Post-processing of quantum instruments

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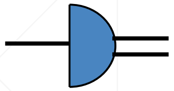
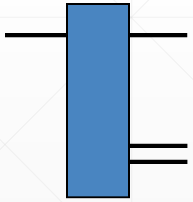
Sequences of measurements

Usefull for:

- **Tomography, estimation, property testing, computation**
- **Q. sequential decoding, joint measurability, sequential state discrimination,**
- **simulation of complex measurements on NISQ devices**

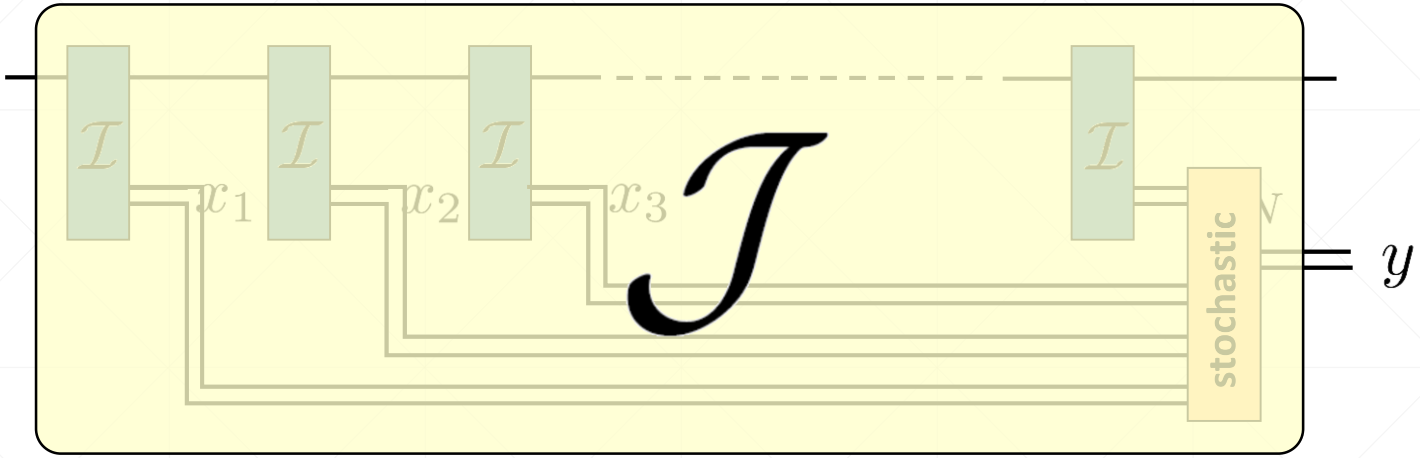


POVM = Positive operator valued measure
- describes only outcome probabilities



Instrument = collection of CPTD maps, which sum up to a channel
- describes also post measurement state

Concatenation of instruments



If $\mathcal{I}$ is **Lüders instrument of a nondegenerate projective measurement**

i.e. $\mathcal{I}_x(\rho) = P_x \rho P_x \quad P_x = |\psi_x\rangle\langle\psi_x|$

then **repetitive use of the same measurement apparatus does not help**

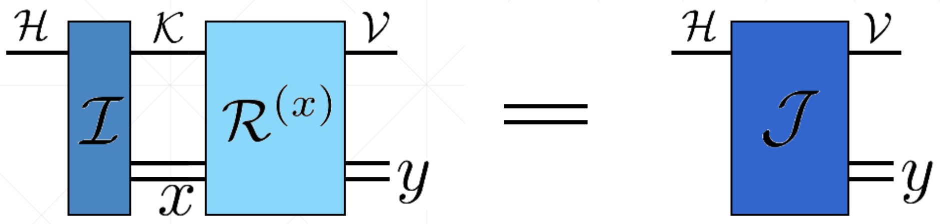
but

If $\mathcal{I}$ is **a noisy version of Lüders instrument of a nondegenerate projective measurement**

then **repetitive use can suppress the noise level**

Haapasalo, Heinosaari and Kuramochi, Saturation of repeated quantum measurements, J. Phys. A: Math. Theor. 49, 33LT01 (2016)

Postprocessing of instruments



$$\mathcal{J}_y(\varrho) = \sum_{x\in\Omega} \mathcal{R}_y^{(x)}(\mathcal{I}_x(\varrho))$$

Suppose that for $\mathcal{I}, \mathcal{J}$ instruments $\mathcal{R}^{(x)}$ exist so that the above Eq. hold, then **we denote it** $\mathcal{I} \to \mathcal{J}$

If also the opposite relation $\mathcal{J} \to \mathcal{I}$ holds, we say $\mathcal{I}$ and $\mathcal{J}$ are postprocessing equivalent $\mathcal{I} \leftrightarrow \mathcal{J}$

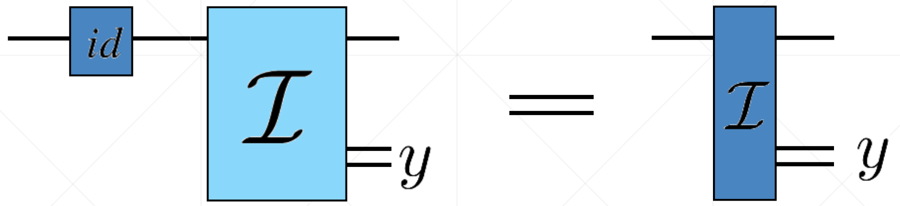
Our main aim:

Characterization of partial order induced on the equivalence classes

Greatest element = [identity]

For any instrument $\mathcal{I}$ we clearly have:

$$id \to \mathcal{I}$$



Equivalence class of the greatest element

$$\mathcal{I} \to id \Rightarrow \boxed{\mathcal{I}_x(\varrho) = \sum_{i=1}^{n_x} p_{xi} V_{xi} \varrho V_{xi}^*}$$

$$p_{xi} \geq 0 \quad \sum_{x,i} p_{xi} = 1$$

$V_{xi} : \mathcal{H} \to \mathcal{K}$ isometry

$V_{xj}^* V_{xi} = 0$ for $i \neq j$

Interpretation: randomly choosing outcome and a Kraus operator mapping into OG subspaces is reversible.

Nontrivial is necessity of above form, **proof = adaptation of method by** A. Nayak, and P. Sen

Indecomposable instruments

$\mathcal{M} = \mathcal{N} + \mathcal{N}' \Rightarrow \quad \mathcal{N} = \mu\mathcal{M}$, $\mathcal{N}' = \mu'\mathcal{M}$

= build from indecomposable q. operations, i.e. all the operations are Kraus rank one

Kraus representation of the instrument $\mathcal{I} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$

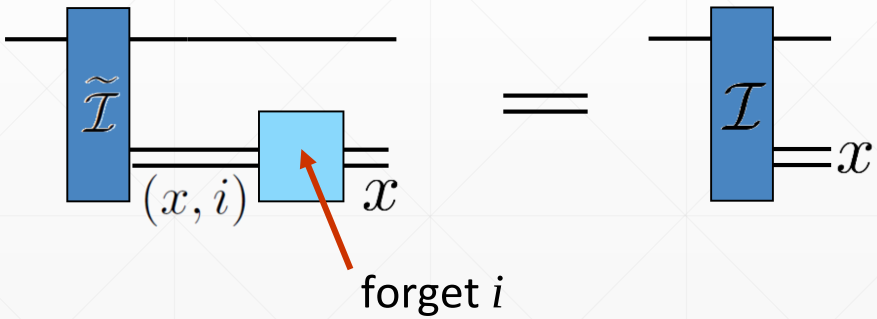
$$\mathcal{I}_x(\rho) = \sum_{i=1}^{n_x} K_{ix} \rho K_{ix}^\dagger \qquad \forall x \in \Omega$$

Let's define **detailed instrument** $\widetilde{\mathcal{I}} \in \mathrm{Ins}(\Delta, \mathcal{H}, \mathcal{K})$

$$\Delta = \{(x,i) : x \in \Omega, i = 1, \ldots, n_x\}$$

$$\widetilde{\mathcal{I}}_{(x,i)}(\rho) = K_{ix} \rho K_{ix}^\dagger$$

Every instrument is a postprocessing of some **detailed instrument**



Indecomposable instruments

(= operations are Kraus rank one)

Sufficient condition for equivalence with indecomposable instrument

if

$$\mathcal{I}_x(\varrho) = \sum_{i=1}^{n_x} K_{ix}\varrho K_{ix}^*$$

$$K_{ix}^* K_{jx} = 0 \quad \text{all } i \neq j \text{ and } x$$

then $\mathcal{I} \to \mathcal{J}$ **for some indecomposable** $\mathcal{J}$

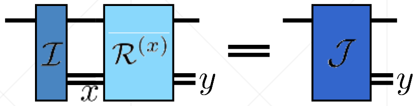
Interpretation: If for each outcome we have Kraus representation in which the output ranges are OG then the instrument can be refined by further measurements

Open question

Is this form also a necessary condition?

Indecomposable instruments

(= operations are Kraus rank one)



$\mathsf{A}^{\mathcal{I}} \longleftarrow \mathsf{A}^{\mathcal{J}}$ is refinement !

Proposition:

Suppose $\mathcal{I} \to \mathcal{J}$ and $\mathcal{J}$ is indecomposable instrument $\Rightarrow$ $\mathsf{A}^{\mathcal{J}} \to \mathsf{A}^{\mathcal{I}}$ for Induced POVMs

Proof:

$$\mathcal{J}_y = \sum_{x\in\Omega} \mathcal{R}_y^{(x)} \circ \mathcal{I}_x$$

$\{A_{ix}\}_i,\ B_y\ \ \{R_{ky}^{(x)}\}_k$ Kraus operators of $\mathcal{I}_x,\ \mathcal{J}_y$, and $\mathcal{R}_y^{(x)}$

$$B_y \varrho B_y^* = \sum_{i,k,x} R_{ky}^{(x)} A_{ix} \varrho A_{ix}^* (R_{ky}^{(x)})^*$$

$$R_{ky}^{(x)} A_{ix} = u_{ikxy} B_y \ \text{ for all } i, k, x, y \qquad \sum_{i,k,x} |u_{ikxy}|^2 = 1$$

$$\sum_i A_{ix}^* A_{ix} = \sum_{i,k,y} A_{ix}^* (R_{ky}^{(x)})^* R_{ky}^{(x)} A_{ix} = \sum_{i,k,y} |u_{ikxy}|^2 B_y^* B_y$$

$$\mathsf{A}^{\mathcal{I}}(x) = \sum_i A_{ix}^* A_{ix}$$

$$\mathsf{A}^{\mathcal{J}}(y) = B_y^* B_y$$

$$\nu_{yx} = \sum_{i,k} |u_{ikxy}|^2 \geqslant 0 \qquad \sum_{x\in\Omega} \nu_{yx} = 1$$

$$\mathsf{A}^{\mathcal{I}}(x) = \sum_{y\in\Lambda} \nu_{yx} \mathsf{A}^{\mathcal{J}}(y)$$

$$\mathsf{A}^{\mathcal{J}} \to \mathsf{A}^{\mathcal{I}}.$$

Indecomposable instruments

How we check if two indecomposable instruments are equivalent?

Previous proposition implies **necessary condition**

$$\mathcal{I} \leftrightarrow \mathcal{J} \quad \begin{cases} \mathcal{I} \to \mathcal{J} \quad \Rightarrow \quad \mathsf{A}^{\mathcal{J}} \to \mathsf{A}^{\mathcal{I}} \\ \mathcal{J} \to \mathcal{I} \quad \Rightarrow \quad \mathsf{A}^{\mathcal{I}} \to \mathsf{A}^{\mathcal{J}} \end{cases} \quad \mathsf{A}^{\mathcal{I}} \leftrightarrow \mathsf{A}^{\mathcal{J}}$$

Proposition:

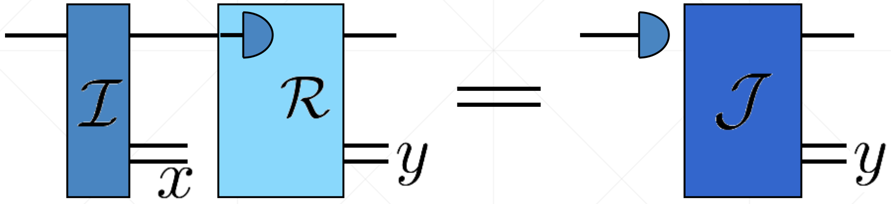
Suppose $\mathcal{I}$ **and** $\mathcal{J}$ **are indecomposable instruments**

$\Rightarrow \quad \mathcal{I} \leftrightarrow \mathcal{J}$ **if and only if** $\mathsf{A}^{\mathcal{I}} \leftrightarrow \mathsf{A}^{\mathcal{J}}$

Least element = [trash&prepare]

Trash and prepare instrument

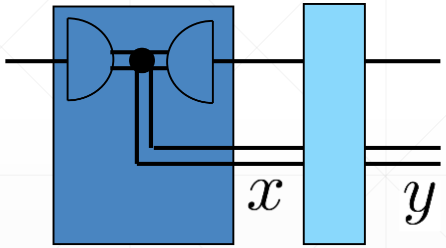
$$\mathcal{T}_y(\varrho) = \mathrm{tr}[\varrho] p_y \xi_y$$



For any instrument $\mathcal{I}$

$$\mathcal{I} \to \mathcal{T}$$

measure & prepare:



$$\mathcal{J}_y(\varrho) = \sum_{x \in \Omega} \mathrm{tr}\left[\nu_{xy} \mathsf{A}(x) \varrho\right] \xi_{xy}$$

$$\mathcal{I} \to \mathcal{J} \quad \Rightarrow \mathsf{A}^{\mathcal{I}} \to \mathsf{A}^{\mathcal{J}}$$

Proposition:

Suppose $\mathcal{I}$ **and** $\mathcal{J}$ **are measure & prepare instruments**

$\Rightarrow \quad \mathcal{I} \leftrightarrow \mathcal{J}$ **if and only if** $\mathsf{A}^{\mathcal{I}} \leftrightarrow \mathsf{A}^{\mathcal{J}}$

Simulation of instruments

= random mixing of different instruments + their postprocessing
(we keep track of the choices)

- analogical to simulation of POVMs

L. Guerini, J. Bavaresco, M. Terra Cunha, and A. Acín, J. Math. Phys. **58**, 092102 (2017)
S. N. Filippov, T. Heinosaari, and L. Leppäjärvi, Phys. Rev. A 97, 062102 (2018)

$$\mathcal{I}_y(\varrho) = \sum_{i=1}^{n} p_i \sum_{x\in\Omega} \mathcal{R}_y^{(i,x)}\left(\mathcal{J}_x^{(i)}(\varrho)\right)$$

Simulation irreducible instruments = equivalence class of identity instrument

Relation of postprocessing to other works and concepts

	special case when	Selected references
Postprocessing of POVMs	**1-dimensional output space**	**H. Martens, W. M. de Muynck,** Nonideal quantum measurements, Found. Phys. **20**, 255 (1990). **F. Buscemi, G. M. D'Ariano, M. Keyl, P. Perinotti, and R. Werner,** Clean POVMs, J. Math. Phys. **46**, 082109 (2005).
Postprocessing of channels	**Single outcome instruments**	**T. Heinosaari and T. Miyadera,** Phys. Rev. A **88**, 042117 **A. Jenčová**, arXiv:2002.04240 (2020). **C. Bény and O. Oreshkov**, Phys. Rev. A **84**, 022333 (2011).

- In these special cases our results are in accordance with the known results

Postprocessing & simulation of devices

	POVMs	*Channels*	*Instruments*
Maximal elements *Greatest element*	Rank-1 POVMs None	Random orthogonal isometric channels	Random orthogonal isometric instruments
Least element	Trivial POVMs	Trash-and-prepare channels	Trash-and-prepare instruments
Indecomposable elements	Rank-1 POVMs	Kraus rank-1 channels	Kraus rank-1 instruments
Extreme simulation irreducible elements	Extreme rank-1 POVMs	Isometric channels	Isometric channels

Summary & Outlook

- **Presented results = Physical Review A 103, 022615 (2021)**
- **postprocessing of instruments was introduced**
- **Greatest and least elements and their equivalence classes found**
- **if and only if conditions for equivalence of indecomposable and measure&prepare instruments**
- **Sufficient condition for equivalence with indecomposable instrument**
- **Simulation irreducible instruments found**

Open questions & future work:

When is an instrument (postprocessing) equivalent with

- an indecomposable instrument?
- one of its detailed instruments?

Thanks for your attention.