Generic formula for dynamical entropy of the ball & point quantum system

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joint work with Wojciech Słomczyński

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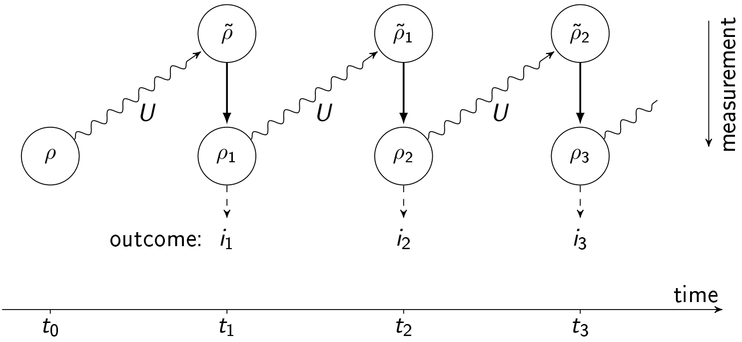
- Successive measurements of a d-dim quantum system.
- There are k possible measurement outcomes.
- Unitary evolution U applied between two subsequent measurements.

$\Rightarrow$ We observe strings of random measurement outcomes.

$\Rightarrow$ We want to quantify the **irreducible randomness** of these outcomes.

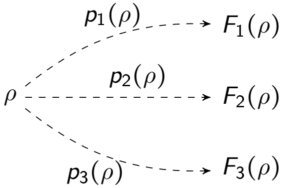
$\Rightarrow$ This is done by **quantum dynamical entropy**.



For an input state ρ, a unitary U, and a POVM $\Pi = \{\Pi_1, \ldots, \Pi_k\}$:

- the probability of obtaining the result i:
  $\mathsf{p}_i(\rho)$
- the post-measurement state (if the outcome i has been obtained):
  $\mathsf{F}_i(\rho)$

Partial Iterated Function System (PIFS) generated by U and Π.



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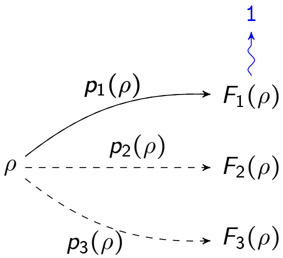

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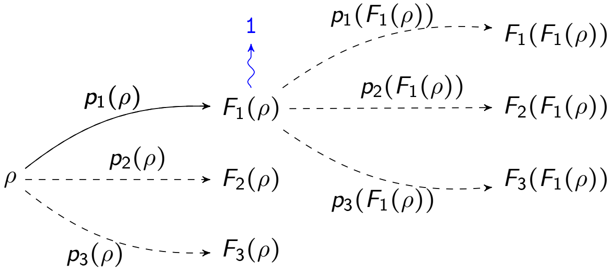

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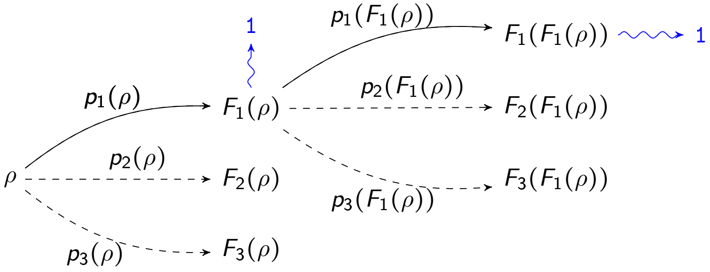

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- the post-measurement state (if the outcome i has been obtained):
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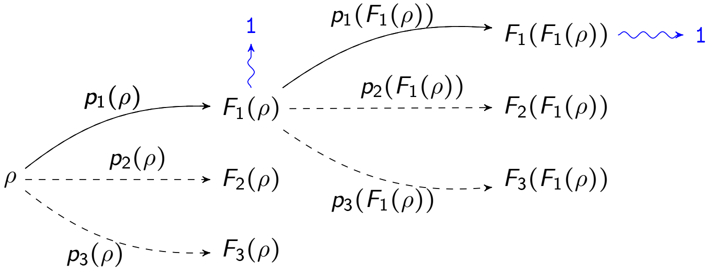

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Partial Iterated Function System (PIFS) generated by U and Π.

$\Rightarrow$ $\mathcal{F}_{U,\Pi}$ induces an aggregated Markov chain with outcomes $\{1, \ldots, k\}$ and hidden state space

$\Rightarrow$ Probability of outputting the string of outcomes $(i_1, \ldots, i_n)$:

$$\mathsf{p}_{i_1,\ldots,i_n}(\rho) = \mathsf{p}_{i_1}(\rho)\mathsf{p}_{i_2}(\mathsf{F}_{i_1}(\rho)) \cdot \ldots \cdot \mathsf{p}_{i_n}(\mathsf{F}_{i_{n-1}} \ldots \mathsf{F}_{i_1}(\rho)).$$

$\Rightarrow$ Evolution of Dirac delta measures on quantum states:

$$\Psi\colon\ \delta_\rho \longmapsto \sum_{\substack{i=1,\ldots,k \\ \mathsf{p}_i(\rho)>0}} \mathsf{p}_i(\rho)\, \delta_{\mathsf{F}_i(\rho)}.$$

n-th partial entropy:

$$H_n := \sum_{i_1,\ldots,i_n=1}^{k} \eta\left(\mathrm{p}_{i_1,\ldots,i_n}(\rho_*)\right) \quad \text{where} \quad \eta(x) := \begin{cases} -x\ln x & x > 0 \\ 0 & x = 0 \end{cases}$$

Quantum dynamical entropy of U with respect to Π:

$$H(U,\Pi) := \lim_{n\to\infty} \frac{H_n}{n} = \lim_{n\to\infty} \left(H_{n+1} - H_n\right)$$

Blackwell integral formula (1957):

$$H(U,\Pi) = \int_{\mathcal{S}(\mathbb{C}^d)} H_1 \, \mathrm{d}\mu_*$$

where μ_* is the weak-* limit of $(\Psi^n(\delta_{\rho_*}) : n \in \mathbb{N})$.

Shannon (1948), Kolmogorov (1958);
Srinivas (1978), Pechukas (1982), Beck & Graudenz (1992) - for projective measurements; Słomczyński & Życzkowski (1994) - for generalized measurements; Crutchfield & Wiesner (2008) - *entropy rate*

rank-1 POVMs

Let Π consist of k one-dim (rescaled) projections: $\Pi_i = \frac{d}{k} |\varphi_i\rangle \langle\varphi_i|$

- Probabilities in the first step: $p_i(\rho_*) = \frac{1}{k}$
- Post-measurement state: $\mathsf{F}_i(\rho) = |\varphi_i\rangle \langle\varphi_i|$ for every ρ and i
- Probabilities in the subsequent steps: $p_j(\mathsf{F}_i(\rho)) = \frac{d}{k} \left|\langle\varphi_j|U|\varphi_i\rangle\right|^2$

$$\Rightarrow \quad \mu_* \ \text{ is uniform,} \qquad H(U, \Pi) = \frac{1}{k} \sum_{i,j=1}^{k} \eta\left(\frac{d}{k} \left|\langle\varphi_j|U|\varphi_i\rangle\right|^2\right)$$

W. Słomczyński, AS, "Quantum Dynamical Entropy, Chaotic Unitaries and Complex Hadamard Matrices", IEEE Trans. Inform. Theory 63 (2017)

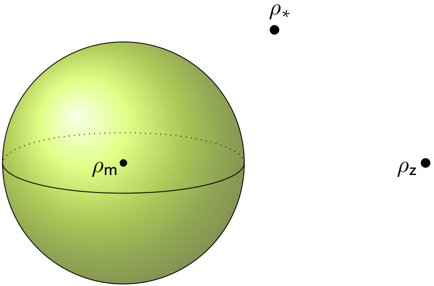
non-rank-1 measurements: simplest case

- rank-1 POVMs: $F_1, \ldots, F_k$ are constant, *one symbol* ~ *one state*.
- In general: *one symbol* ~ *many states*.

$$U \in \mathcal{U}(\mathbb{C}^3),\ \Pi = \{\underbrace{|z\rangle\langle z|}_{\text{symbol } 0},\ \underbrace{\mathbb{I} - |z\rangle\langle z|}_{\text{symbol } 1}\}$$

$$F_0 \colon \mathcal{S}(\mathbb{C}^3) \to \text{point}$$

$$F_1 \colon \mathcal{S}(\mathbb{C}^3) \to \text{Bloch ball}$$



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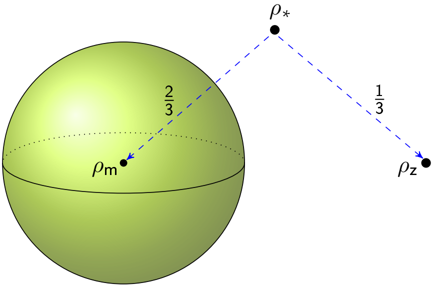

Initial distribution:
$\frac{1}{3}\delta_{\rho_z} + \frac{2}{3}\delta_{\rho_m}$

Limiting distribution
$\mu_* = ?$

Dynamical entropy
$H(U, \Pi) = ?$

Ball dynamics

From now on we put $P := \mathbb{I} - |z\rangle\langle z|$ and $\Theta := \mathrm{Im}(P) = \mathrm{span}\{z\}^{\perp}$,

i.e., P projects on Θ, $\mathcal{P}(\Theta) \sim \mathbb{CP}^1$ is the Bloch sphere, $\mathcal{S}(\Theta)$ is the Bloch ball.

$$\text{Ball dynamics:} \quad \mathcal{S}(\Theta) \ni \rho \longmapsto \frac{PU\rho U^* P}{\mathrm{tr}(PU\rho U^* P)} \in \mathcal{S}(\Theta)$$

$$\text{Sphere dynamics:} \quad \mathcal{P}(\Theta) \ni |w\rangle\langle w| \longmapsto \frac{PU|w\rangle\langle w|U^* P}{||PUw||^2} \in \mathcal{P}(\Theta)$$

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$$\text{For state vectors:} \quad \Theta \ni w \longmapsto \frac{PUw}{\|PUw\|} \in \Theta, \quad \text{where} \quad \|w\| = 1$$

$$\text{For rays:} \quad \mathbb{P}\Theta \ni [w] \longmapsto [PUw] \in \mathbb{P}\Theta$$

This is a **Möbius map**, i.e., an orientation-preserving and angle-preserving automorphism of the Riemann sphere.

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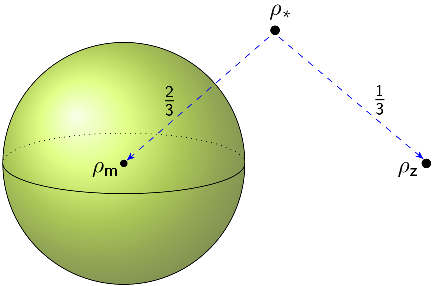

Assume that z is an eigenvector of U. Then:

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State space:

Transition matrix:

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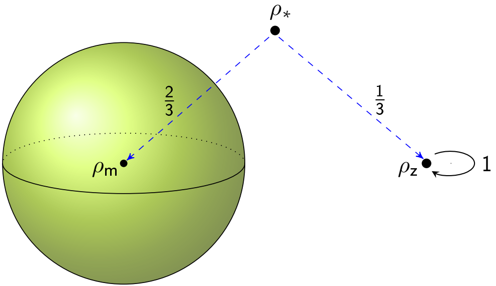

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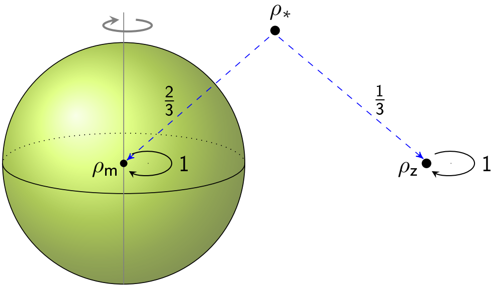

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- Ball dynamics is rotation (it is generated by a unitary operator),
  - $\Rightarrow$ ρ_m is a fixed point,
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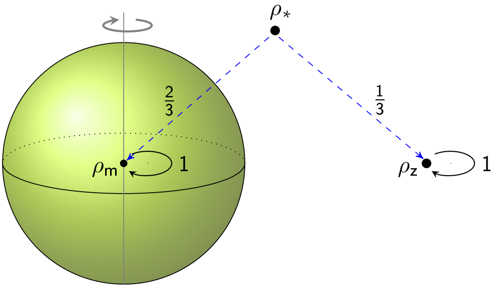

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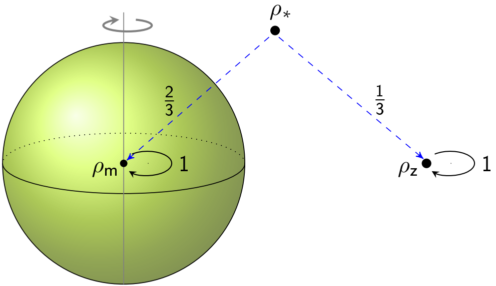

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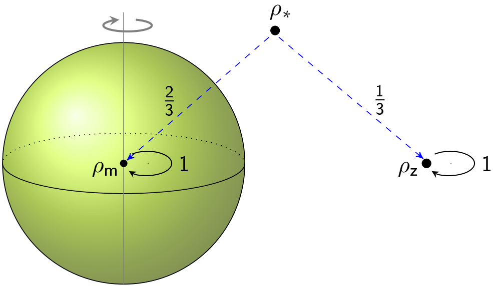

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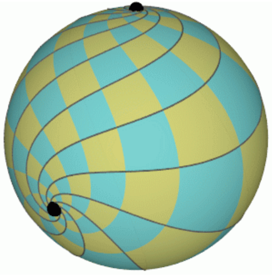
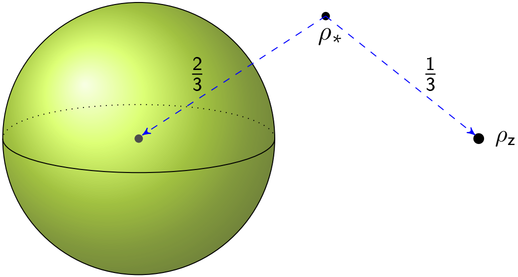


Two fixed points: attractive ρ_{atr} & repulsive ρ_{rep}

Pictures' source: *Hyrodium's Graphical MathLand* `http://hyrodium.tumblr.com/`

Generic chain



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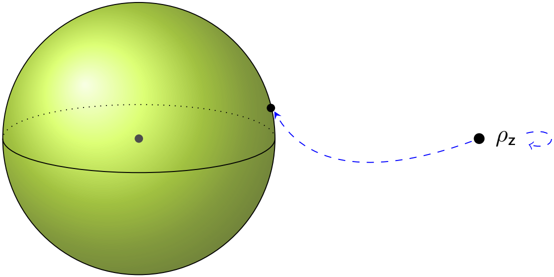

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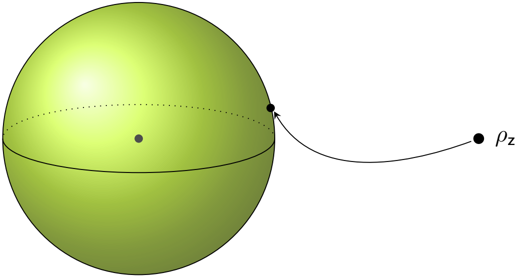

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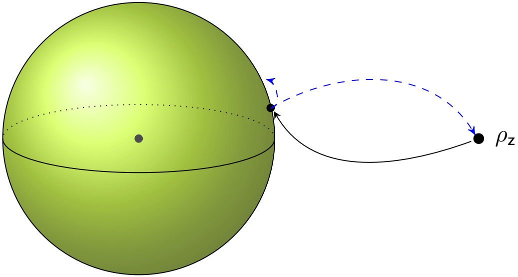

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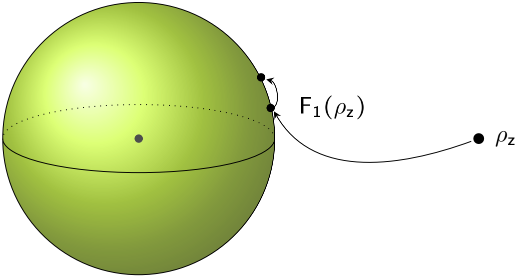

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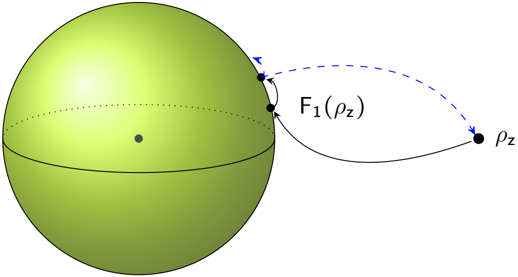

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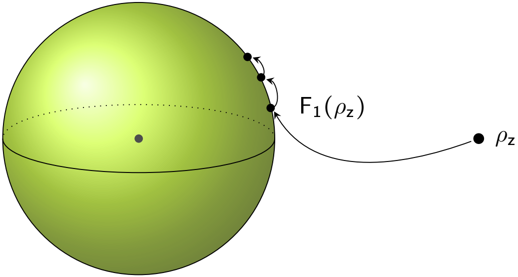

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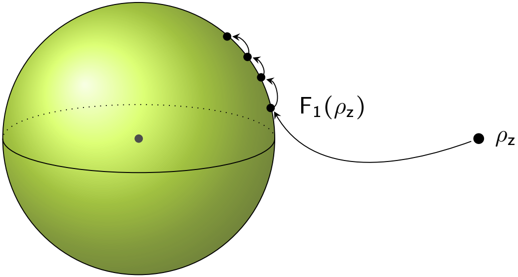

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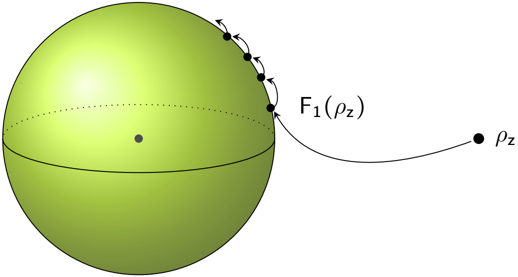

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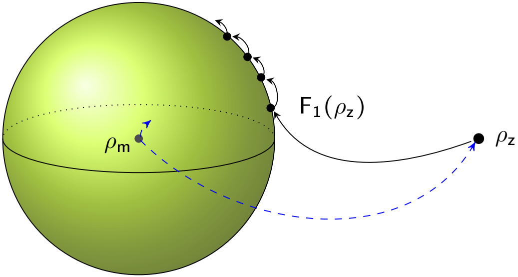

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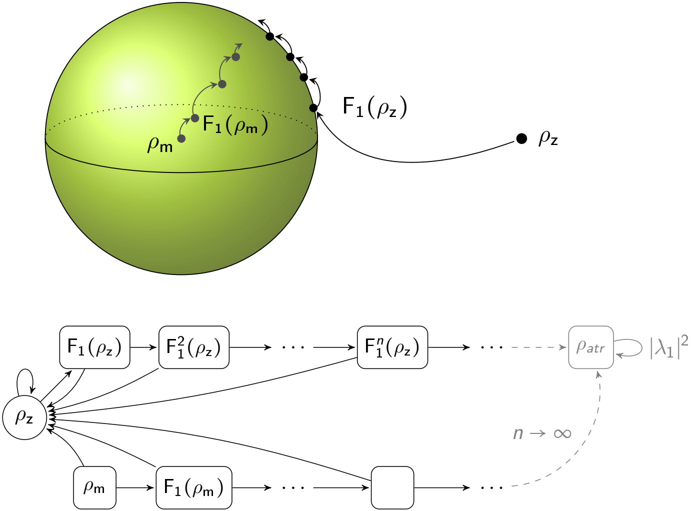
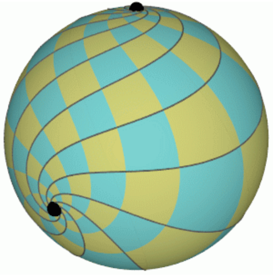


One of non-generic chains



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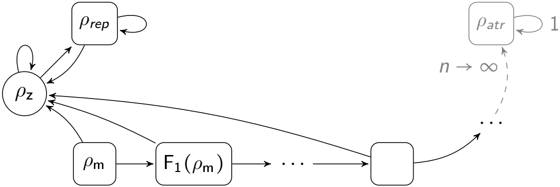

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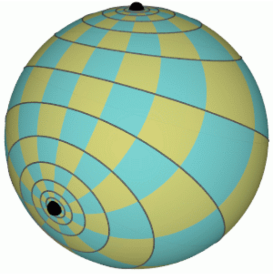

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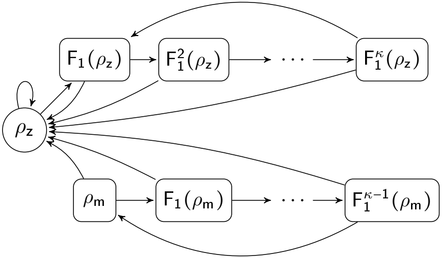
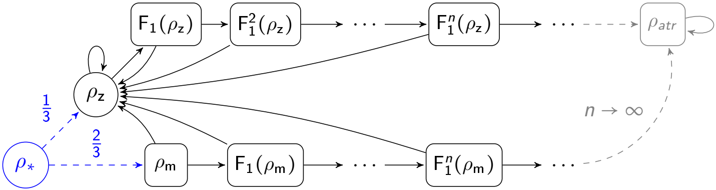


Limiting measure μ_* of the generic chain



Recall that $\Psi^n(\delta_{\rho_*}) \xrightarrow{w^*} \mu_*$.
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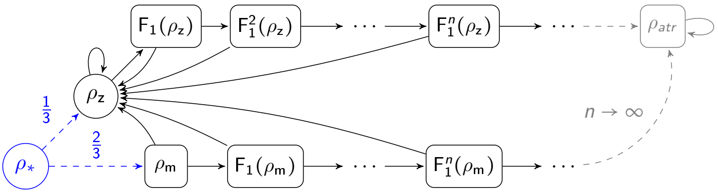

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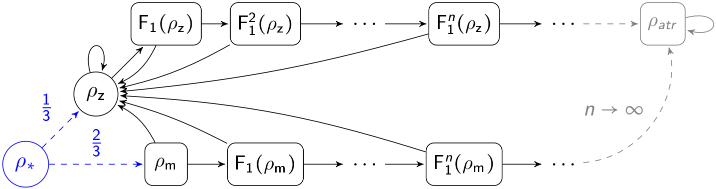
Trajectory of ρ_m:

- $\Psi(\delta_{\rho_m}) = \mathsf{p}_1(\rho_m)\delta_{\mathsf{F}_1(\rho_m)} + \mathsf{p}_0(\rho_m)\delta_{\rho_z}$
- $\Psi^n(\delta_{\rho_m}) = \frac{1}{2}\operatorname{tr}((PU)^n(PU)^{*n})\delta_{\mathsf{F}_1^n(\rho_m)} +$ the part on the trajectory of ρ_z

Limiting measure μ_* of the generic chain



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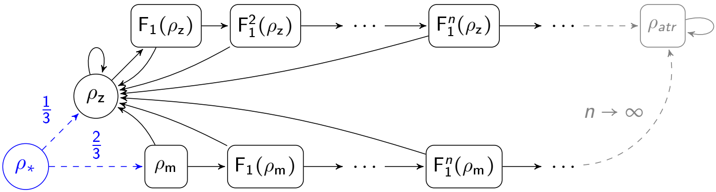
- $\Psi(\delta_{\rho_m}) = p_1(\rho_m)\delta_{F_1(\rho_m)} + p_0(\rho_m)\delta_{\rho_z}$
- $\Psi^n(\delta_{\rho_m}) = \frac{1}{2}\,\mathrm{tr}((PU)^n(PU)^{*n})\delta_{F_1^n(\rho_m)} +$ the part on the trajectory of ρ_z
- Generically: $\lim_{n\to\infty} \mathrm{tr}((PU)^n(PU)^{*n}) = 0$.
- So μ_* is supported on the trajectory of ρ_z.

Limiting measure μ_* of the generic chain



Recall that $\Psi^n(\delta_{\rho_*}) \xrightarrow{w^*} \mu_*$.

Trajectory of ρ_z: the *success-run* chain (or *reliability* chain).

- it can be positive recurrent or null recurrent or transient, depending on the transition probabilities.
- in our case it's positive recurrent, so μ_* does not reach ρ_{atr}.
- we have:

$$\mu_* = \frac{1}{c}\sum_{n=0}^{\infty} \|(PU)^n z\|^2 \delta_{F_1^n(\rho_z)} \quad \text{where} \quad c := \sum_{n=0}^{\infty} \|(PU)^n z\|^2$$

Theorem

Let U be a unitary operator on $\mathbb{C}^d$ and let z be a unit vector in $\mathbb{C}^d$. We put Θ for the orthogonal complement of z in $\mathbb{C}^d$ and P for the orthogonal projection on Θ. Then we have

$$\sum_{n=0}^{\infty} \|(PU)^n z\|^2 = d - \sum_{\lambda \in \sigma(U)} \dim(\Theta \cap \mathrm{Ker}(U - \lambda \mathbb{I}))$$

$$= d - \#\{\text{lin. indep. eigenvectors of } U \text{ orthogonal to } z\}$$

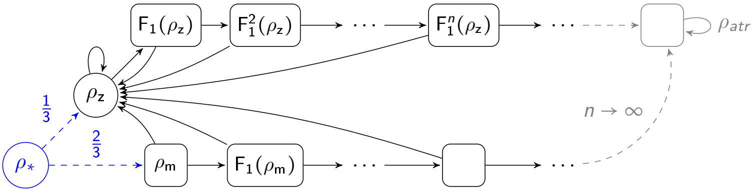
Corollary

For a generic choice of U and z:

$$\sum_{n=0}^{\infty} \|(PU)^n z\|^2 = d$$

W. Słomczyński, AS, "Orthogonal Projections on Hyperplanes Intertwined with Unitaries", J Phys A 54 (2021)

Dynamical entropy of the generic chain



$$\mathrm{c} = \sum_{n=0}^{\infty} \|(PU)^n \mathrm{z}\|^2 = 3$$

$$\mu_* = \frac{1}{3} \sum_{n=0}^{\infty} \|(PU)^n \mathrm{z}\|^2 \delta_{F_1^n(\rho_z)} = \frac{1}{3} \sum_{n=0}^{\infty} \mathrm{p}_{0\underbrace{1\cdots1}_{n}}(\rho_*)\delta_{F_1^n(\rho_z)}$$

$$H(U,\Pi) = \frac{1}{3}\mathrm{h}(\|PU\mathrm{z}\|^2) + \frac{1}{3} \sum_{n=1}^{\infty} \|(PU)^n \mathrm{z}\|^2\, \mathrm{h}\left(\frac{\|(PU)^{n+1}\mathrm{z}\|^2}{\|(PU)^n \mathrm{z}\|^2}\right)$$

$$\text{where } \mathrm{h}(x) := \mathrm{H}_2(x, 1-x) = \eta(x) + \eta(1-x)$$

- 8 different chain types can be generated:
  - Invertible ball dynamics (Möbius maps): 5 chain types,
  - Non-invertible ball dynamics: 3 chain types.
  - Max. 6 chain types possible for a given unitary.
- For each chain type we know the formula for μ_* and $H(U, \Pi)$.
- Chain types are characterized by the eigenvalues of PU.
- Chain types are also characterized by the position of $\langle z|U|z\rangle$ in the numerical range of U.

  `AS, "Classification of Markov chains generated by the 'ball & point' quantum system", soon on arXiv`

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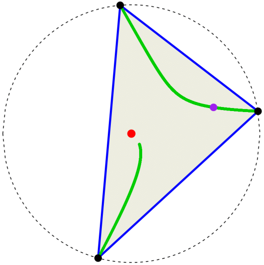

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- Generically, we have $\sum_{n=0}^{\infty} \|(PU)^n z\|^2 = d$

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  "A new dimension witness, or the Adventures of Alice, an Addicted Traveller"

  Talk by Wojciech Słomczyński - tomorrow at 17:30 JPN, 10:30 CET