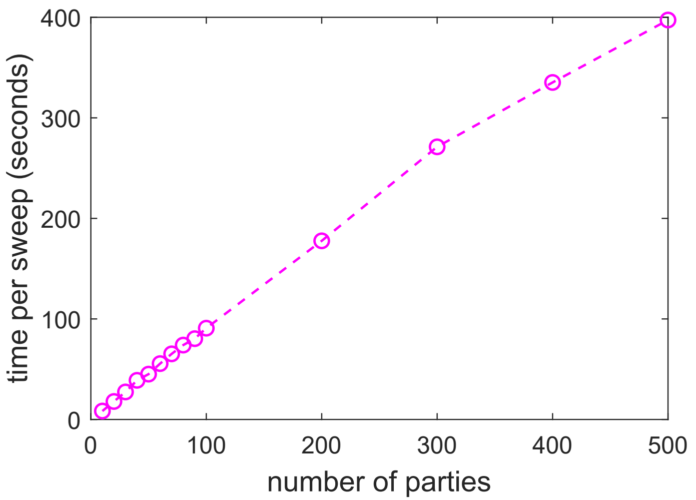


100 parties with small and scalable effort.



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Similar considerations apply to entanglement, but now:

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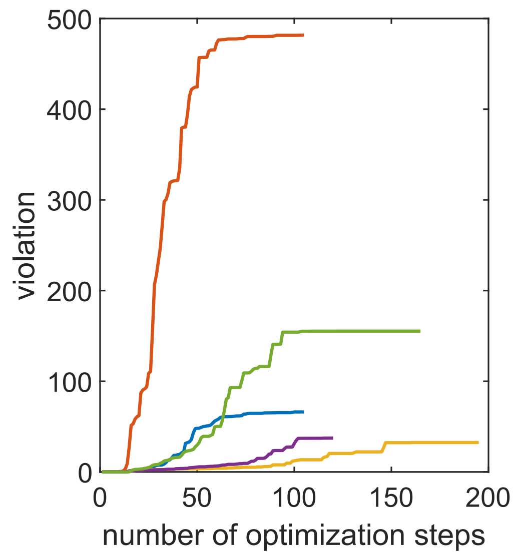
Examples:

- We apply the considerations to six-qubit PPT states constructed from UPB bases and detect them.
- We also consider mixed states of 60 qubits with efficient MPDO decomposition.

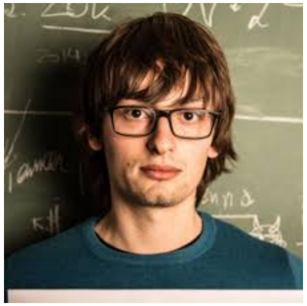


Entanglement marginal problems

Miguel Navascués, Flavio Baccari and Antonio Acín

arxiv:2006.09064

Miguel



Flavio

Entanglement marginal problem

Consider a system, denoted by $\mathcal{A}$. Subsets of this systems are denoted by I. Finally, we denote by $\mathcal{I}$ a set of subsets of $\mathcal{A}$ or, in other words a subset of its power set, $\mathcal{I} \subset \mathcal{P}(\mathcal{A})$.

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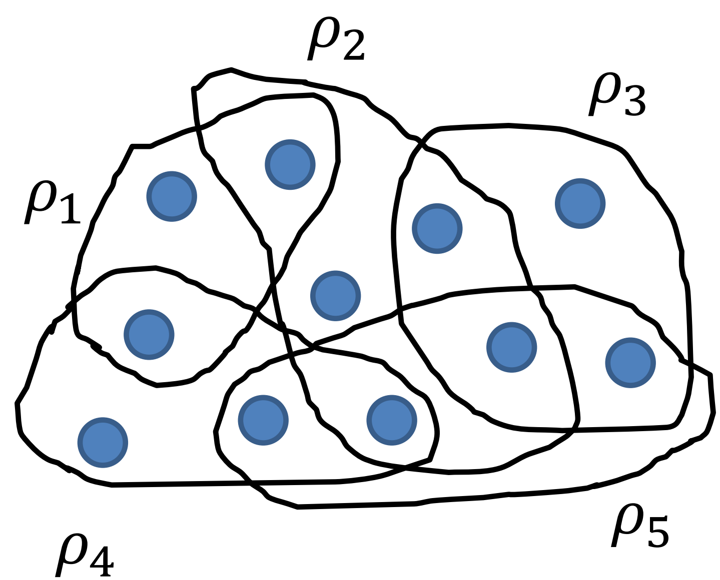
Entanglement marginal problem: given $\{\rho_I\}_{I\in\mathcal{I}}$ a set of marginals of a system $\mathcal{A}$, are they compatible with a separable state σ on $\mathcal{A}$?

The problem is non-trivial if the marginals:

- are separable and
- overlap, in which case they satisfy the **compatibility conditions**:

$$\mathrm{tr}_{I\setminus J}(\rho_I) = \mathrm{tr}_{J\setminus I}(\rho_J).$$



Classical marginal problem

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Classical marginal problem: is an ensemble of locally compatible measures $\{p_I\}_{I\in\mathcal{I}}$ compatible with a global measure p?

$$\int d\phi_{I\setminus J} p_I(\phi_I) = \int d\phi_{J\setminus I} p_J(\phi_J), \forall I, J \in \mathcal{I}$$

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The problem is non-trivial, in the sense that local compatibility is not sufficient.

If the entanglement marginal problem has a solution, then:

$$\sigma = \int p(\phi) d\phi \bigotimes_{\alpha\in A} |\phi_\alpha\rangle\langle\phi_\alpha| \quad \Longrightarrow \quad \rho_I \equiv \int p_I(\phi_I) d\phi_I \bigotimes_{\alpha\in I} |\phi_\alpha\rangle\langle\phi_\alpha|$$

where p_I are the reduced measures of p.

Hierarchy of necessary conditions

$$\sigma = \int p(\phi) d\phi \bigotimes_{\alpha \in A} |\phi_\alpha\rangle\langle\phi_\alpha| \quad \Rightarrow \quad \rho_I \equiv \int p_I(\phi_I) d\phi_I \bigotimes_{\alpha \in I} |\phi_\alpha\rangle\langle\phi_\alpha|$$

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They satisfy the following conditions, valid for any L:

(i) $\mathrm{tr}_{I^{L-1}}(\rho_I^{(L)}) = \rho_I$.

(ii) $\rho_I^{(L)}$ is Positive under Partial Transposition (PPT) [18] across all bipartitions of its $|I|L$ systems.

(iii) $\rho_I^{(L)} \in B(\bigotimes_{\alpha \in I} \mathcal{H}_{\mathrm{sym}}(L, d_\alpha))$, where $\mathcal{H}_{\mathrm{sym}}(L, d)$ denotes the symmetric space of L d-dimensional particles.

Doherty-Parrilo-Spedalieri

(iv) $\mathrm{tr}_{(I \setminus J)^L}(\rho_I^{(L)}) = \mathrm{tr}_{(J \setminus I)^L}(\rho_J^{(L)})$ (after appropriately reordering the systems of one of the sides).

Hierarchy of necessary conditions

Now, given the reduced states $\{\rho_I\}_{I\in\mathcal{J}}$ we check whether they satisfy the previous conditions for any value of L. This can be done by SDP.

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We denote each condition by $\mathbb{H}^L$. All together they define a hierarchy $\mathbb{H}$. If a test in the hierarchy is not satisfied, the reduced states must come from an entangled state.

Convergence of the hierarchy

Proposition 1. *Let* $\{\rho_I\}_I \in \mathbb{H}^L$ *or* $\{\rho_I\}_I \in \bar{\mathbb{H}}^L$*. Then, there exists an ensemble of fully separable states* $\{\tilde{\rho}_I\}_I$*, such that* $\|\rho_I - \tilde{\rho}_I\|_1 \leq O\left(\frac{\sum_{\alpha \in I} d_\alpha^2}{L^2}\right)$ *for all* $I \in \mathcal{I}$*. Moreover, the separable states* $\{\tilde{\rho}_I\}_I$ *are generated by an ensemble* $\{\tilde{p}_I(\phi_I)\}_I$ *of locally compatible distributions.*

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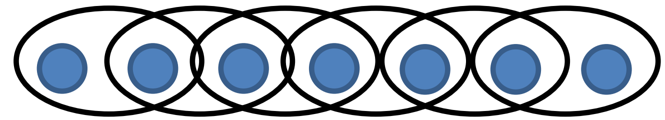
The hierarchy $\mathbb{H}$ tends to the set of separable states with compatible measures.

The hierarchy is complete for the problem if, and only if the classical marginal problem is trivial, that is, it follows from local compatibility.

Example

Nearest-neighbour states in the line: $\rho_{j,j+1}$, $\mathcal{A} = \{1, \dots, n\}$, $\mathcal{I} = \{I_j\}_{j=1}^{n-1}$ and $I_j = \{j, j+1\}$.



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\exists & \{\rho_{a_1 \ldots a_L b_1 \ldots b_L | j} \in B(\mathcal{H}_j)\}_{j=1}^{n-1}, \\
\text{s.t.} & \forall\, j \in \{1, \ldots, n-1\} : \\
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\text{(iii)} & \rho_{a_1 \ldots a_L b_1 \ldots b_L | j}^{T_{a_1 \ldots a_k} T_{b_1 \ldots b_l}} \succeq 0 \quad k, l = 1, \ldots, L \\
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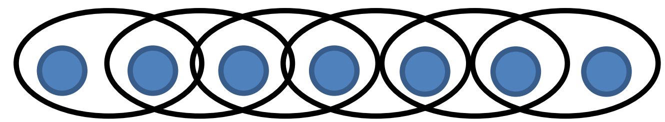

$\mathcal{H}_j = \mathcal{H}_{\mathrm{sym}}(d_j, L) \otimes \mathcal{H}_{\mathrm{sym}}(d_{j+1}, L)$

Every step $\mathbb{H}^L$ is polynomial in n.

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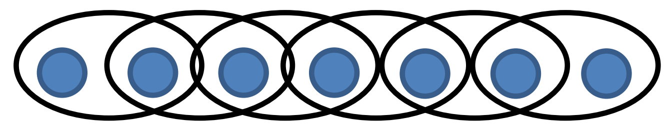

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The hierarchy is complete because the classical problem has a trivial solution:

$$p(X_1, X_2, \dots, X_{n-1}, X_n) = \frac{p(X_1, X_2) p(X_2, X_3) \dots p(X_{n-1}, X_n)}{p(X_2) p(X_3) \dots p(X_{n-1})}$$

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The hierarchy is now not complete because the previous trick does not work.

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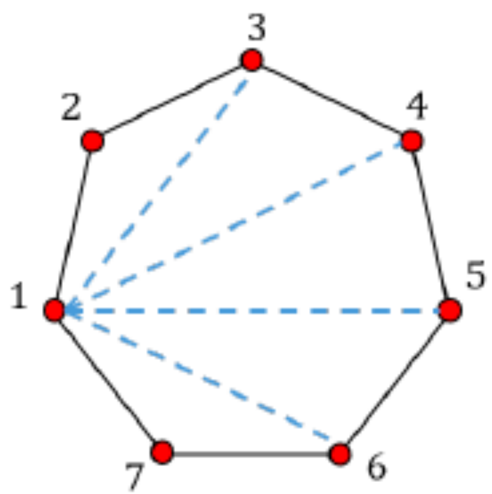

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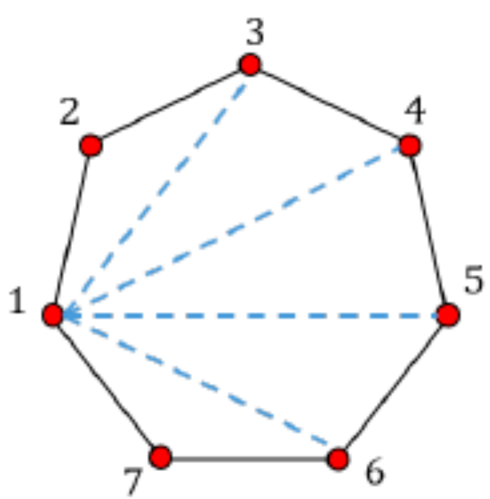

Convergent hierarchy

$$\exists\{\bar{\rho}_I\}_{I\in\mathcal{I}} \in \mathbb{H}^L(\bar{\mathcal{I}}) \text{ s.t.}$$
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For the chain, the corresponding set of states is given by the set of separable states satisfying $\rho_{12} = \rho_{21}$.

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Reduced k-particle state of an infinite chain with translational invariance (TI).
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The same techniques apply and the hierarchy converges to separable states with measures that satisfy LTI. The classical problem is known to be trivial, LTI necessary and sufficient, hence the hierarchy converges.

Similar hierarchies can be defined for 2D, although without convergence.

Conclusions

We presented two methods to detect relevant quantum features of many-body systems with good scalability properties.

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Connector tensor networks:

- General method applicable to the detection of many properties: entanglement, non-locality, supra-quantumness, steering,…
- Especially useful when the input state has an efficient tensor-network structure: 60 qubits, or 100 boxes.
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Entanglement marginal problem:

- Hierarchies for entanglement detection from reduced states.
- Polynomial scaling in many relevant scenarios. E.g.: application to nearest neighbour states of 1D chains of 100 particles.
- Connection to the classical marginal problem.
- Applicable to infinite systems with symmetries.