Isoentangled Mutually Unbiased Bases and Mixed-states $t$–designs

Karol Życzkowski
Jagiellonian University, Cracow, & Polish Academy of Sciences, Warsaw

in collaboration with

Jakub Czartowski (Cracow)
Dardo Goyeneche (Antofagasta)
Markus Grassl (Gdańsk)

Yukawa Institute,
Kyoto University, September 19, 2020
What is this talk about? (1)

we analyze **discrete** structures like **combinatorial designs**
and look for generalizations motivated by **Quantum Information**

What are **combinatorial designs**?

*finite sets arranged with balance and symmetry*

*for instance:*

- Magic square
- Latin square
other important examples include

**Fano plane**
- 3 points at each line
- 3 lines from each point
- any two lines cross in a single point

**Greaco–Latin square**
- all pairs are different!
- all $N^2$ combinations used
- **Euler square** = 2 orthogonal LS
we analyze **discrete** structures in the finite **Hilbert space** $\mathcal{H}_N$. relevant for the standard **Quantum Theory**, for instance:

- **Mutually Unbiased Bases** (MUBs)
- **Symmetric Informationally Complete** generalized quantum measurements (SIC POVMs)
- **Quantum orthogonal Latin** squares & orthogonal arrays (OA)
- Complex **projective t-designs** formed of pure quantum states and their generalizations:
  - selected constellations of mixed states which form **mixed states t-designs**.

**Why we do it ?** Because we
a) do not fully understand these structures relevant for **quantum theory** !
b) wish to construct novel schemes of **generalized measurements** and
c) design techniques averaging over the set of **density matrices** of size $N$
Quantum combinatorial designs

What are they?

*finite sets of states/operators arranged with balance and symmetry*

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**Mermin-Peres Magic Square** *(1990)*

operators in each column do commute $\Rightarrow$ compatible measurements

operators in each row do commute $\Rightarrow$ compatible measurements
Quantum Latin Squares

Introduced by Vicary, Musto (2016): Example of order $N = 4$

\[
\begin{array}{cccc}
|0\rangle & |1\rangle & |2\rangle & |3\rangle \\
|3\rangle & |2\rangle & |1\rangle & |0\rangle \\
|\chi_-\rangle & |\xi_-\rangle & |\xi_+\rangle & |\chi_+\rangle \\
|\chi_+\rangle & |\xi_+\rangle & |\xi_-\rangle & |\chi_-\rangle
\end{array}
\]

where $|\chi_{\pm}\rangle = \frac{1}{\sqrt{2}} (|1\rangle \pm |2\rangle)$ denote Bell states, while $|\xi_+\rangle = \frac{1}{\sqrt{5}} (i|0\rangle + 2|3\rangle)$ $|\xi_-\rangle = \frac{1}{\sqrt{5}} (2|0\rangle + i|3\rangle)$ other entangled states.

Four states in each row & column form an orthogonal basis in $\mathcal{H}_4$

Standard combinatorics: discrete set of symbols, $1, 2, \ldots, N$,
+ permutation group

generalized ("Quantum") combinatorics: continuous family
of states $|\psi\rangle \in \mathcal{H}_N$ + unitary group $U(N)$. 
Classical combinatorial designs... include: Orthogonal Arrays (OA), Latin Squares (LS), Latin Cubes (LC)

More general quantum combinatorial designs include: Quantum Orthogonal Arrays (QOA), Quantum Latin Squares (QLS) and Quantum Latin Cubes (QLC)

Two orthogonal bases consisting of \( n \) vectors each in \( \mathcal{H}_N \) are called **mutually unbiased** (MUB) if

\[
|\langle \phi_i | \psi_j \rangle|^2 = \frac{1}{N}, \quad \text{for} \quad i, j = 1, \ldots, N.
\]

Such bases provide maximally different quantum measurements.

For a complex Hilbert space of dimension \( N \) there exist at most \( N + 1 \) such bases.

Example \( N = 2 \), complex space:

3 eigenbases of \( \sigma_x, \sigma_y, \sigma_y \)
Mutually Unbiased Bases & Hadamard matrices

- Full sets of \((N + 1)\) MUB’s are known if dimension is a **power of prime**, \(N = p^k\).
  
  For \(N = 6 = 2 \times 3\) only 3 < 7 MUB’s are known!

- A transition matrix \(H_{ij} = \langle \phi_i | \psi_j \rangle\) from one **unbiased** basis to another forms a **complex Hadamard** matrix, which is
  a) **unitary**, \(H^\dagger = H^{-1}\),
  b) has ”**unimodular”** entries, \(|H_{ij}|^2 = 1/N\), \(i, j = 1, \ldots, N\).

- **Classification** of all **complex Hadamard matrices** is complete for \(N = 2, 3, 4, 5\) only. (Haagerup 1996)
  
  see Catalog of **complex Hadamard matrices**, at
  
  \[http://chaos.if.uj.edu.pl/~karol/hadamard\]
Standard set of 2-qubit MUBs, \((N = 2 \times 2 = 4)\)

consists of 3 separable bases + 2 maximally entangled bases in \(H_4\)

- Reduced states \(\rho_A\) and \(\rho_B\) form 6 (doubly degenerated) vertices of the regular octahedron within the Bloch ball
  (eigenvectors of \(\sigma_x, \sigma_y, \sigma_z\) = 3 MUBs for \(N = 2\))
  and
  8-fold degenerated maximally mixed state \(\mathbb{1}/2\) in the centre.
Symmetric Informationally Complete POVM

- Symmetric informationally complete (SIC) POVM is such a set of $N^2$ vectors $\{|\psi_i\rangle\}$ in $\mathcal{H}_N$, that

$$|\langle \psi_i | \psi_j \rangle|^2 = \frac{1}{N + 1}$$


- They may be thought as equiangular structures in the Hilbert space.

- SIC POVM are found analytically for $N = 2, \ldots, 24$ and numerically up to 151 + some special cases: $N = 844$ Grassl & Scott (2017)

4 pure states at the Bloch sphere forming a SIC for $N = 2$. 
**projective t-designs = discrete set of pure states**

**Definition**

Any ensemble $|\psi_i\rangle_{i=1}^{M}$ of pure states in $\mathcal{H}^{N}$ is called **complex projective t-design** if for any polynomial $f_t$ of degree at most $t$ in both components of the states and their conjugates the average over the ensemble coincides with the average over the space $\mathbb{C}P^{N-1}$

$$\frac{1}{M} \sum_{i=1}^{M} f_t\{\psi_i\} = \int_{\mathbb{C}P^{N-1}} f_t(\psi) d\psi_{FS}.$$  

with respect to the unitarily–invariant Fubini–Study measure $d\psi_{FS}$.

- **Complex projective** $t$–designs are used for quantum state tomography, quantum fingerprinting and quantum cryptography.
- Examples of **2-designs** include maximal sets of mutually unbiased bases (MUB) and symmetric informationally complete (SIC) POVM.
- the **larger** $t$ the **better** design approximates the set of states.
Interesting case – isoentangled SIC-POVM

- Averaging property implies a condition for the average entanglement (measured by the purity of partial trace) of vectors in a 2-design in $H_N \otimes H_N$

$$\langle \text{Tr}[(\text{Tr}_A |\psi_i\rangle\langle\psi_i|)^2] \rangle = \frac{2N}{N^2 + 1}$$

**Zhu & Englert** (2011) found an interesting constellation of $4^2 = 16$ states in $H_2 \otimes H_2$ forming a SIC for two-qubit system, such that entanglement of all states is constant,

$$\text{Tr}[(\text{Tr}_A |\psi_i\rangle\langle\psi_i|)^2] = \frac{4}{5}, \text{ for } i = 1, \ldots, 16.$$  

Such a set of states can be obtained from a single *fiducial* state $|\phi_0\rangle$ by **local unitary** operations, $|\phi_j\rangle = U_j \otimes V_j |\phi_0\rangle$. 
Isoentangled MUBs for 2 qubits?

Question:

Is there a similar configuration for the full set of 5 iso-entangled MUBs for 2 qubits?

the standard MUB solution for $N = 4$ consists of 3 separable bases and 2 maximally entangled...
The answer is positive!

\[ |\phi_0 \rangle = \frac{1}{20} (a_+ |00 \rangle - 10i |01 \rangle + (8i - 6) |10 \rangle + a_- |11 \rangle), \]

where \( a_{\pm} = -7 \pm 3\sqrt{5} + i(1 \pm \sqrt{5}) \)

and other states are **locally equivalent**, \( |\phi_j \rangle = U_j \otimes V_j |\phi_0 \rangle \)

Each of \( 5 \times 4 = 20 \) pure states \( |\psi_j \rangle \) in \( \mathcal{H}_2 \otimes \mathcal{H}_2 \)

will be represented by its partial trace, \( \rho_j = \text{Tr}_B |\psi_j \rangle \langle \psi_j| \)

belonging to the Bloch ball of one-qubit mixed states.

Each basis is represented by a regular **tetrahedron** inside the Bloch ball.

Each colour corresponds to a single basis.

Entire five-color set forms a **regular 5-tetrahedra compound**.

Its convex hull forms a **regular dodecahedron**, different from the one of Zimba and Penrose...
Jakub Czartowski and his sculpture
Mixed states t-designs
Generalized designs

In quantum theory one uses

- **projective designs** formed by pure states, $|\psi_i\rangle \in \mathcal{H}_N$
- **unitary designs** formed by unitary matrices, $U_i \in U(N)$ (which induce designs in the set of maximally entangled states, $|\phi_j\rangle = (U_j \otimes 1)|\psi_+\rangle$
- **spherical designs** - sets of points *evenly distributed* at the sphere $S^k$
- related notions, e.g. **conical designs**, Graydon & Appleby (2016), mixed designs by Brandsen, Dall’Arno, Szymusiak (2016)

These examples for special case of a general construction of **averaging sets** by Seymour and Zaslavsky (1984). It concerns a collection of $M$ points $x_j$ from an arbitrary measurable set $\Omega$ with measure $\mu$ such that

$$\frac{1}{M} \sum_{i=1}^{M} f_t(x_i) = \int_{\Omega} f_t(x) d\mu(x),$$

where $f_t(x)$ denote selected continuous functions, e.g. $f_t(x) = x^t$. 
Mixed states t-designs

We apply this idea for a compact set of mixed states $\Omega_N \subset \mathbb{R}^{N^2-1}$ endowed with the flat Hilbert-Schmidt measure $d\rho_{HS}$

**Definition**

Any ensemble $\{\rho_i\}_{i=1}^M$ of $M$ density matrices of size $N$ is called a **mixed states t-design** if for any polynomial $g_t$ of degree $t$ in the eigenvalues $\lambda_j$ of the state $\rho$ the average over the ensemble is equal to the mean value over the space of mixed states $\Omega_N$ with respect to the **Hilbert-Schmidt** measure $d\rho_{HS}$,

$$\frac{1}{M} \sum_{i=1}^{M} g_t(\rho_i) = \int_{\Omega_N} g_t(\rho) d\rho_{HS}. \quad (1)$$
Method of generating mixed states $t$-designs

**Proposition 1.**

Any complex **projective $s$-design** $\{|\psi_i\rangle\}_{i=1}^M$ in the composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ of size $N \times N$ induces, by partial trace, a **mixed states $t$-design** $\{\rho_i\}_{i=1}^M$ in $\Omega_N$ with $\rho_i = \text{Tr}_B |\psi_i\rangle\langle \psi_i|$ and $t \geq s$.

The same property holds also for the dual set $\{\rho'_i : \rho'_i = \text{Tr}_A |\psi_i\rangle\langle \psi_i|\}$.

**Construction for** $t = s$ **is based on the fact that Fubini–Study measure** in the space of pure states of size $N^2$ induces, by partial trace, the flat **HS measure** in the space $\Omega_N$ of mixed states of size $N$.

**Observation 1.**

Every positive operator-valued measurement (POVM) induces a mixed states **1-design**, as its barycenter coincides with the maximally mixed state $\mathbb{1}/N$. 
Proposition 2.

A set \( \{ \rho_j \}_{j=1}^M \) of density matrices of size \( N \) forms a \textbf{mixed states} \( t \)-design \textbf{if and only if} it saturates the inequality

\[
2 \text{Tr} \left( \frac{1}{M} \sum_{i=1}^M \rho_i \otimes^t \int_{\Omega_N} \rho \otimes^t \, d\rho_{HS} \right) - \frac{1}{M^2} \sum_{i,j=1}^M \text{Tr}(\rho_i \rho_j)^t \leq \gamma_{N,t}
\]

where \( \gamma_{N,t} := \text{Tr} \omega_{N,t}^2 \) and \( \omega_{N,t} := \int_{\Omega_N} \rho \otimes^t \, d\rho_{HS} \)

Observation 2.

Due to the theorem of Seymour and Zaslavsky, \textbf{mixed–states} \( t \)-\textbf{designs} exists for any order \( t \) and matrix size \( N \).
Isoentangled 2-qubit SIC-POVM formed of 16 pure states$^1$

both partial traces form a constellation of 8 (doubly degenerated) points inside Bloch ball

\[ \text{In Alice reduction SIC-POVM yields a Platonic solid - the cube. The constellation in the reduction of Bob is not as regular as for Alice.} \]

$^1$Zhu & Englert (2011)
Hoggar provides an example (no. 24) of projective 3-design in $\mathcal{H}_4$ attained by considering particular complex polytope that consists of 60 states.

Both reductions yield the same structure inside the Bloch ball as the one generated by the standard MUB for 2 qubits.

This implies that reducing 20 states forming the standard set of MUBs for 2 qubits induces mixed 3-design.

---

5 \times 4 = 20 \text{ mixed states obtained by partial trace, } \\
\rho_j = \text{Tr}_B \langle \psi_j \rangle \langle \psi_j | \\
of 20 \text{ pure states } | \psi_j \rangle \\
\text{from iso-entangled MUB in } \mathcal{H}_2 \otimes \mathcal{H}_2 \\
\text{form a mixed states } 2\text{-design} \\
\text{inside the Bloch ball.} \\
\text{In fact they form a } 3\text{-design}! \\
\text{so that } t = 3 \geq s = 2.
Example: $t$-designs in the interval
(= averaging sets)

- Consider a measure $\mu(x)$ defined on the interval $[0, 1]$ and a minimal sequence of points $\{x_i : x_i \in [0, 1]\}_{i=1}^{M}$ such that

$$\frac{1}{M} \sum_{i=1}^{M} x_i^t = \int_{0}^{1} x^t \mu(x) \, dx.$$ (2)

- Such structures may find use in approximate integration using Taylor expansion

$$\int_{0}^{1} f(x) \, dx = \left( \sum_{i=0}^{t} \sum_{j=1}^{M} \frac{1}{i!} \left. \frac{d^i f(x)}{dx^i} \right|_{x=x_0} (x_j - x_0)^t \right) + O(x^{t+1})$$ (3)
Lebesgue measure on an interval

$n$ – number of points on interval  \quad t$ – degree of the design

- $t = n = 1$
- $t = 3, n = 2$
- $t = 5, n = 4$
- $t = 7, n = 6$
- $t = 7$

- $\mu(x) = 1$ defines flat measure.
- Configurations have been found up to $t = 7$. 
Consider the Hilbert-Schmidt measure on eigenvalues of density matrices,

\[ P(\lambda_1, \lambda_2) \sim (\lambda_1 - \lambda_2)^2 \]

which leads to the flat measure inside the Bloch Ball

\[ \mu_{HS}(x) = 3(2x - 1)^2 \]

with radius \( r = |2x - 1| \)

For \( t = 5 \) we found \( n = 4 \) points.
Projection of projective designs onto the simplex

Pure states $t$–design $\{|\psi_j\rangle\}$ in $\mathcal{H}_N$ cover the set of pure states. Their projection on the simplex due to decoherence, $p_j = \text{diag}(|\psi_j\rangle\langle\psi_j|)$ gives a $t$–design in the simplex $\Delta_N$ according to the flat measure:

(example for $N = t = 2$ and Bloch sphere).

Approximate integration rules:

**Simpson** $1 : 4 : 1$

**Gauss–Legendre** $3 : 3 \equiv 1 : 1$
a) Pure states $t$–design $\{|\psi_j\rangle\}$ in $\mathcal{H}_N$ cover the set of pure states. Their projection on the simplex due to decoherence, $p_j = \text{diag}(|\psi_j\rangle\langle\psi_j|)$ gives a $t$–design in the simplex $\Delta_N$ according to the flat measure: (example for $N = t = 2$ and the Bloch sphere).

b) Mixed states $t$–design $\{\rho_j\}$ cover the set $\Omega_N$ of mixed states. Their projection on the simplex related to spectrum, $p_j = \text{eig}(\rho_j)$, gives a $t$–design in the simplex $\Delta_N$ according to the HS measure: (example for $N = t = 2$ and the Bloch ball).
Bipartite quantum measurements with optimal single-sided distinguishability

**Problem**: Which orthonormal basis \( \{ |\psi_i\rangle \in \mathcal{H}^\otimes 2 \}_{i=1}^{N^2} \) gives optimal single-sided distinguishability?

**Maximize** trace distance between reduced states \( \rho_i = \text{Tr}_B (|\psi_i\rangle \langle \psi_i|) \) and \( \sigma_i = \text{Tr}_A (|\psi_i\rangle \langle \psi_i|) \) in both subsystems:

\[
D_{\text{max}} = \max_{\{ |\psi_i\rangle \}} \{ D : \forall i,j \ D_{\text{tr}}(\rho_i, \rho_j) = D_{\text{tr}}(\sigma_i, \sigma_j) = D \}.
\]

**Solution for** \( N = 2 \) (yields a tetrahedral structure)

\[
U_4 = \begin{pmatrix}
\langle \psi_1 \rangle \\
\langle \psi_2 \rangle \\
\langle \psi_3 \rangle \\
\langle \psi_4 \rangle
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{1}{2} + \sqrt{\frac{3}{16}}} & 0 & 0 & -\sqrt{\frac{1}{2} - \sqrt{\frac{3}{16}}} \\
\frac{1}{6} \sqrt{6 - 3\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{6} \sqrt{6 + 3\sqrt{3}} \\
\frac{1}{6} \sqrt{6 - 3\sqrt{3}} & \omega^2 & \omega & \frac{1}{6} \sqrt{6 + 3\sqrt{3}} \\
\frac{1}{6} \sqrt{6 - 3\sqrt{3}} & \omega^2 & \omega & \frac{1}{6} \sqrt{6 + 3\sqrt{3}}
\end{pmatrix}
\]

and coincides with **Elegant Joint Measurement**

Tetrahedral constellations in the Bloch ball

\[ \langle \text{Tr} (\rho^2) \rangle = \begin{cases} 1 & \text{for SIC-POVM} \\ \frac{7}{8} & \text{for Elegant Joint Measurement} \\ \frac{4}{5} & \text{for mixed 2-designs} \end{cases} \]
with the largest single-sided distinguishability
written as rows of a unitary matrix $U$ of size $N^2 = 9$ rescaled by $1/6\sqrt{3}$

$$
\begin{pmatrix}
10 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -2 \\
1 & -3i\sqrt{3} & 0 & 3i\sqrt{3} & 7 & 0 & 0 & 0 & -2 \\
1 & 3i\sqrt{3} & 0 & -3i\sqrt{3} & 7 & 0 & 0 & 0 & -2 \\
1 & 3 & 3\sqrt{2} & 3 & 1 & 3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} & 4 \\
1 & -3 & 3\sqrt{2} & -3 & 1 & -3\sqrt{2} & 3\sqrt{2} & -3\sqrt{2} & 4 \\
1 & 3 & -3\omega\sqrt{2} & 3 & 1 & -3\omega\sqrt{2} & 3\omega^2\sqrt{2} & 3\omega^2\sqrt{2} & 4 \\
1 & -3 & -3\omega\sqrt{2} & -3 & 1 & 3\omega\sqrt{2} & 3\omega^2\sqrt{2} & -3\omega^2\sqrt{2} & 4 \\
1 & 3 & 3\omega^2\sqrt{2} & 3 & 1 & 3\omega^2\sqrt{2} & -3\omega\sqrt{2} & -3\omega\sqrt{2} & 4 \\
1 & -3 & 3\omega^2\sqrt{2} & -3 & 1 & -3\omega^2\sqrt{2} & -3\omega\sqrt{2} & 3\omega\sqrt{2} & 4 \\
\end{pmatrix}
$$

with $\omega = \exp(2i\pi/3)$. 
arbitrary $N$: Optimal basis in $\mathcal{H}_N \otimes \mathcal{H}_N$

Solution found if there exist a SIC POVM $\{|i_0\rangle\}_{j=1}^{N^2}$ in $\mathcal{H}_N$.

**Optimal basis** in $\mathcal{H}_N \otimes \mathcal{H}_N$ reads

$$|\psi_i\rangle = \sqrt{\lambda_{\text{max}}} |i_0\rangle |i_0^*\rangle - \sqrt{\frac{1 - \lambda_{\text{max}}}{N - 1}} \sum_{j=1}^{N-1} |i_j\rangle |i_j^*\rangle , \quad i = 1, \ldots, N^2$$

Vectors $|i_j^*\rangle \in \mathcal{H}^B$ are obtained by conjugate components of $|i_j\rangle$ for $i = 1, \ldots, N^2$ and $j = 0, \ldots, N - 1$.

Dominating Schmidt coefficient $\lambda_{\text{max}}$ is

$$\lambda_{\text{max}} = \frac{N^3 - N^2 - N + 2(N - 1)\sqrt{N + 1} + 2}{N^3}.$$  

and tends to unity for $N \rightarrow \infty$.  

Concluding Remarks

- An invitation to quantum combinatorics: a search for discrete structures in Hilbert space...
- Configuration of 20 pure states in $\mathcal{H}_4$ which form the full set of 5 iso-entangled MUBs for 2 qubits is constructed.
- Notion of mixed states $t$-design is introduced and necessary and sufficient conditions for $\{\rho_j\}$ to be a $t$-design are established.
- Projective $t$-designs on composite spaces $\mathcal{H}_N \otimes \mathcal{H}_N$ induce, by partial trace, mixed states $t$-designs in the set $\Omega_N$ of mixed states.
- Simplicial $t$–designs in the simplex $\Delta_n$ obtained from projective $t$–designs $\{|\psi_j\rangle\}$ in $\mathcal{H}_N$ by decoherence, $p_j = \text{diag}(|\psi_j\rangle\langle\psi_j|)$.
- Bi-partite orthogonal basis with optimal single side distinguishability: reduced states for $N = 2$ gives rescaled tetrahedron of SIC equivalent to Elegant Joint Measurement (EJM)
- Analitical form of EJM found for higher $N$, for which SIC is known.