Data-driven inference of measurements

Based on arXiv:1812.08470 and arXiv:1905.04895

Michele Dall'Arno (Kyoto University and Waseda University)
Francesco Buscemi (Nagoya University)
Alessandro Bisio and Alessandro Tosini (Pavia University)
Asaph Ho and Valerio Scarani (NUS)

QICF20, September 2020

Table of contents

Framework of probabilistic theories, quantum theory as an instance.

Any system is associated with a dimension $\ell \in \mathbb{N}$ and a state space $\mathbb{S} \subseteq \mathbb{R}^\ell$.

For any $\ell, n \in \mathbb{N}$, any $\mathbb{S} \subseteq \mathbb{R}^\ell$, and any $\mathbb{P} \subseteq \mathbb{R}^n$, let

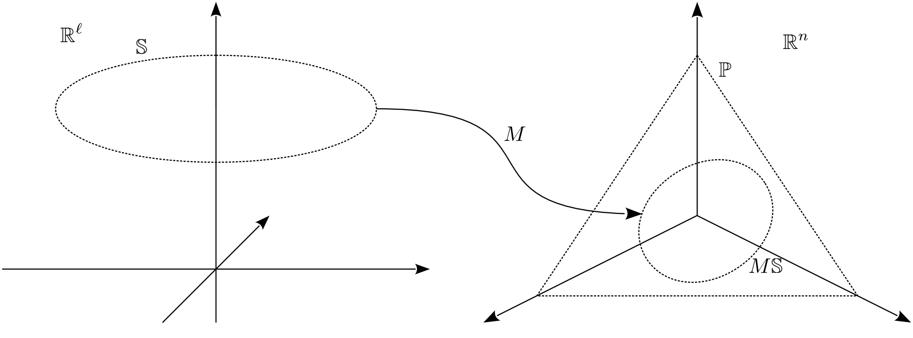
$$\texttt{Lin}\left(\mathbb{S}, \mathbb{P}\right) := \left\{ L : \mathbb{S} \to \mathbb{P} \;\middle|\; \exists\, M : \mathbb{R}^\ell \to \mathbb{R}^n \;\middle|\; M \text{ is linear}, \;\; M|_{\mathbb{S}} = L \right\}.$$

For any $\ell, n \in \mathbb{N}$, any $\mathbb{S} \subseteq \mathbb{R}^\ell$, any $L \in \texttt{Lin}(\mathbb{S}, \mathbb{P})$ is an n outcome **measurement** on $\mathbb{S}$ iff $\mathbb{P} \subseteq \mathbb{R}^n$ is the $(n-1)$ probability simplex.

Measurements: real qubit example

Let $\ell = n = 3$, let $\mathbb{S}$ be a ball, and let $\mathbb{P}$ be the $(n-1)$ probability simplex.

Any map $M \in \mathtt{Lin}(\mathbb{P}, \mathbb{S})$ is a three outcome measurement of a real qubit.



Tomography

Definition (Tomography)

For any $\ell, n \in \mathbb{N}$, *any* $\mathbb{S} \subseteq \mathbb{R}^\ell$, *any* $\mathbb{P} \subseteq \mathbb{R}^n$, *and any* $\mathcal{S} \subseteq \mathbb{S}$, *the tomographic map* $\mathtt{tg}_\mathbb{S}$ *is given by*

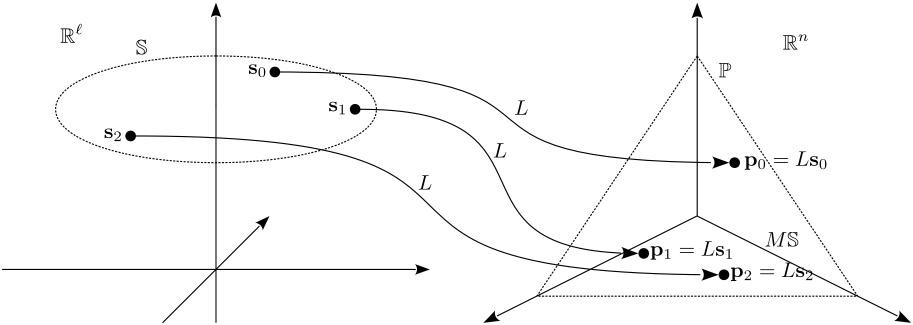
$$\mathtt{tg}_\mathbb{S} : \left\{ L \;\middle|\; L : \mathcal{S} \to \mathbb{P} \right\} \to \mathtt{Pow}\left(\mathtt{Lin}\left(\mathbb{S}, \mathbb{P}\right)\right)$$
$$L \mapsto \left\{ M \in \mathtt{Lin}\left(\mathbb{S}, \mathbb{P}\right) \;\middle|\; M|_\mathcal{S} = L \right\}$$

Tomography: real qubit example

Let $\ell = n = 3$, let $\mathbb{S}$ be the 2 ball, and let $\mathbb{P}$ be the 2 probability simplex.

Let $\mathcal{S} = \{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2\}$ and let $L : \mathcal{S} \to \mathbb{P}$. The protocol $\mathtt{tg}_{\mathbb{S}}$ is such that

$$\mathtt{tg}_{\mathbb{S}}(L) = \left\{ M \in \mathtt{Lin}(\mathbb{S}, \mathbb{P}) \;\middle|\; M(s_k) = p_k,\ k = 0, 1, 2 \right\}$$



Tomography: computation

For any $\ell, n \in \mathbb{N}$, any $\mathbb{S} \in \mathbb{R}^{\ell}$, any finite $\mathcal{S} \subseteq \mathbb{S}$, and any $L : \mathcal{S} \to \mathbb{P}$, where $\mathbb{P} \subseteq R^{n}$ is the affine subspace generated by the $(n-1)$ probability simplex, one has that

$$\mathtt{tg}_{\mathbb{S}}(L) = \left\{ M \;\middle|\; MS = P \right\},$$

where $S := \|_{\mathbf{s} \in \mathcal{S}} \mathbf{s}$, $P := \|_{\mathbf{s} \in \mathcal{S}} L\mathbf{s}$.

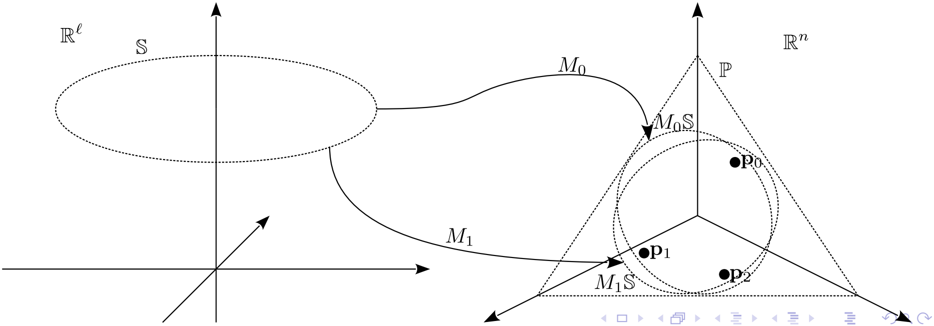
Consistency: real qubit example

For any sets $\mathbb{S}$ and $\mathbb{P}$, any $M : \mathbb{S} \to \mathbb{P}$ is **consistent** with any $\mathcal{P} \subseteq \mathbb{P}$ iff

$$\mathcal{P} \subseteq M\mathbb{S}.$$

Let $\ell = n = 3$, let $\mathbb{S}$ be the 2 ball, and let $\mathbb{P}$ be the 2 probability simplex.

Let $\mathcal{P} = \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$, and let $M_0, M_1 : \mathbb{S} \to \mathbb{P}$ such that $\mathcal{P} \subseteq M_k\mathbb{S}$ for $k = 0, 1$. Then M_0, M_1 are consistent with $\mathcal{P}$.



Data-driven inference

For any $\ell, n \in \mathbb{N}$, any $\mathbb{S} \subseteq \mathbb{R}^\ell$, and any $\mathbb{P} \subseteq \mathbb{R}^n$, let

$$\mathtt{Inv}\left(\mathbb{S}, \mathbb{P}\right) := \left\{ L \in \mathtt{Lin}\left(\mathbb{S}, \mathbb{P}\right) \;\middle|\; \exists\, L^+ \in \mathtt{Lin}\left(\mathbb{P}, \mathbb{S}\right) \;\middle|\; L^+ L = \mathtt{id} \right\}.$$

Definition (Data-driven inference)

For any $\ell, n \in \mathbb{N}$, any $\mathbb{S} \subseteq \mathbb{R}^\ell$, any $\mathbb{P} \subseteq \mathbb{R}^n$, the data-driven inference map $\mathtt{ddi}_{\mathbb{S}}$ *is given by*

$$\begin{aligned} \mathtt{ddi}_{\mathbb{S}} : \mathtt{Pow}\left(\mathbb{P}\right) &\to \mathtt{Pow}\left(\mathtt{Inv}\left(\mathbb{S}, \mathbb{P}\right)\right) \\ \mathcal{P} &\mapsto \operatorname*{argmin}_{\substack{M \in \mathtt{Inv}(\mathbb{S}, \mathbb{P}) \\ \mathcal{P} \subseteq M\mathbb{S}}} \mathtt{vol}\left(M\mathbb{S}\right) \end{aligned}$$

For any $\ell_0, \ell_1, \dots, n \in \mathbb{N}$, any $\mathbb{S}_0 \subseteq \mathbb{R}^{\ell_0}, \mathbb{S}_1 \subseteq \mathbb{R}^{\ell_1}, \dots, \mathbb{P} \subseteq \mathbb{R}^n$, we set

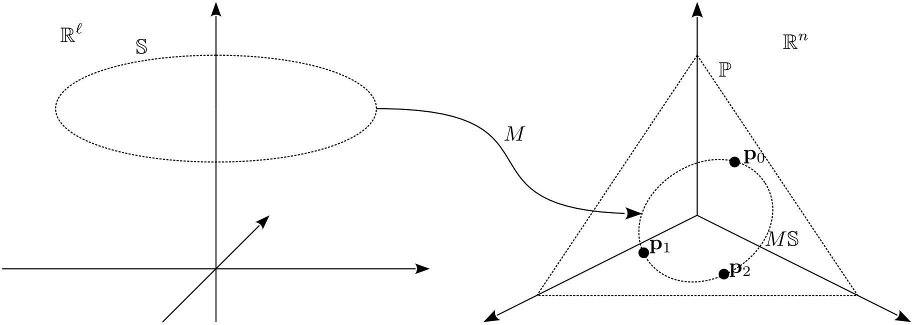
$$\mathtt{ddi}_{\{\mathbb{S}_0, \mathbb{S}_1, \dots\}} := \bigcup_{\mathbb{S} \in \{\mathbb{S}_0, \mathbb{S}_1, \dots\}} \mathtt{ddi}_{\mathbb{S}}\,.$$

Data-driven inference: real qubit example

Let $\ell = n = 3$, let $\mathbb{S}$ be the 2 ball, and let $\mathbb{P}$ be the 2 probability simplex.

Let $\mathcal{P} = \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$. The protocol $\mathtt{ddi}_{\mathbb{S}}$ is such that

$$\mathtt{ddi}_{\mathbb{S}}(\mathcal{P}) = \underset{\substack{M \in \mathtt{Inv}(\mathbb{S},\mathbb{P}) \\ \{\mathbf{p}_0,\mathbf{p}_1,\mathbf{p}_2\} \subseteq M\mathbb{S}}}{\operatorname{argmin}} \mathtt{vol}(M\mathbb{S})$$



Data-driven inference: computation for $(\ell - 1)$ ball $\mathbb{S}$

Definition (Minimum volume enclosing ellipsoid)

For any $\ell, n \in \mathbb{N}$, *let* $\mathbb{S} \subseteq \mathbb{R}^\ell$ *be the* $(\ell - 1)$ *ball and let* $\mathbb{P} \subseteq \mathbb{R}^n$ *be the affine subspace generated by the probability simplex. We define*

$$\begin{aligned} \mathtt{mvee} : \mathtt{Pow}\left(\mathbb{P}\right) &\to \mathtt{Pow}\left(\mathbb{P}\right) \\ \mathcal{P} &\mapsto \mathtt{ddi}_{\mathbb{S}}\left(\mathcal{P}\right)\mathbb{S}. \end{aligned}$$

Theorem (John, 1948)

For any $n \in \mathcal{N}$, *let* $\mathbb{P} \subseteq \mathbb{R}^n$ *be the affine subspace generated by the probability simplex. One has that* $\mathtt{mvee}$ *is a convex program and* $|\mathtt{mvee}(\mathcal{P})| = 1$ *for any* $\mathcal{P} \in \mathbb{P}$.

Data-driven inference: computation for $(\ell - 1)$ ball $\mathbb{S}$

Theorem (Dall'Arno, Brandsen, Buscemi, et al., 2017)

For any $\ell, n \in \mathbb{N}$, any $L \in \mathtt{Lin}(\mathbb{S}, \mathbb{P})$, where $\mathbb{S} \subseteq \mathbb{R}^\ell$ is the $(\ell - 1)$ ball and $\mathbb{P} \subseteq \mathbb{R}^n$ is the affine subspace generated by $(n-1)$ probability simplex, and any $\mathbf{p} \in \mathbb{P}$, one has $\mathbf{p} \in L\mathbb{S}$ if and only if

$$(\mathbf{p} - \mathbf{t})^T Q^+ (\mathbf{p} - \mathbf{t}) \leq 1, \quad \text{and} \quad (I - QQ^+)(\mathbf{p} - \mathbf{t}) = 0,$$

where

$$Q = \frac{1}{2} L L^T - \mathbf{t}\mathbf{t}^T, \quad \text{and} \quad \mathbf{t} = \frac{1}{2} L \mathbf{u}.$$

Equivalence

For any $\ell, n \in \mathbb{N}$, any $\mathbb{S} \subseteq \mathbb{R}^\ell$, any $\mathbb{P} \subseteq \mathbb{R}^n$, and any $L, M : \mathbb{S} \to \mathbb{P}$, we say

$$L \equiv M \;\Leftrightarrow\; L\mathbb{S} = M\mathbb{S}.$$

Theorem (Dall'Arno, Buscemi, Bisio, Tosini, 2018)

For any $\ell, n \in \mathbb{N}$, any $\mathbb{S} \subseteq \mathbb{R}^\ell$, any $\mathbb{P} \subseteq \mathbb{R}^n$, and any $L, M \in \mathtt{Lin}(\mathbb{S}, \mathbb{P})$

$$L \equiv M \;\Leftrightarrow\; \exists O \in \mathtt{Lin}(\mathbb{S}, \mathbb{S}) \;\Big|\; L = MO, \;\; O\mathbb{S} = \mathbb{S}.$$

Observational completeness

Definition (Observational completeness)

For any $\ell \in \mathbb{N}$, any $\mathbb{S} \subseteq \mathbb{R}^\ell$, and any $\mathcal{S} \subseteq \mathbb{S}$, we say that $\mathcal{S}$ is observationally complete (OC) for $\mathbb{S}$ iff

$$\mathtt{tg}_{\mathbb{S}}(L) \equiv \mathtt{ddi}_{\mathbb{S}}(L\mathcal{S}),$$

for any $n \in \mathbb{N}$, any $\mathbb{P} \subseteq \mathbb{R}^n$, and any $L \in \mathtt{Inv}(\mathcal{S}, \mathbb{P})$.

Theorem (Dall'Arno, Buscemi, Bisio, Tosini, 2018)

For any $\ell \in \mathbb{N}$ and any $\mathbb{S} \subseteq \mathbb{R}^\ell$, an $\mathcal{S} \subseteq \mathbb{S}$ is observationally complete for $\mathbb{S}$ iff

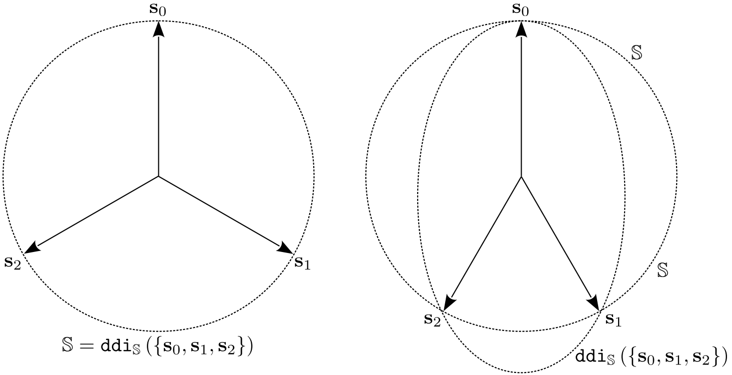
$$\mathtt{ddi}_{\mathbb{S}}(\mathcal{S}) = \left\{ L \in \mathtt{Inv}(\mathbb{S}, \mathbb{S}) \;\middle|\; L\mathbb{S} = \mathbb{S} \right\}.$$

Observational completeness: real qubit example

Let $\ell = 3$ and let $\mathbb{S} \subseteq \mathbb{R}^\ell$ be the 2 ball.

(Left) Regular simplex $\{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2\}$ is observationally complete for $\mathbb{S}$.

(Right) Irregular simplex $\{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2\}$ is not observationally complete for $\mathbb{S}$.



Observational completeness for 2 ball $\mathbb{S}$

For any $\ell \in \mathbb{N}$, an $\mathcal{S} \subseteq \mathbb{R}^\ell$ supports a **spherical 2 design** iff there exists a probability distribution $p : \mathcal{S} \to [0,1]$ such that

$$\sum_{\mathbf{s} \in \mathcal{S}} p(\mathbf{s})\, s^{\otimes 2} = \int \mathbf{s}^{\otimes 2} d\mathbf{s},$$

where $d\mathbf{s}$ denotes the uniform measure on the $(\ell - 1)$ sphere.

Theorem (Dall'Arno, Ho, Buscemi, Scarani, 2019)

For any $\ell \in \mathbb{N}$, an $\mathcal{S} \subseteq \mathbb{S}$ is observationally complete for the $(\ell - 1)$ ball $\mathbb{S} \subseteq \mathbb{R}^\ell$ iff $\mathcal{S}$ supports a spherical 2-design.

Corollary (Dall'Arno, Buscemi, Bisio, Tosini, 2018)

For any $\ell \in \mathbb{N}$, a simplex $\mathcal{S}$ is observationally complete for the $(\ell - 1)$ ball $\mathbb{S} \subseteq \mathbb{R}^\ell$ iff $\mathcal{S}$ is regular.

Summary

- A measurement is a linear map from a given state space to the probability simplex.
- Upon the input of a map L from a set $\mathcal{S}$ of states to the probability simplex, tomography returns the set of linear extensions of L.
- Upon the input of a set $\mathcal{P}$ of probability distributions, data driven inference returns the set of measurements whose range has minimal volume under the constraint that it contains $\mathcal{P}$.
- A set $\mathcal{S}$ of states is observationally complete if and only if, for any measurement, its restriction L to $\mathcal{S}$ is such that the tomography of L equals the data-driven inference of the range of L.
- We proved that a set $\mathcal{S}$ of states is observationally complete if and only if the minimum-volume linear transformation of the state space that contains $\mathcal{S}$ is the state space itself.

References

- F. John, *Extremum problems with inequalities as subsidiary conditions*, in *Studies and Essays Presented to R. Courant on his 60th Birthday*, 187–204, (Interscience Publishers, New York, 1948).
- M. Dall'Arno, S. Brandsen, F. Buscemi, et al., *Device-independent tests of quantum measurements*, Phys. Rev. Lett. **118**, 250501 (2017).
- M. Dall'Arno , F. Buscemi, A. Bisio, and A. Tosini, *Data-Driven Inference, Reconstruction, and Observational Completeness of Quantum Devices*, arXiv:1812.08470.
- M. Dall'Arno, A. Ho, F. Buscemi, and V. Scarani, *Data-driven inference and observational completeness of quantum devices*, arXiv:1905.04895.

Thank you!