Causal and compositional structure of higher order quantum maps

Alessandro Bisio

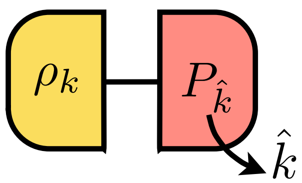
Third Kyoto Workshop on Quantum Information, Computation, and Foundations

YITP, Kyoto 18th October 2022

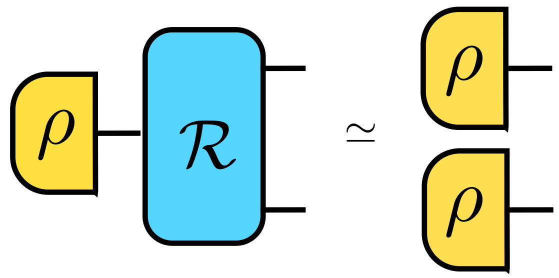
AB, L. Apadula, P. Perinotti arXiv:2202.10214

Prelude

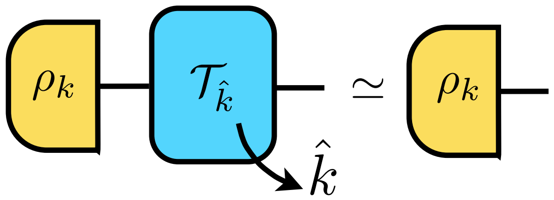
Quantum states as carriers of information



state estimation



cloning

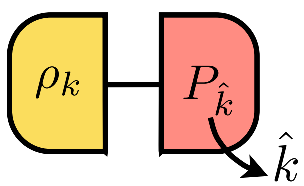


information-disturbance tradeoff

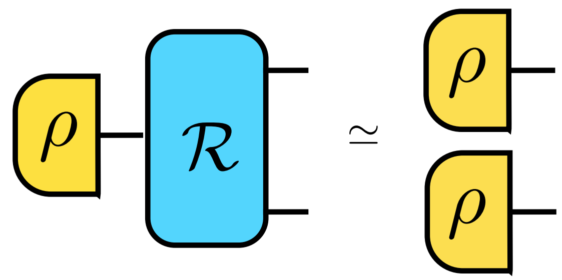
The "best" state trasformations allowed by quantum theory ?

Prelude

Quantum states as carriers of information



state estimation



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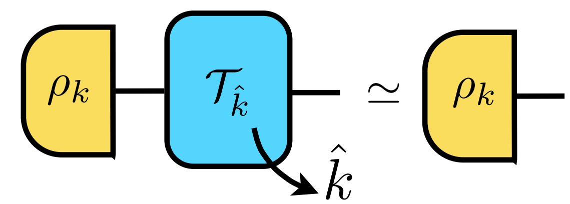
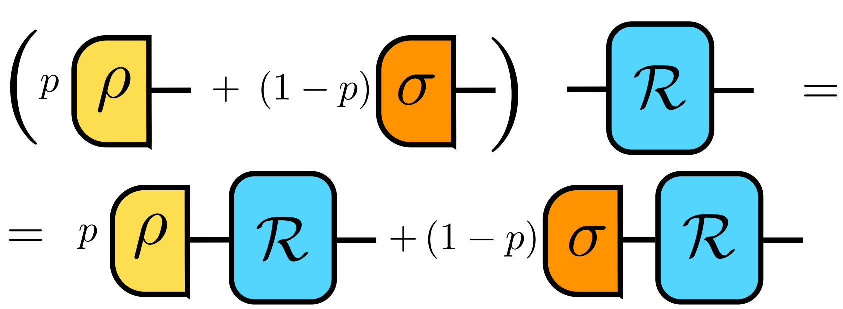


information-disturbance tradeoff

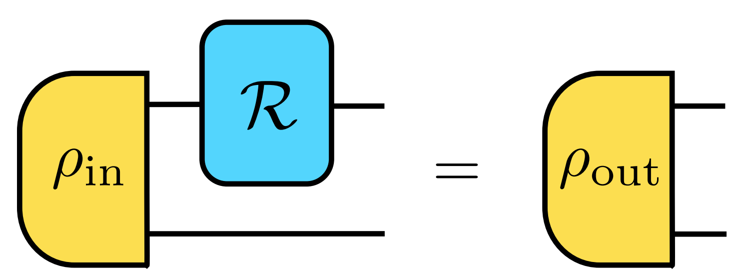
The “best” state trasformations allowed by quantum theory ?

Admissibility conditions

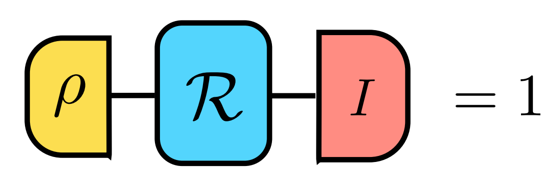
linearity



complete positivity

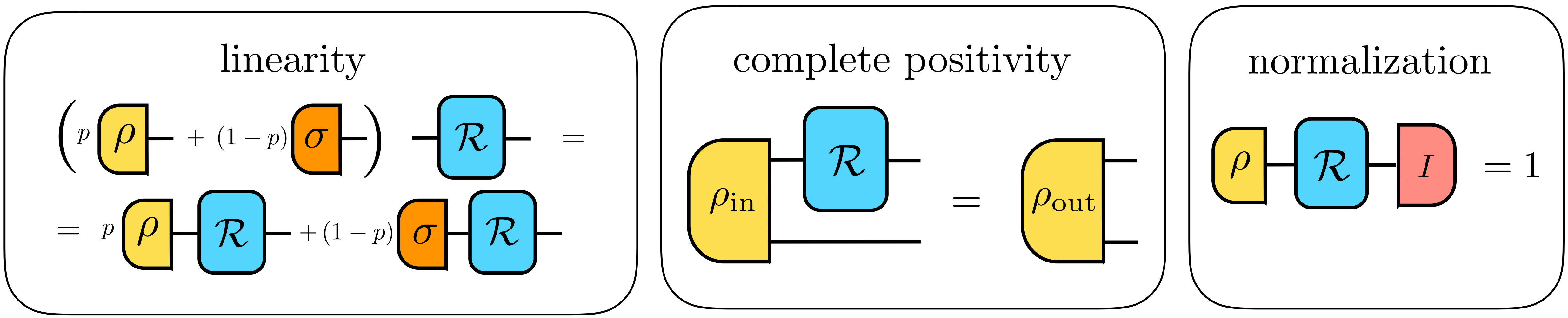


normalization

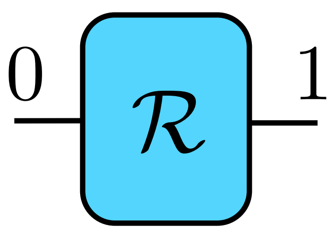


Prelude

Admissibility conditions



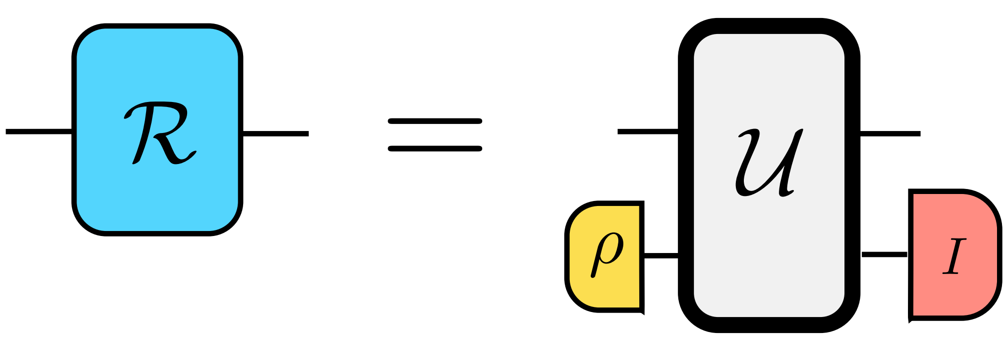
The most general (deterministic) transformation



Choi representation

$$R \in \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_0), \quad R \geqslant 0\,, \qquad \mathrm{Tr}_1[R] = I_0$$

Realisation theorem

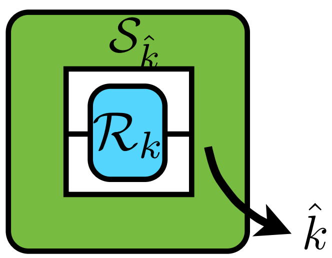


Stinespring dilation

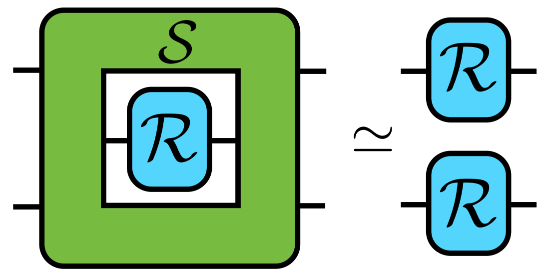
Prelude

Quantum transformations as carriers of information

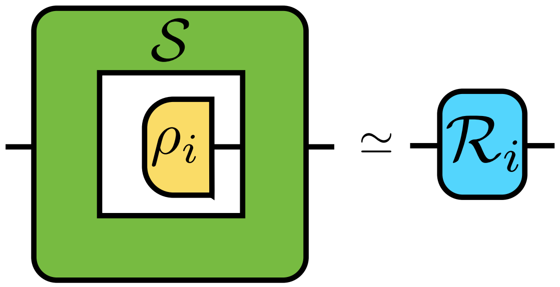
Channel estimation



Cloning



Programmable channel



Prelude

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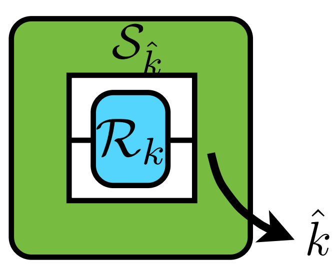

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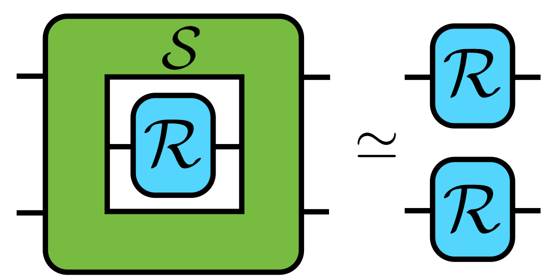

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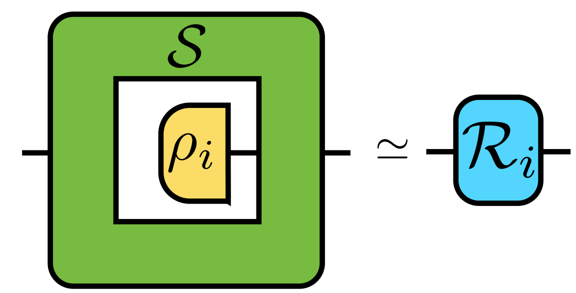
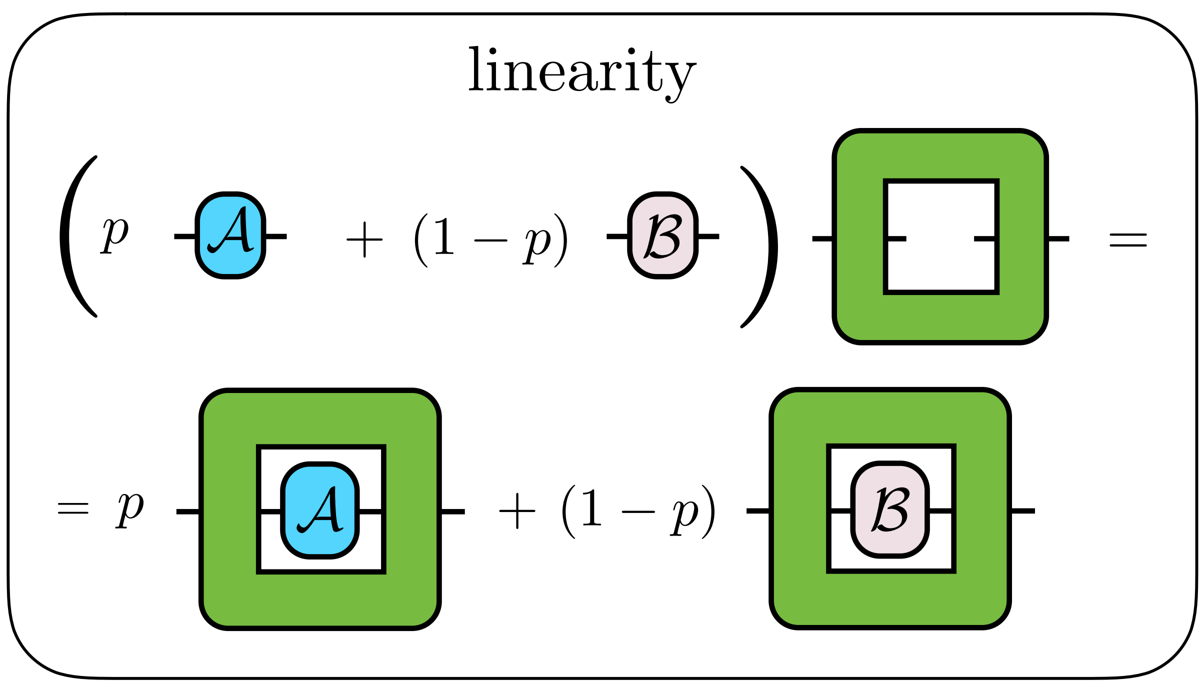
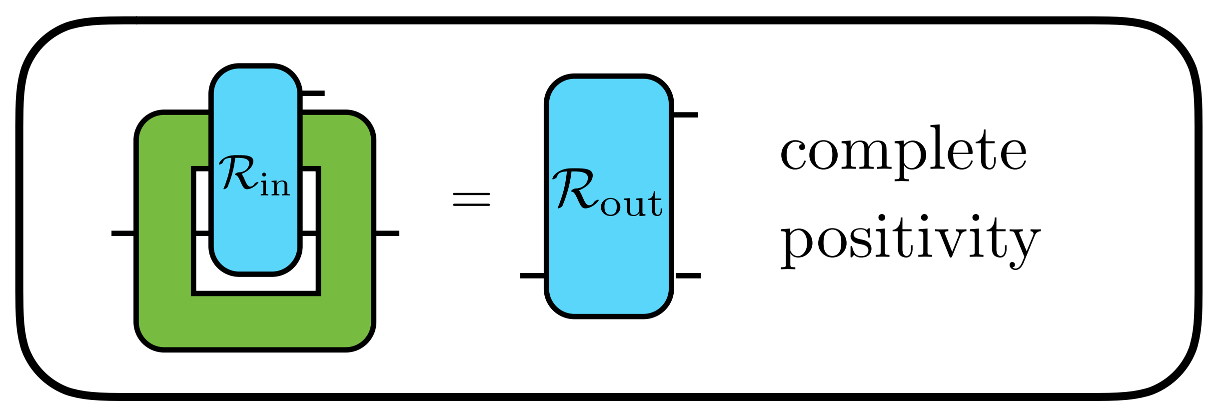
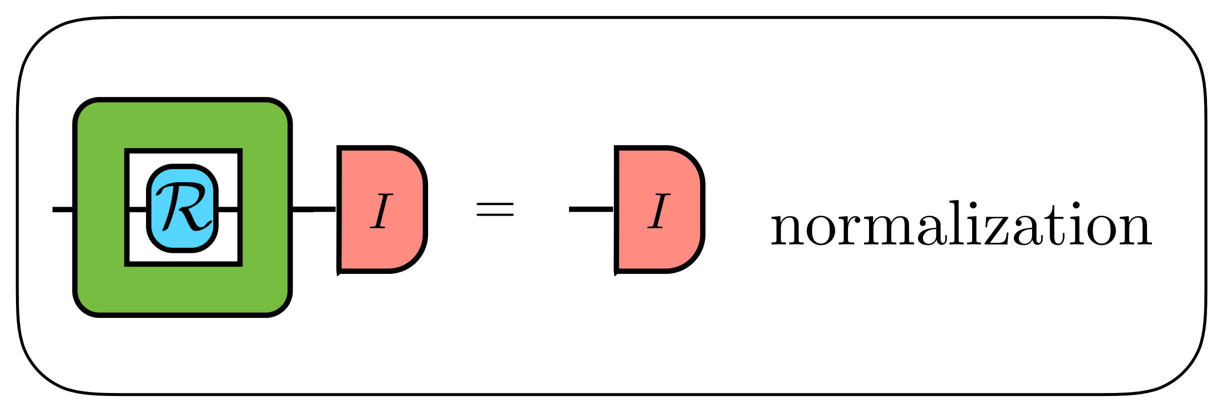


Admissibility conditions







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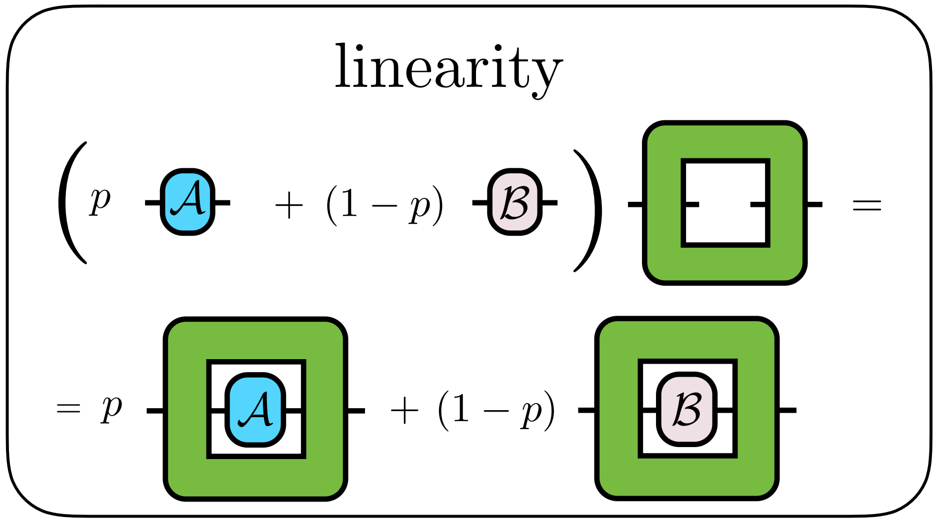
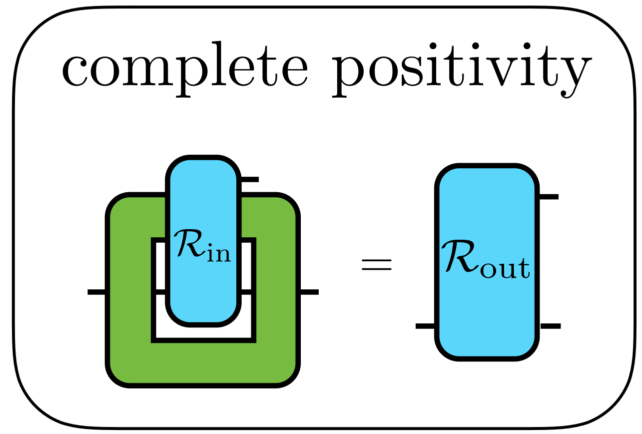
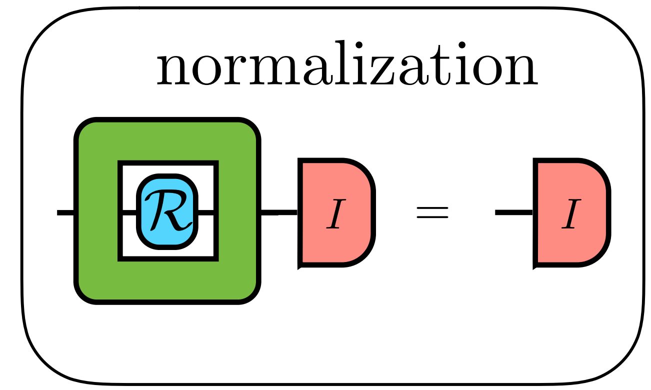
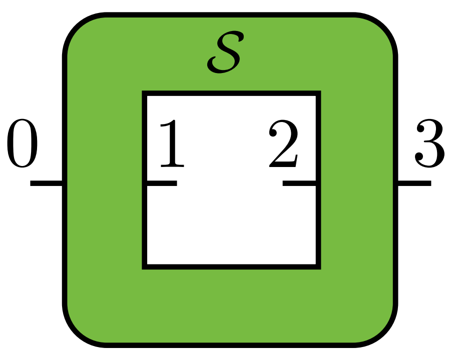






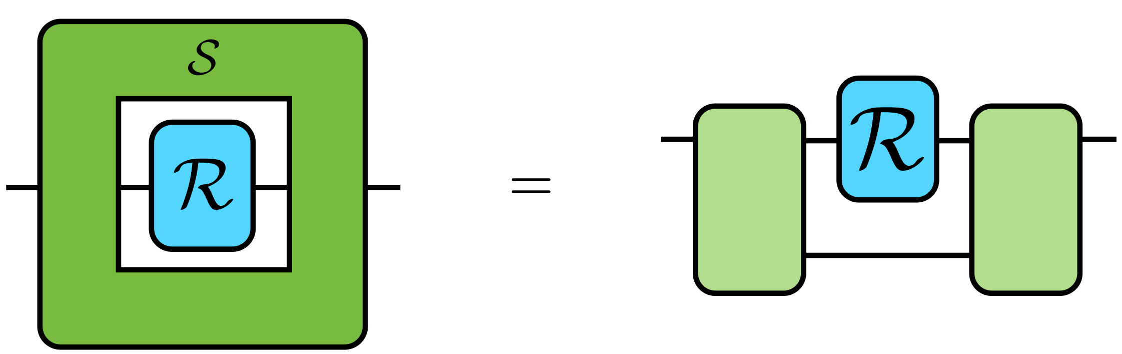
The most general supermap



Choi representation

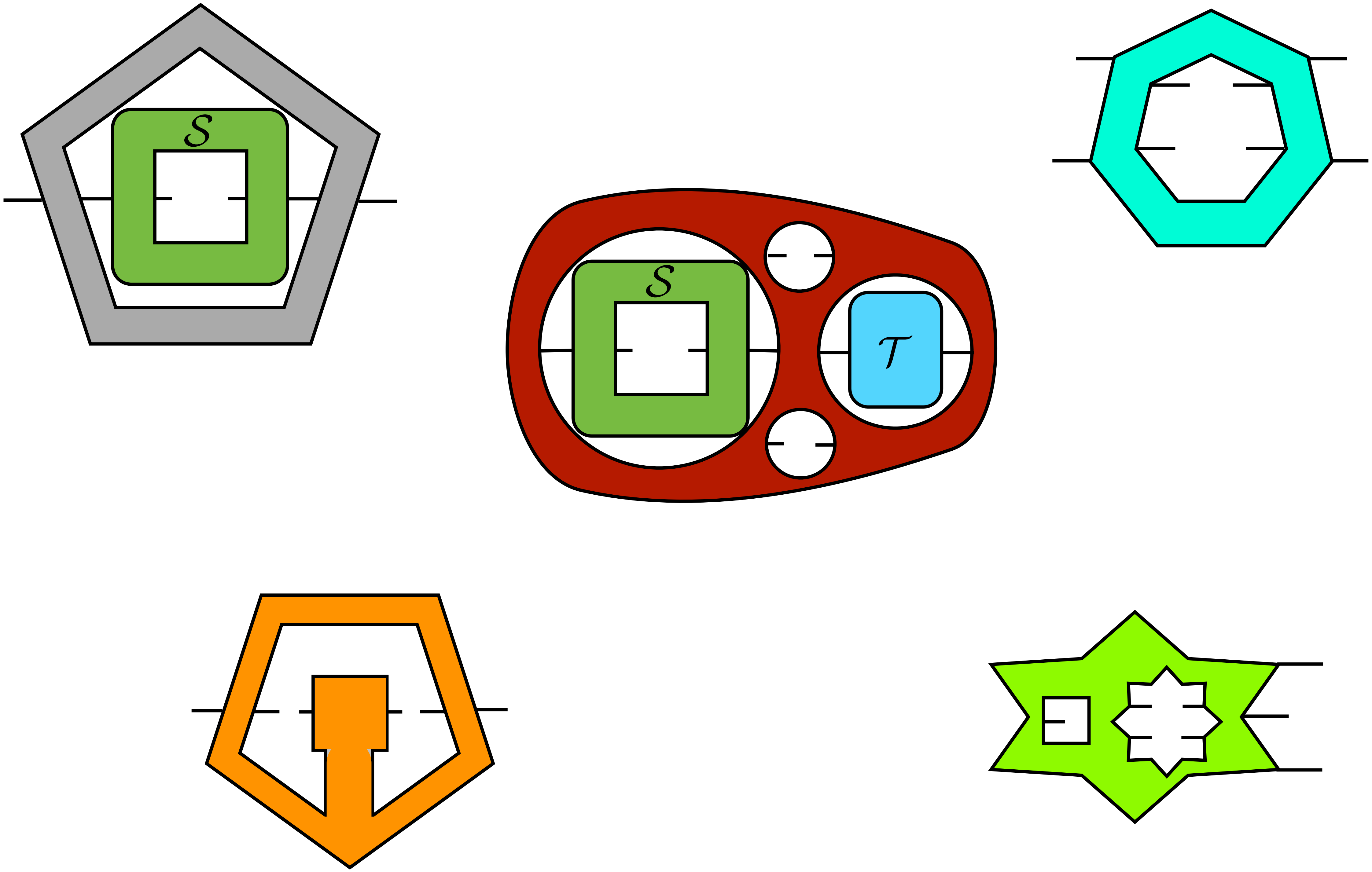
$$S \in \mathcal{L}\Big(\bigotimes_{i=0}^{3} \mathcal{H}_i\Big), \quad S \geqslant 0\,, \quad \mathrm{Tr}_3\, S = I_2 \otimes S', \quad \mathrm{Tr}_1\, S' = I_0$$

Realisation theorem



Quantum circuit
with a open slot

The hierarchy of higher order quantum maps



The hierarchy of higher order quantum maps

Type system

$$x = y \to z$$

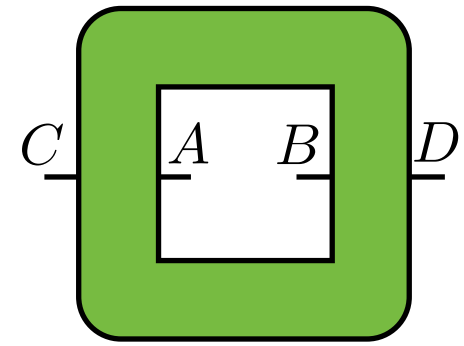
input output

The hierarchy of higher order quantum maps

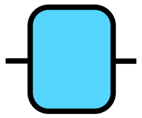
Type system

$x = y \rightarrow z$

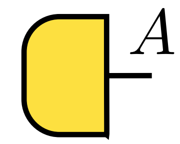
input output



$(A \rightarrow B) \rightarrow (C \rightarrow D)$



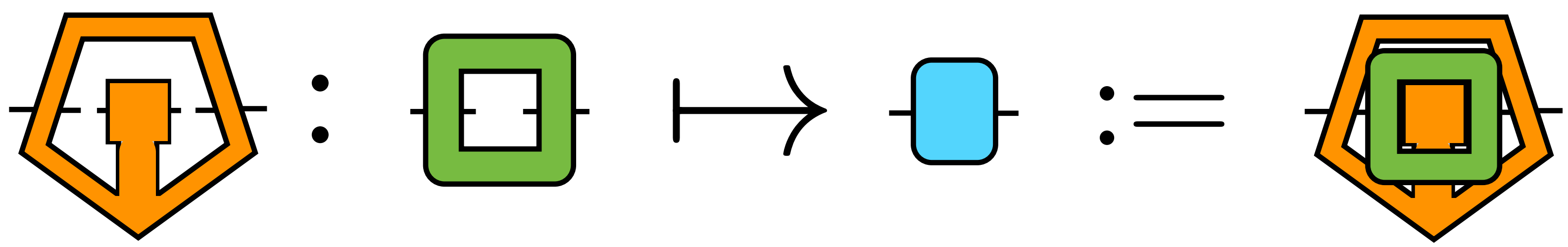
$A \rightarrow B$



elementary types

⬍

quantum systems



$((A \rightarrow B) \rightarrow (C \rightarrow D)) \rightarrow (E \rightarrow F)$

The hierarchy of higher order quantum maps

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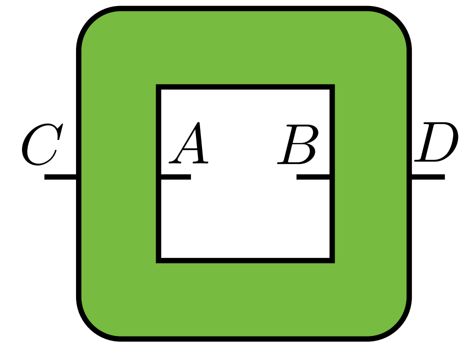
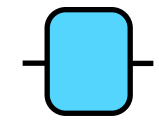
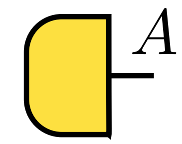


$(A \to B) \to (C \to D)$



$A \to B$



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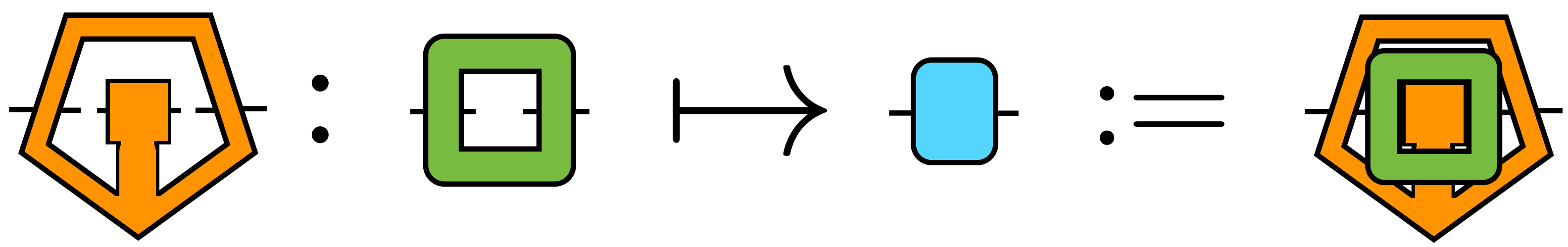
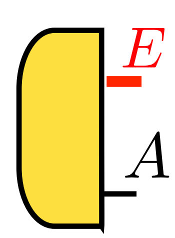


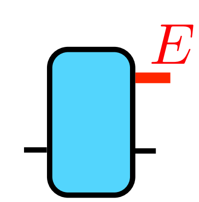
$((A \to B) \to (C \to D)) \to (E \to F)$

<u>Type extension:</u>

$(y \to z) \parallel E = y \to z \parallel E$

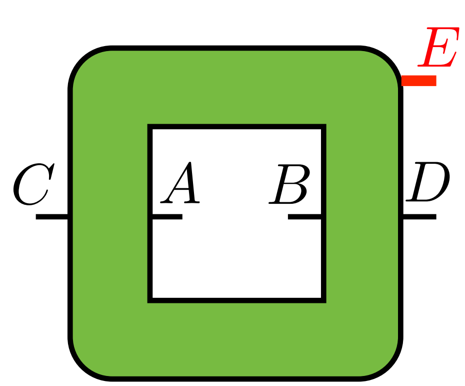


$A \parallel E = AE$



$(A \to B) \parallel E$

$= A \to B \parallel E$



$(A \to B) \to (C \to D) \parallel E$

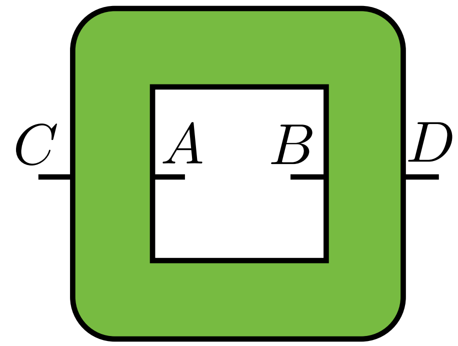
$= (A \to B) \to (C \to DE)$

The hierarchy of higher order quantum maps

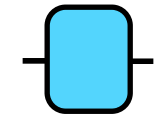
Type system

$$x = y \to z$$

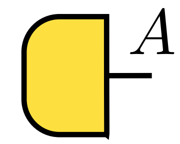
input | output



$(A \to B) \to (C \to D)$



$A \to B$



elementary types ⬍ quantum systems

Type extension:

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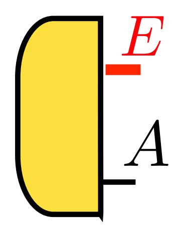
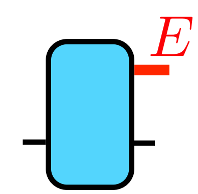


$A \parallel E = AE$



$(A \to B) \parallel E$

$= A \to B \parallel E$

Admissibility conditions

(Linearity)

(Generalised complete positivity) $x \parallel E \to y \parallel E$ admissible to admissible for any extension

(Normalisation)

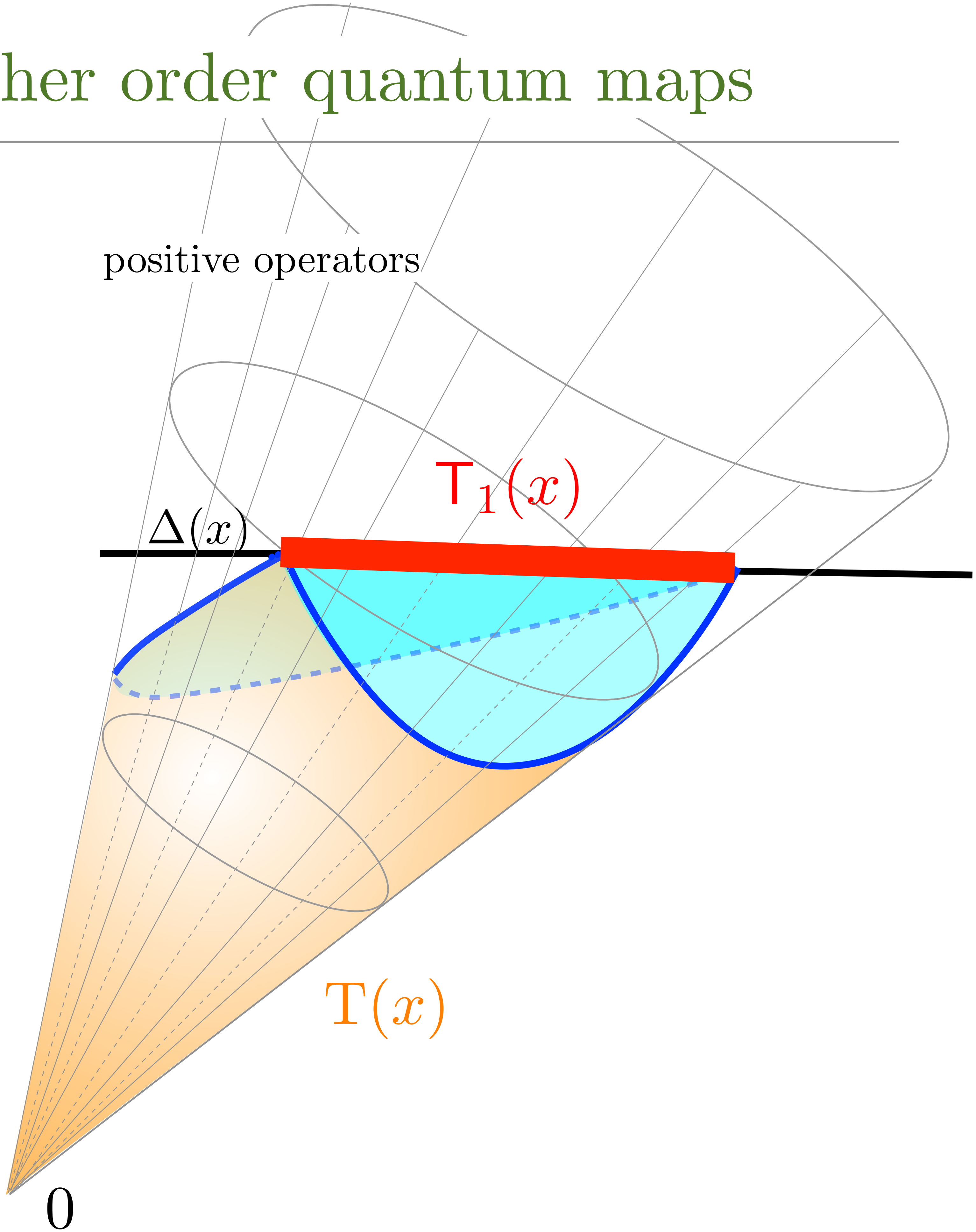
The hierarchy of higher order quantum maps

The most general higher order map of type x

$\mathsf{T}(x)$ probabilistic maps of type x

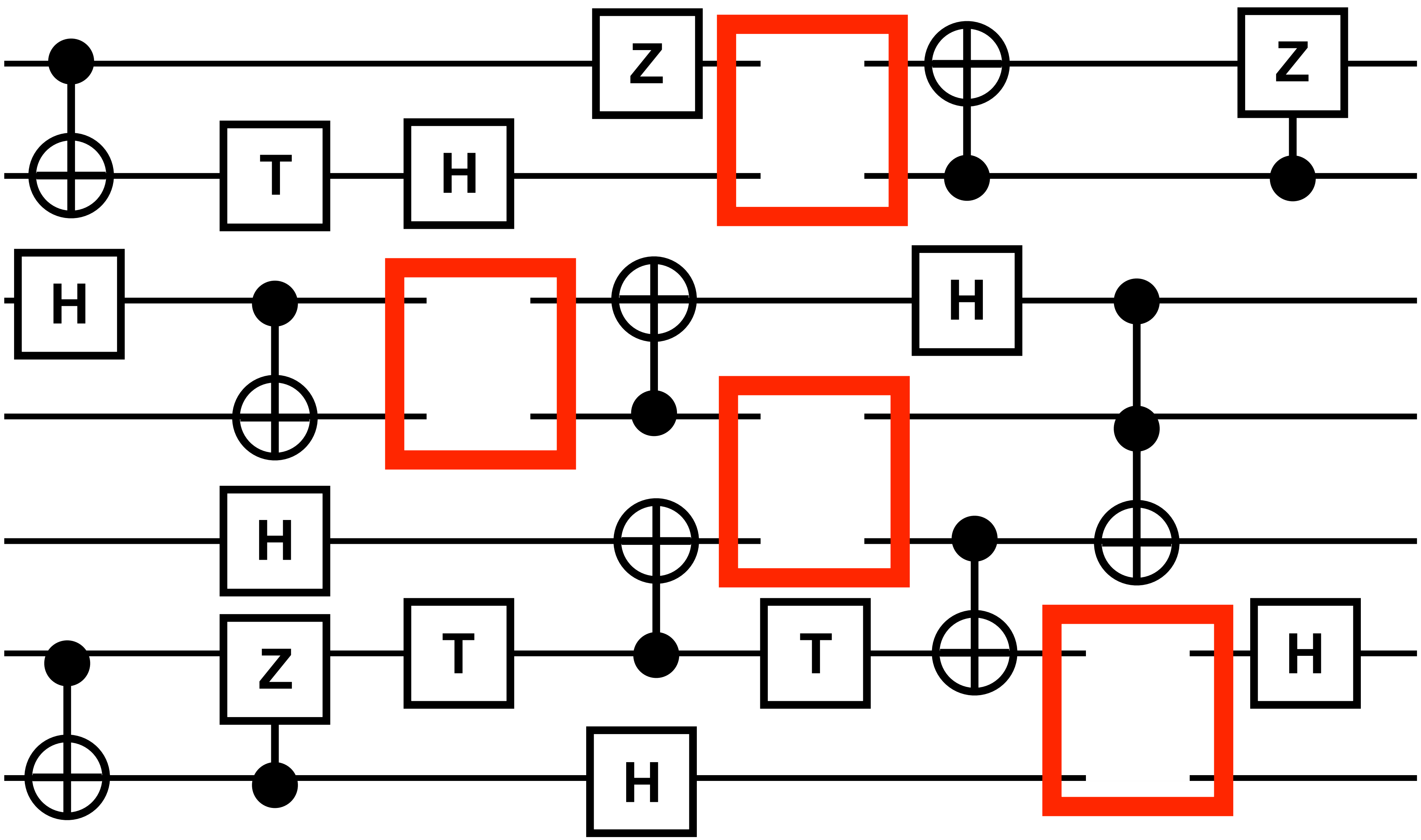
$\mathsf{T}_1(x)$ deterministic maps of type x

$\Delta(x)$ linear constraint



Causal structure

Quantum circuits with open slots (quantum network)



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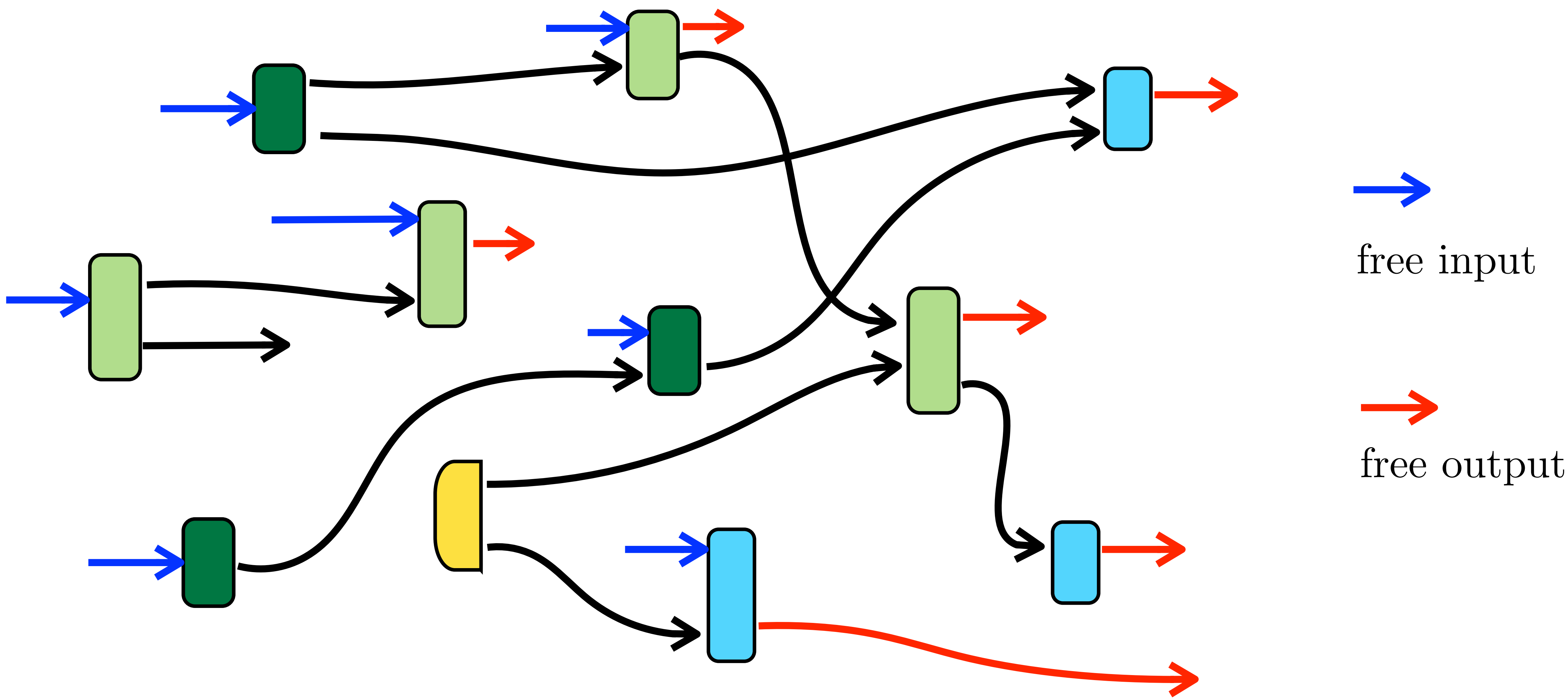

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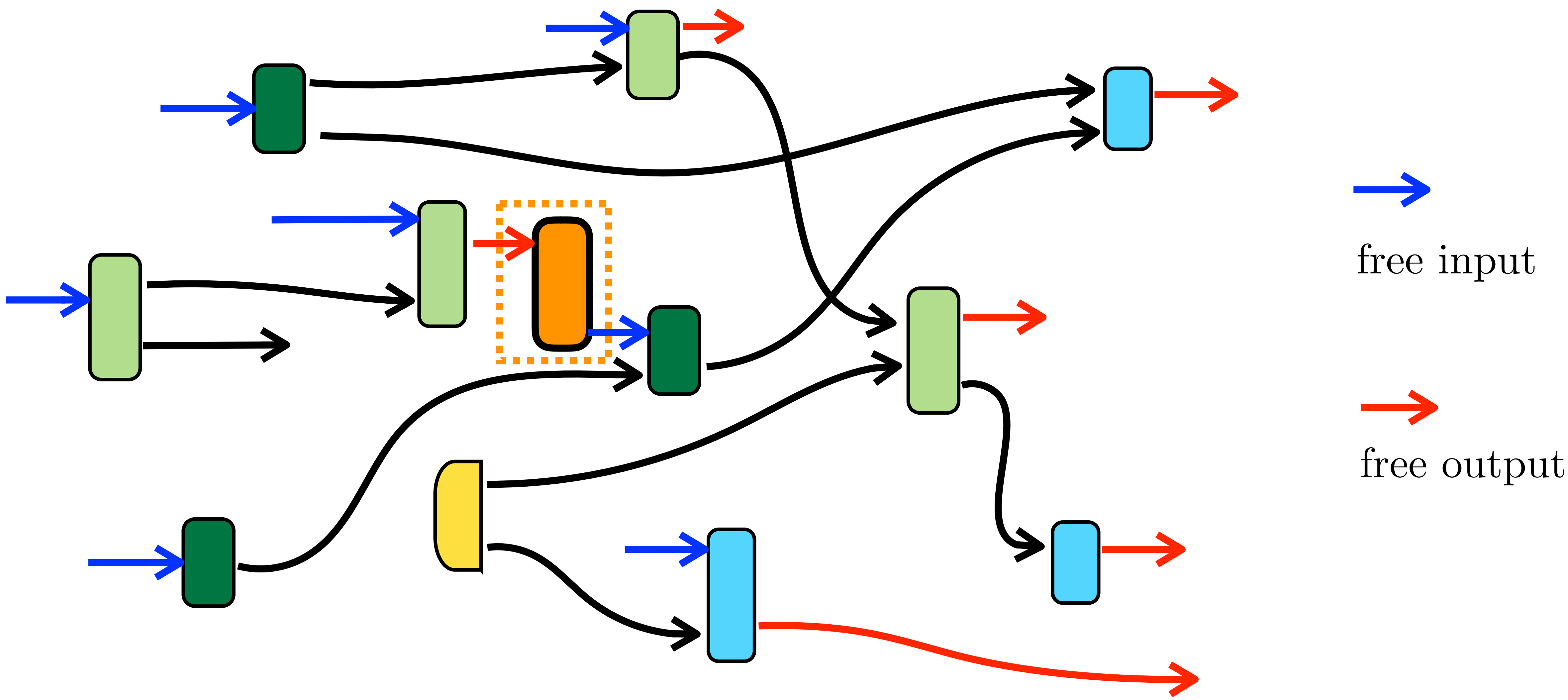

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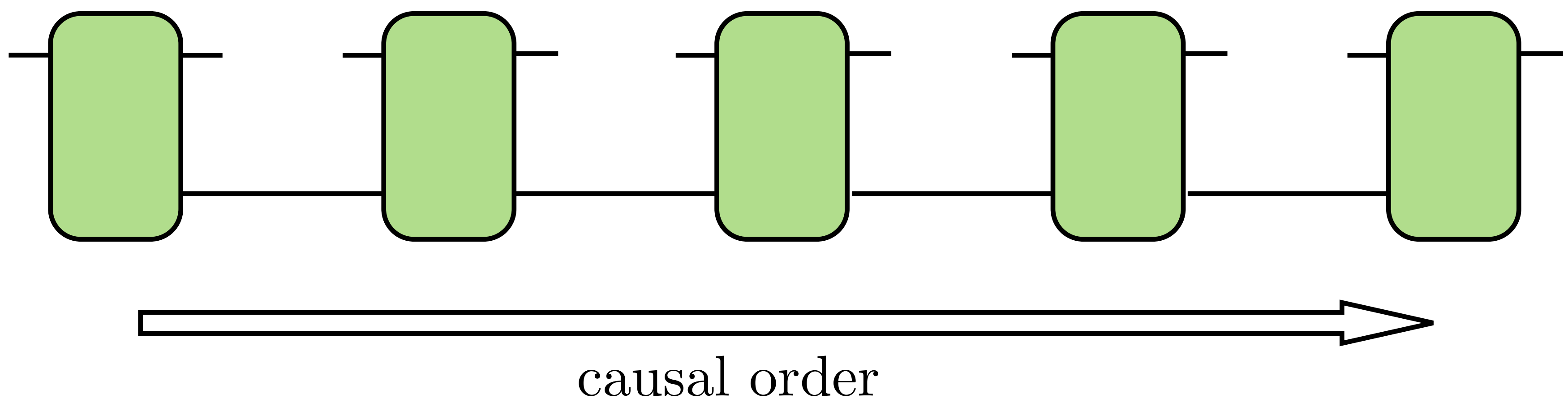

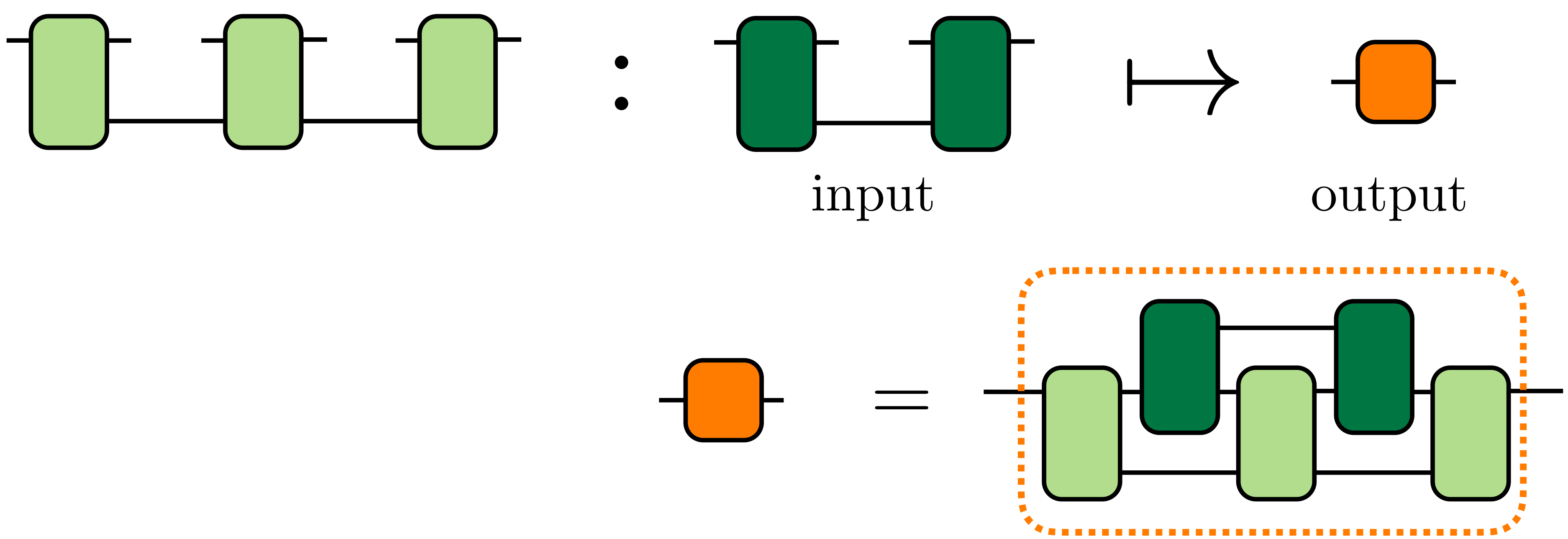

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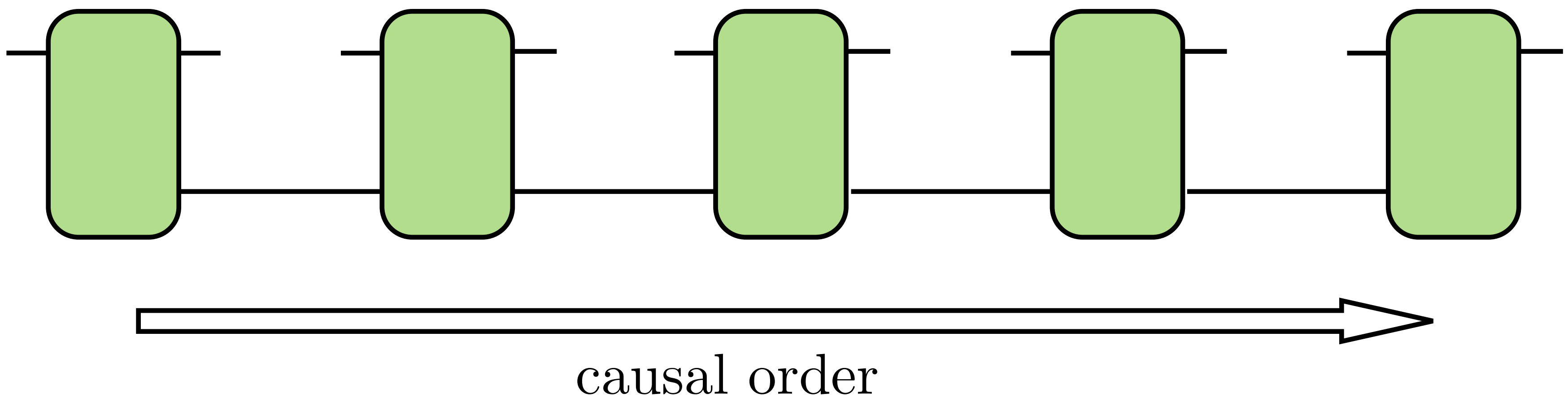
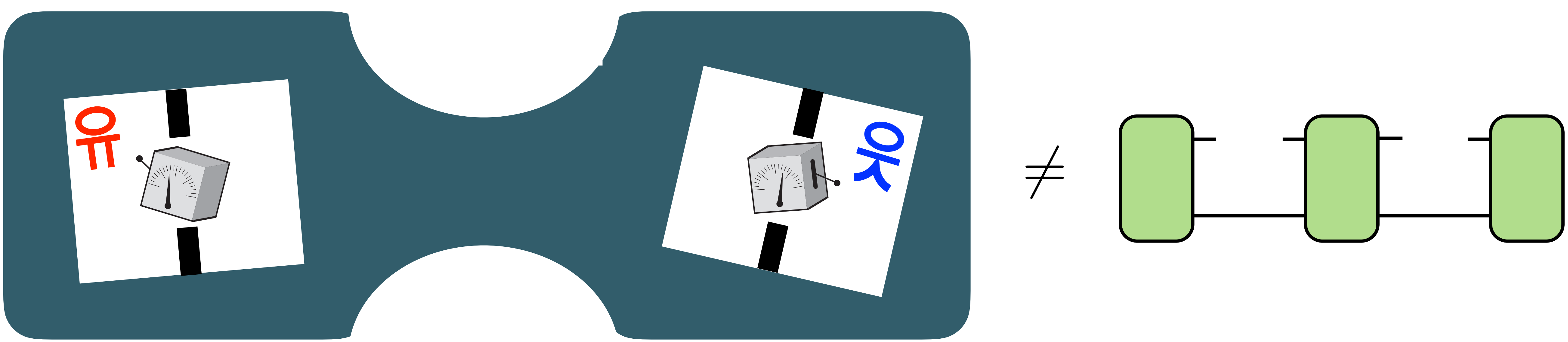


Indefinite causal structure



Causal structure

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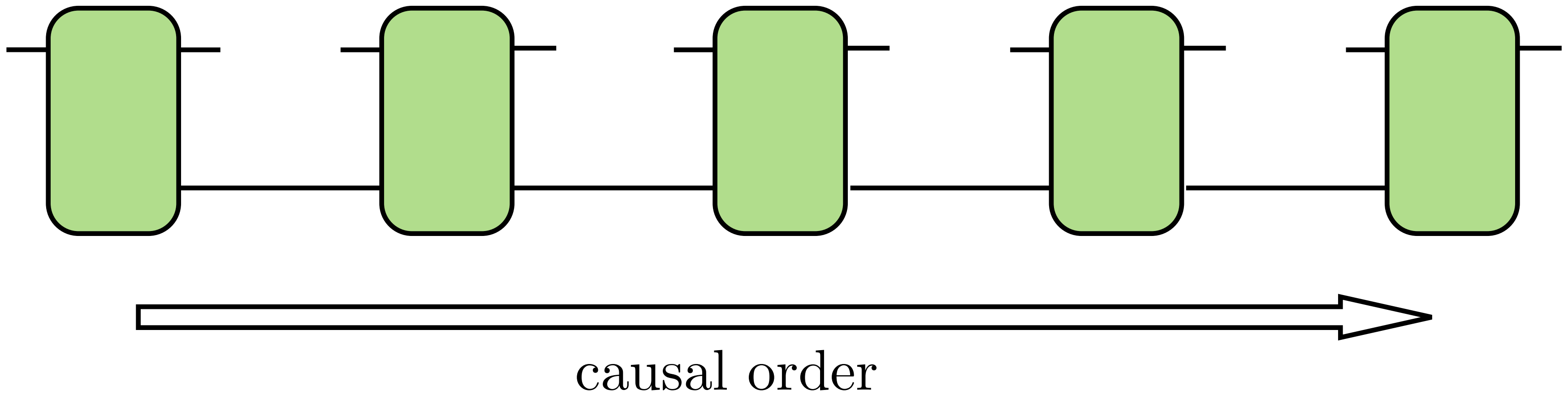
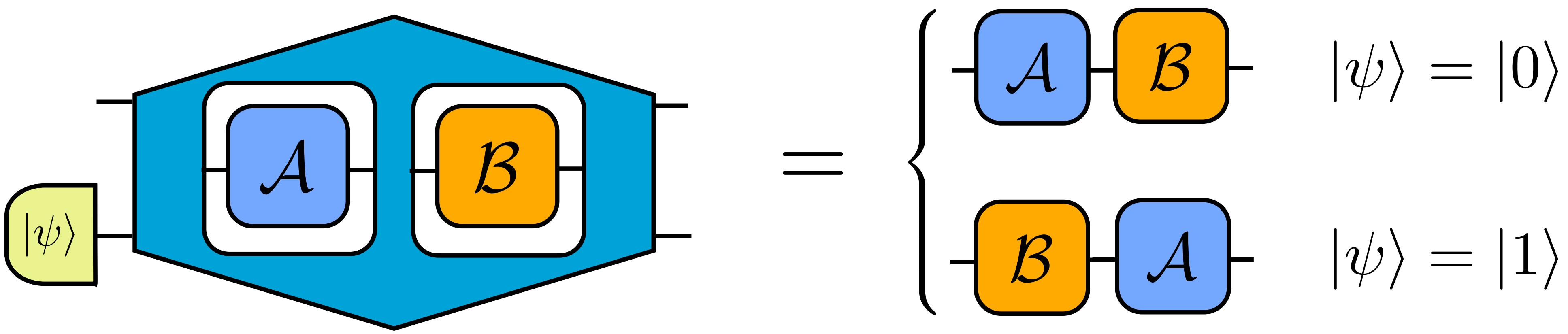


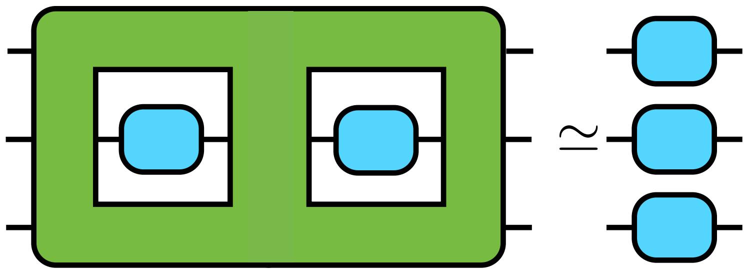
Indefinite causal structure

Quantum switch



Higher order quantum computation

- Higher order maps are a convenient framework for Quantum Information Processing when the carriers of information are quantum objects more general than states (transformations, quantum network...)



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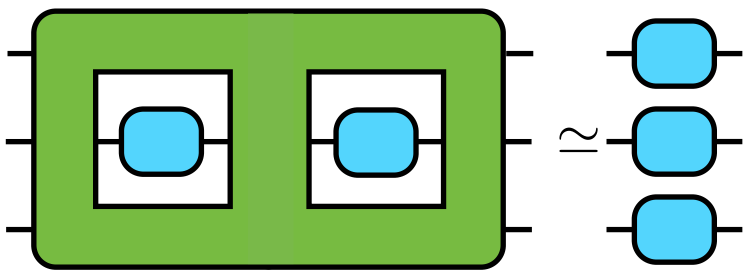

-  The framework of higher order maps encompasses quantum networks and quantum processes with indefinite causal order.

Higher order quantum computation

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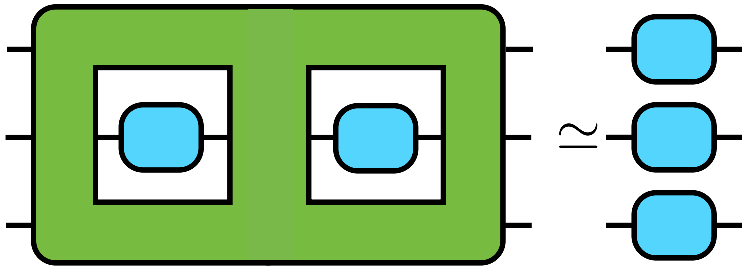


- The framework of higher order maps encompasses quantum networks and quantum processes with indefinite causal order.

- Quantum processes with indefinite causal order may outperform circuital strategies:

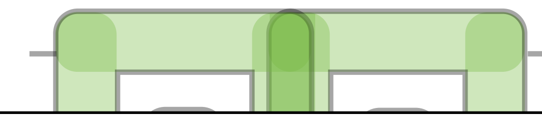
M. Araújo, F Costa, C. Brukner
Phys. Rev. Lett. 113 250402 (2014) — computational speedup

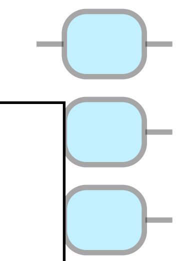
D. Ebler, S. Salek, G. Chiribella
Phys. Rev. Lett. 120, 120502 (2018) — enhance channel capacity

J.Bavaresco, M. Murao, M. T. Quintino
Phys. Rev. Lett. 127, 200504 (2021) — channel discrimination

Higher order quantum computation

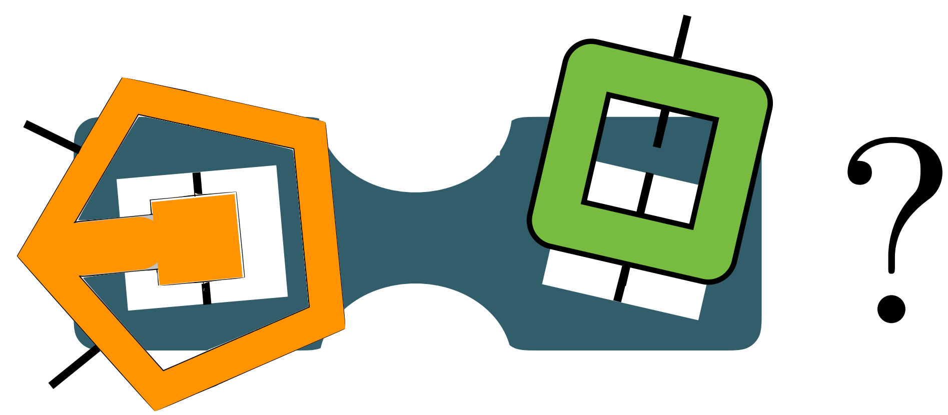
- Higher order maps are a convenient framework for Quantum Information Processing when the carriers





Quantum computation with higher order maps is powerful. But what are the <u>rules</u> of this computation?

What are the rules for <u>composing</u> <u>higher order maps?</u>



um

order.

edup

- with indefinite causal order may <u>outperform</u> <u>circuital strategies</u>:

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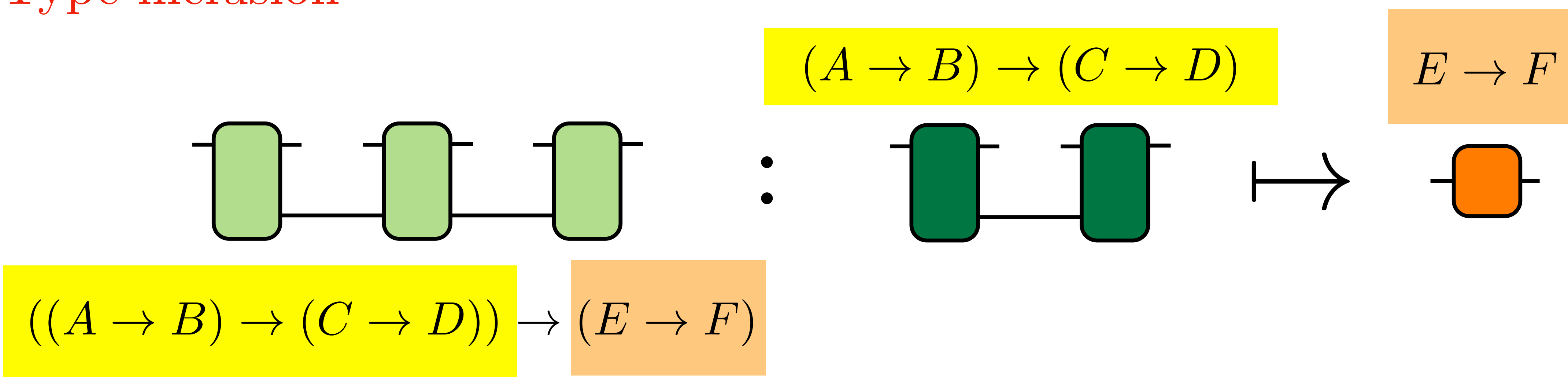
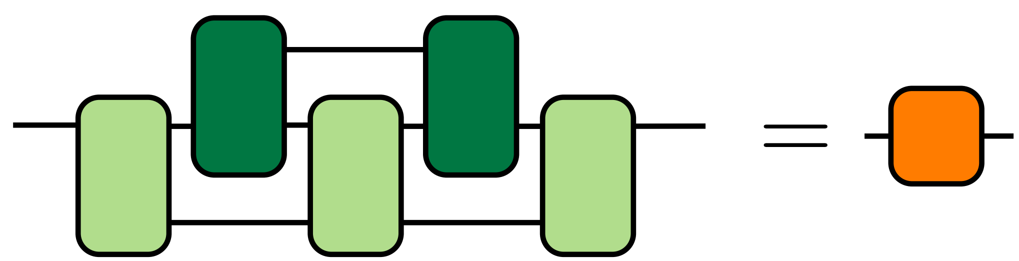
channel discrimination

One type, many uses

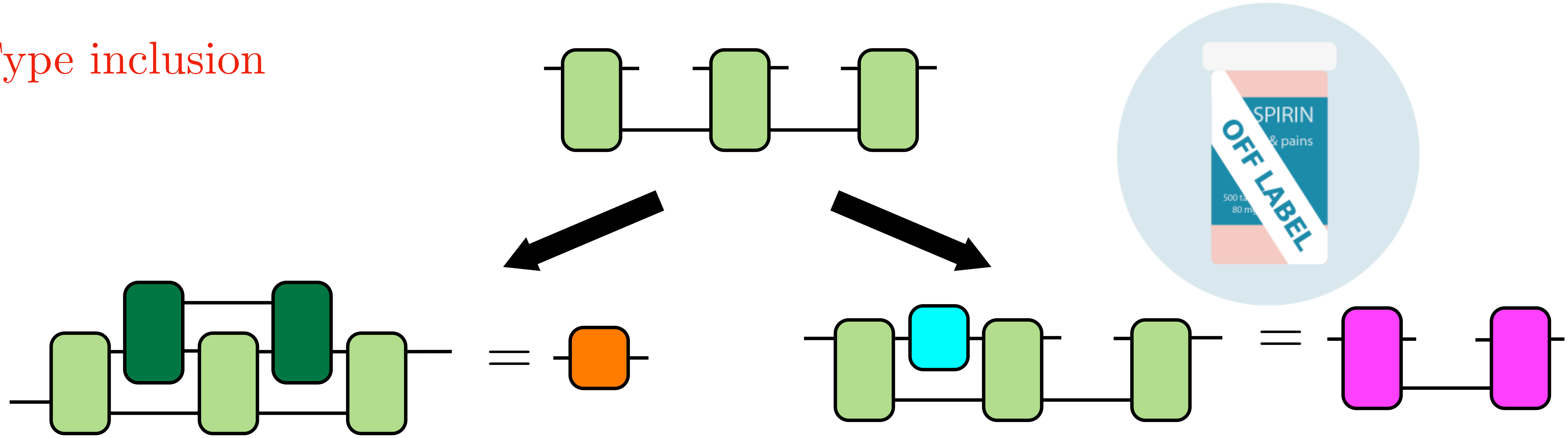
Type inclusion





One type, many uses

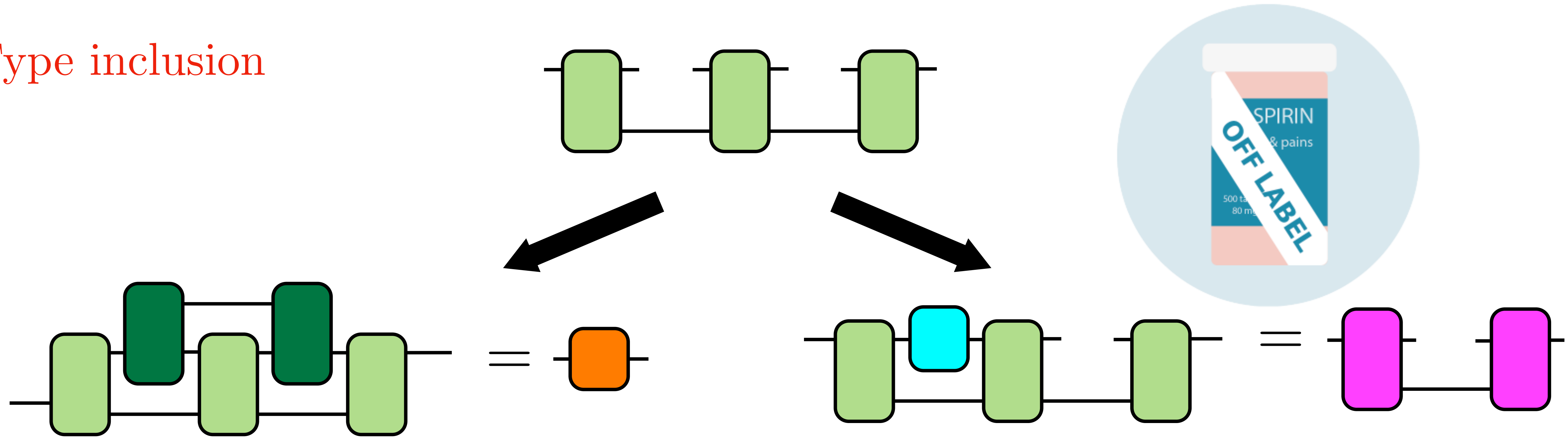
Type inclusion



$$((A \to B) \to (C \to D)) \to (E \to F) \quad \subseteq \quad (C \to A) \to ((B \to D) \to (E \to F))$$

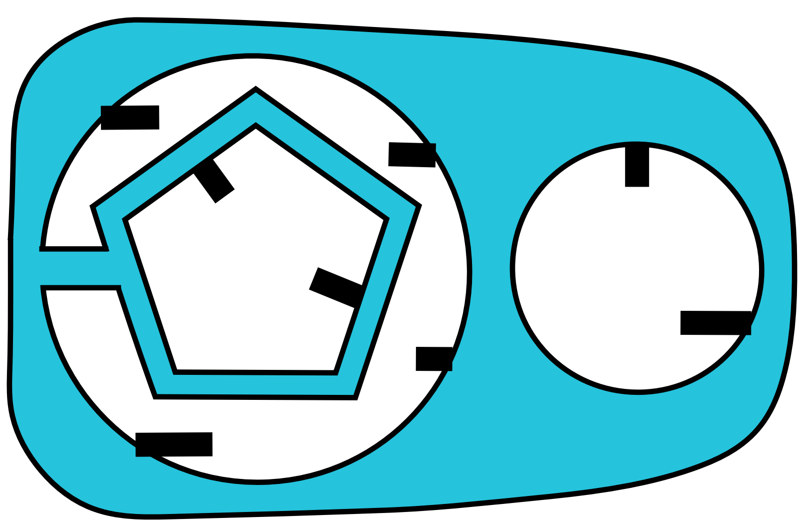
One type, many uses

Type inclusion



$$((A \to B) \to (C \to D)) \to (E \to F) \quad \subseteq \quad (C \to A) \to ((B \to D) \to (E \to F))$$

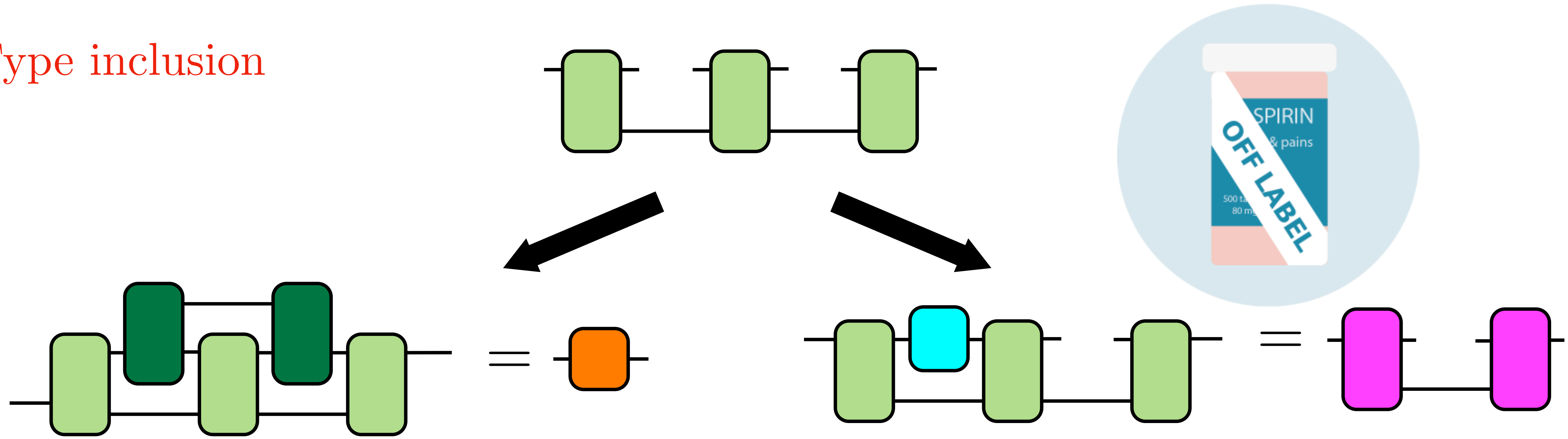
Input and output



All elementary systems
are **not** created equal

One type, many uses

Type inclusion



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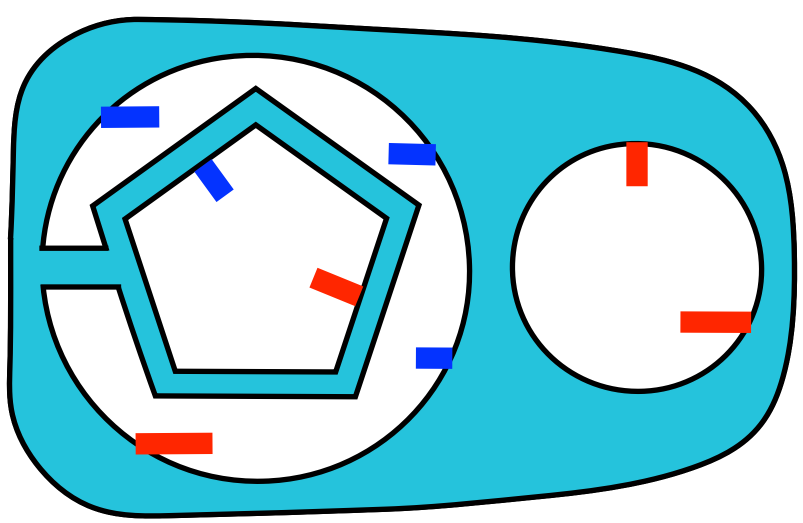

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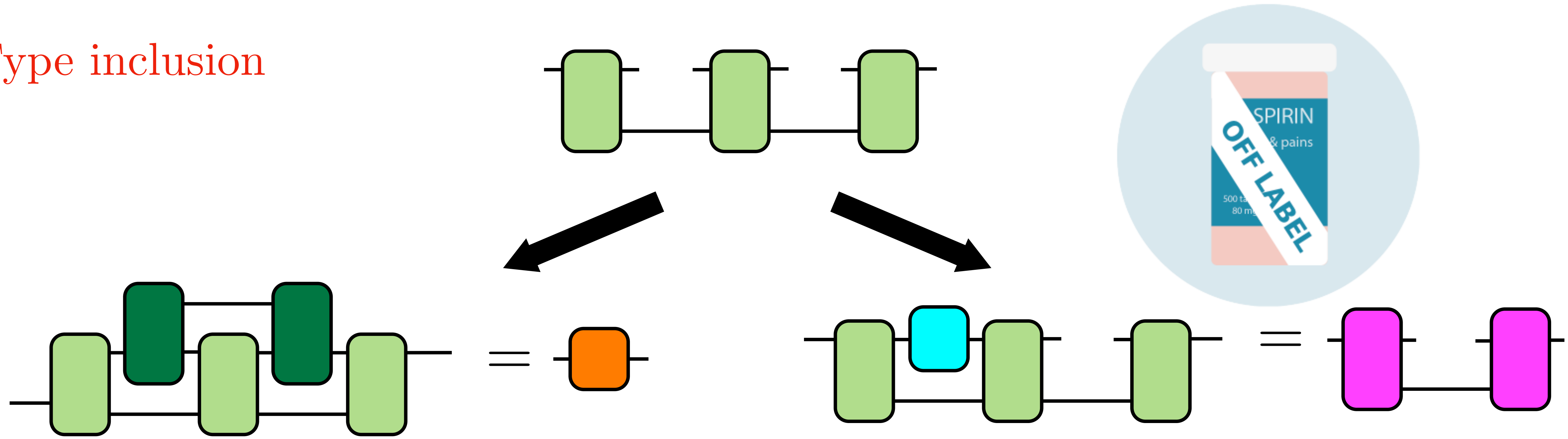

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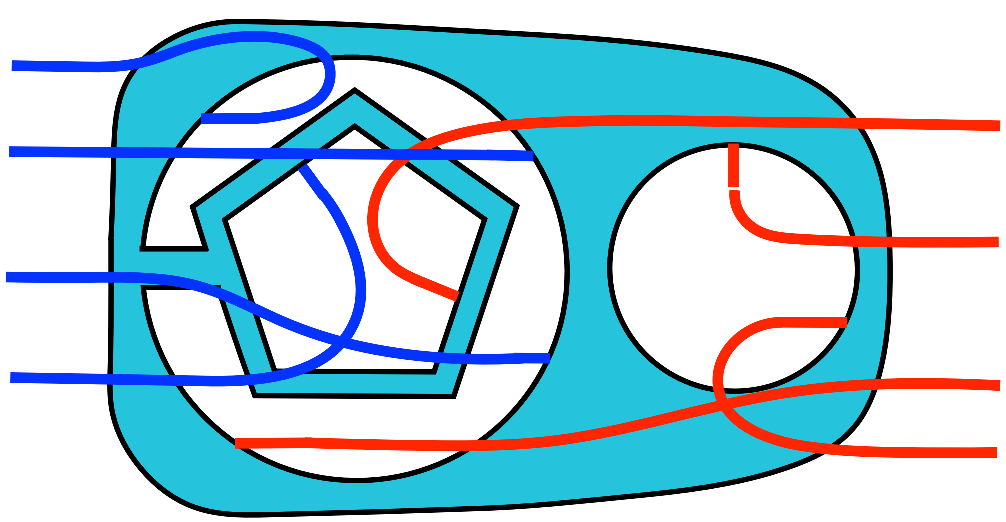

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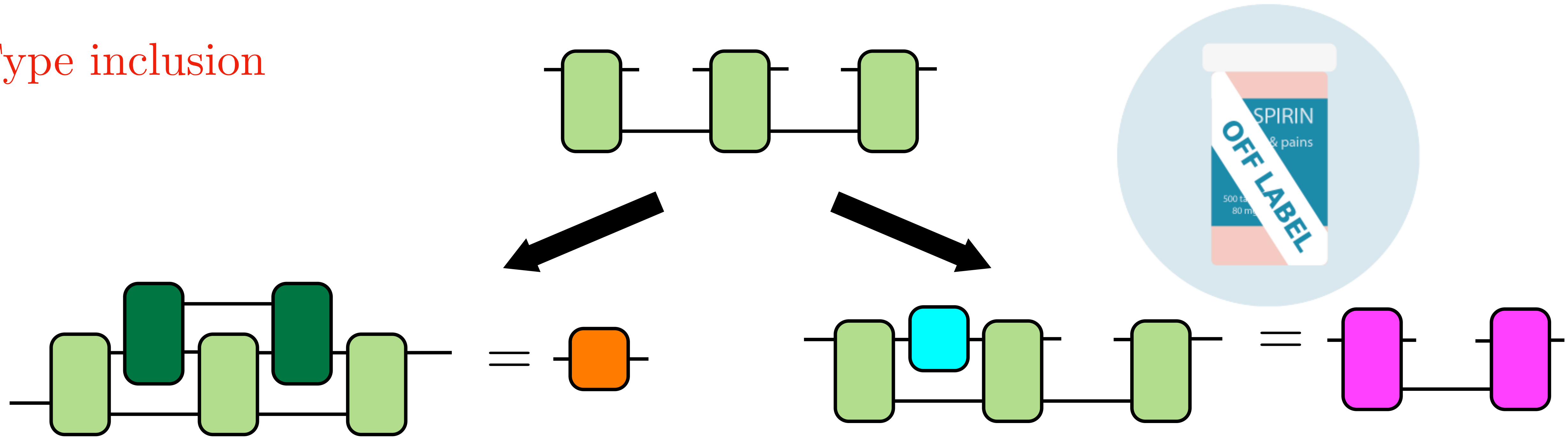

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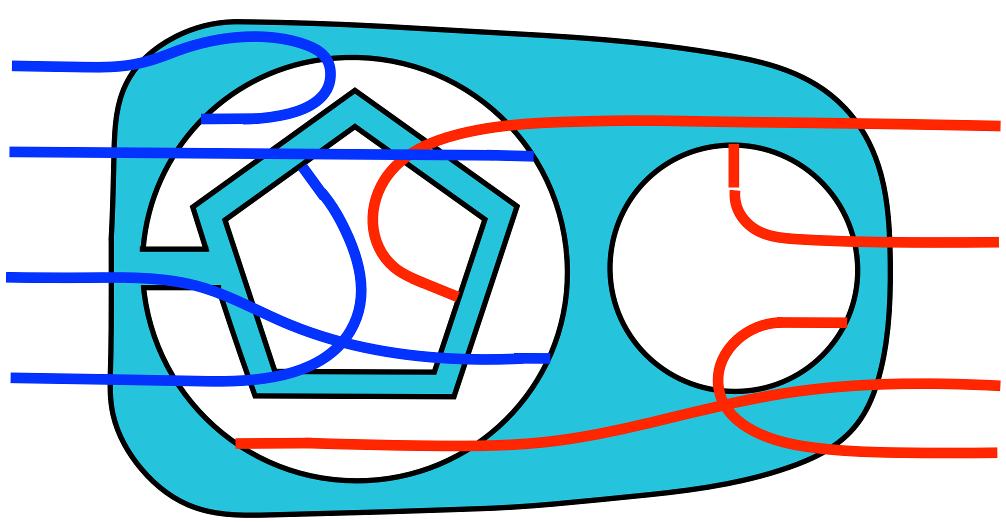

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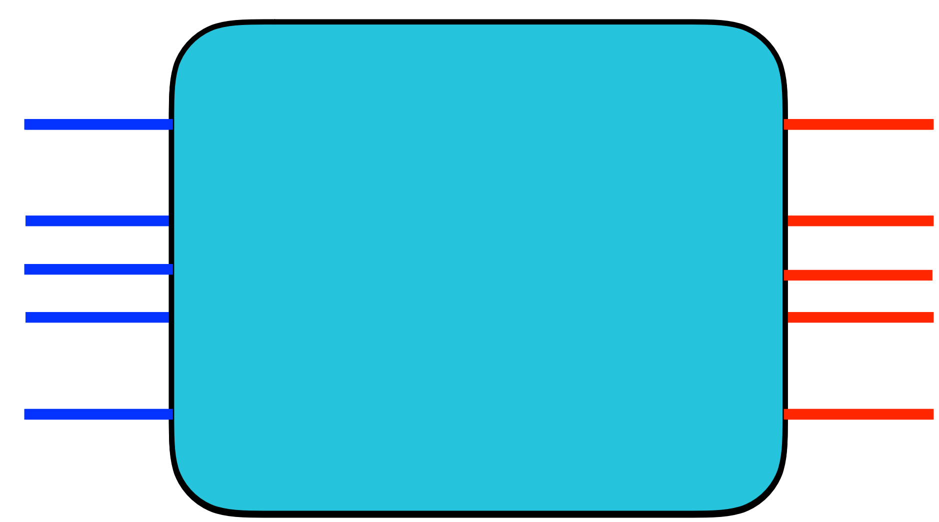

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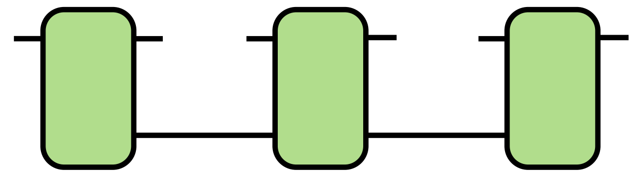

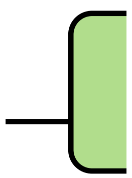

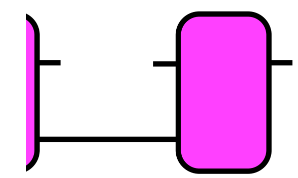

$((A$ $\to F))$

Every

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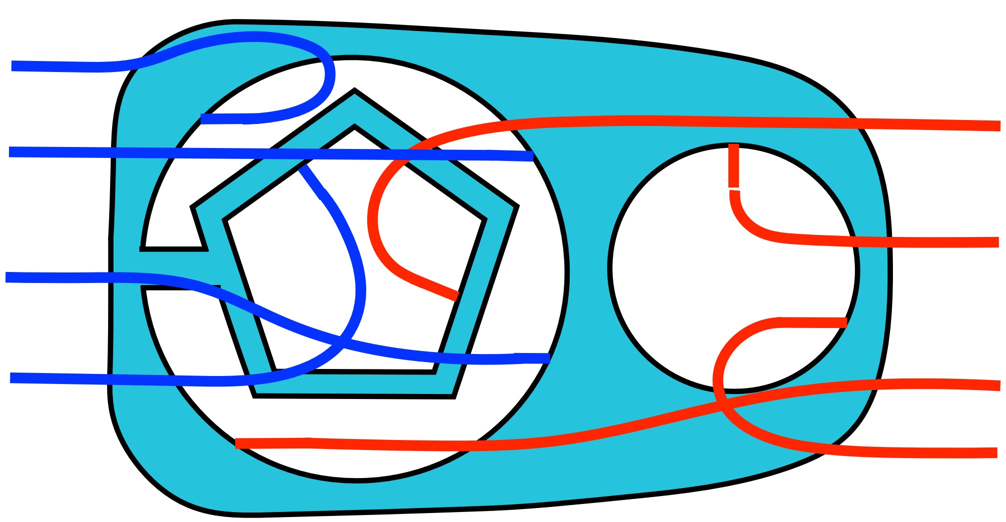
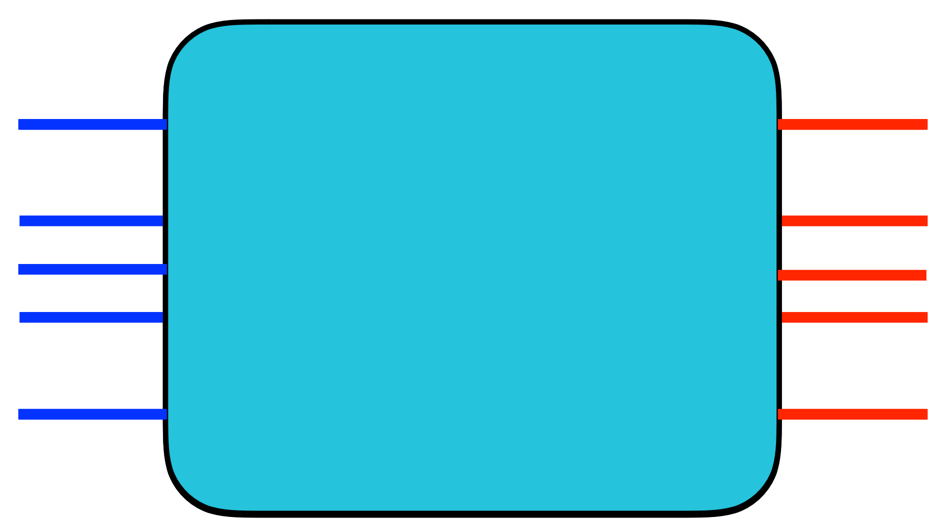


$\subseteq$



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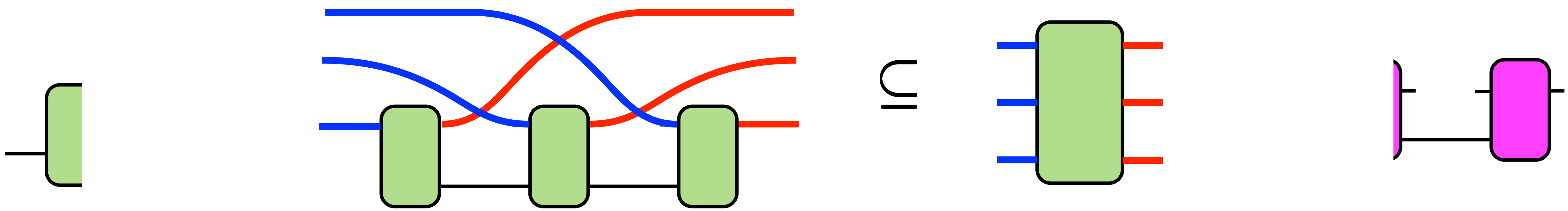

$((A$ … $\to F))$

Every…

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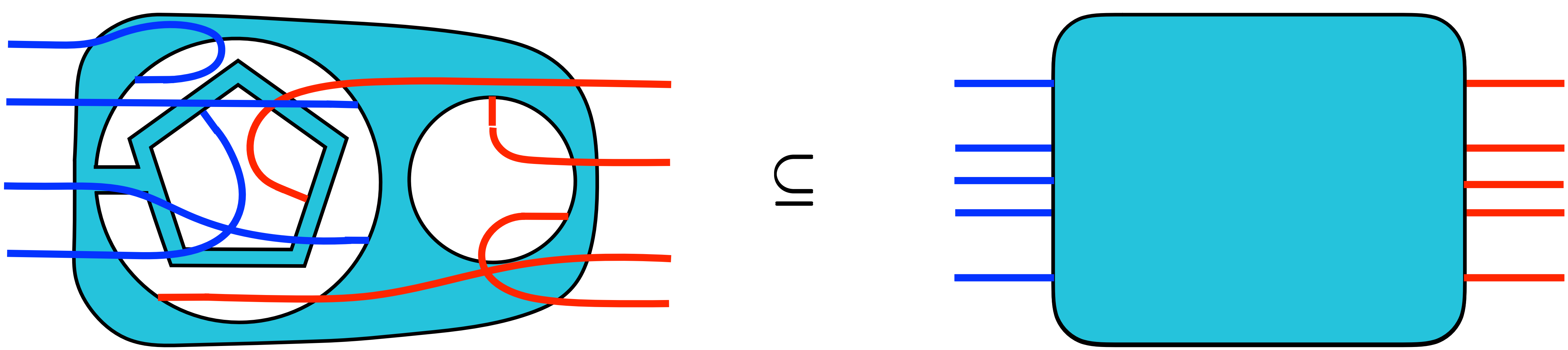
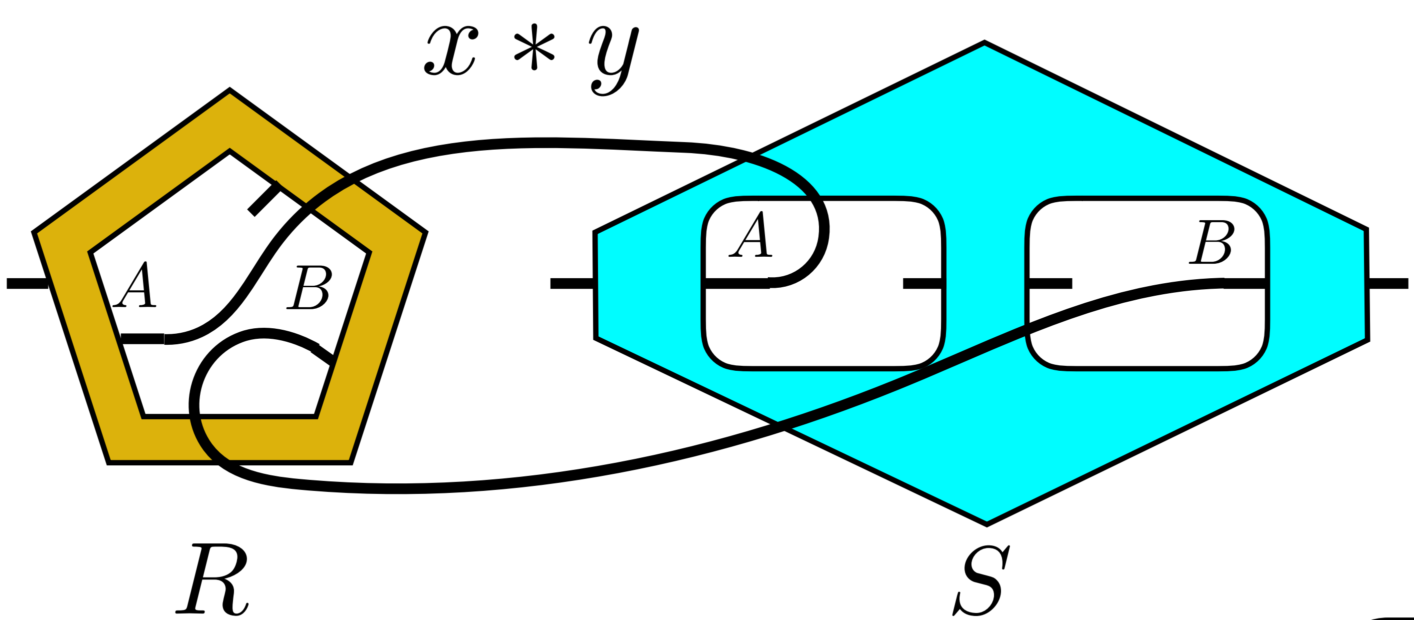


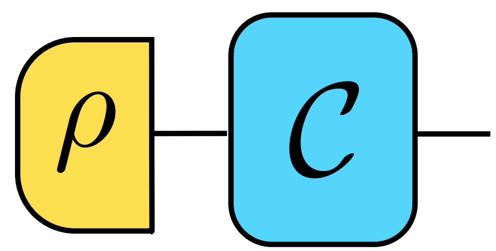
Compositional structure

Connecting wires



link product

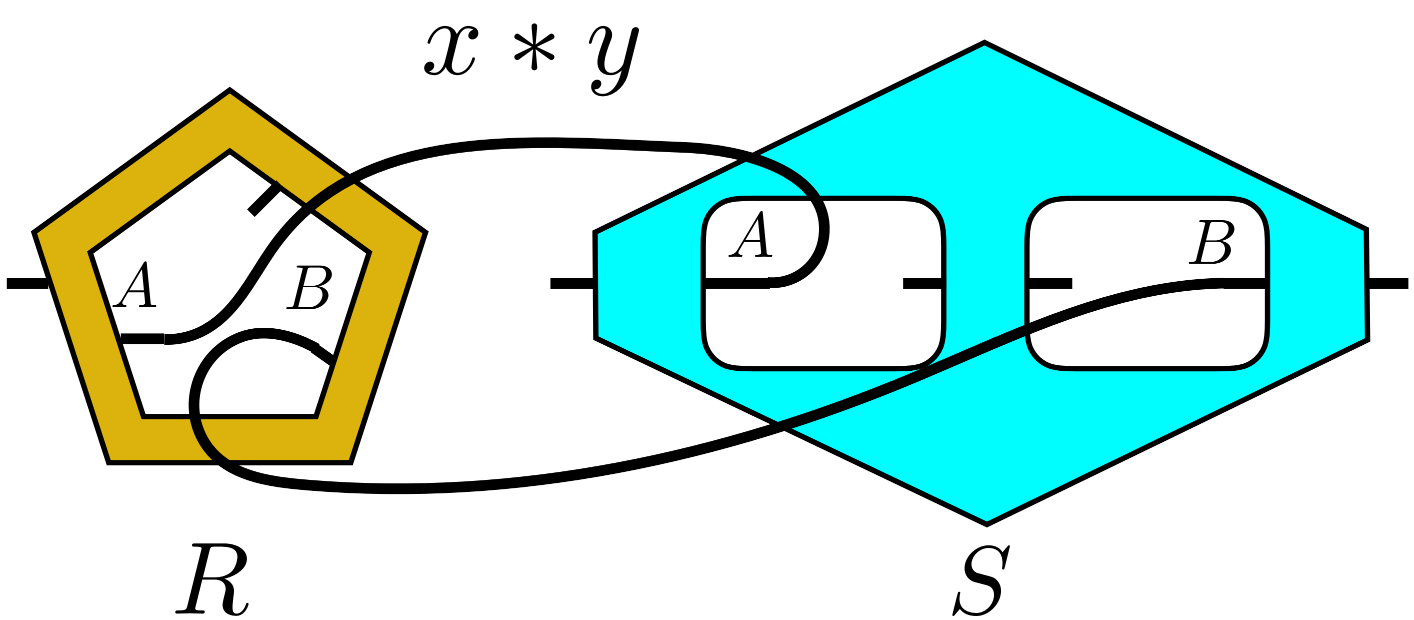
$$R * S = \mathrm{Tr}_{AB}[R\, S^{T_{AB}}]$$



$$\mathcal{C}(\rho) = C * \rho = \mathrm{Tr}[C(I \otimes \rho^{T})]$$

Compositional structure

Connecting wires



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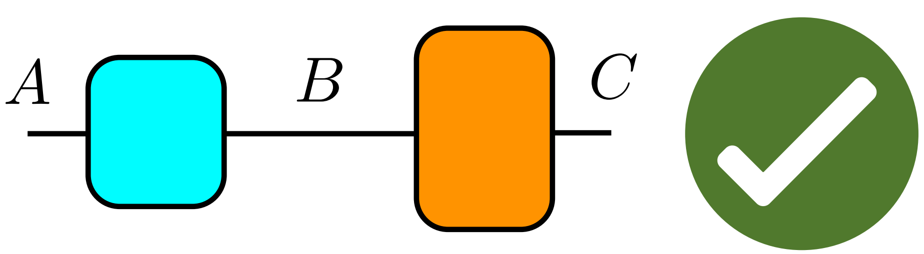
Admissible type composition

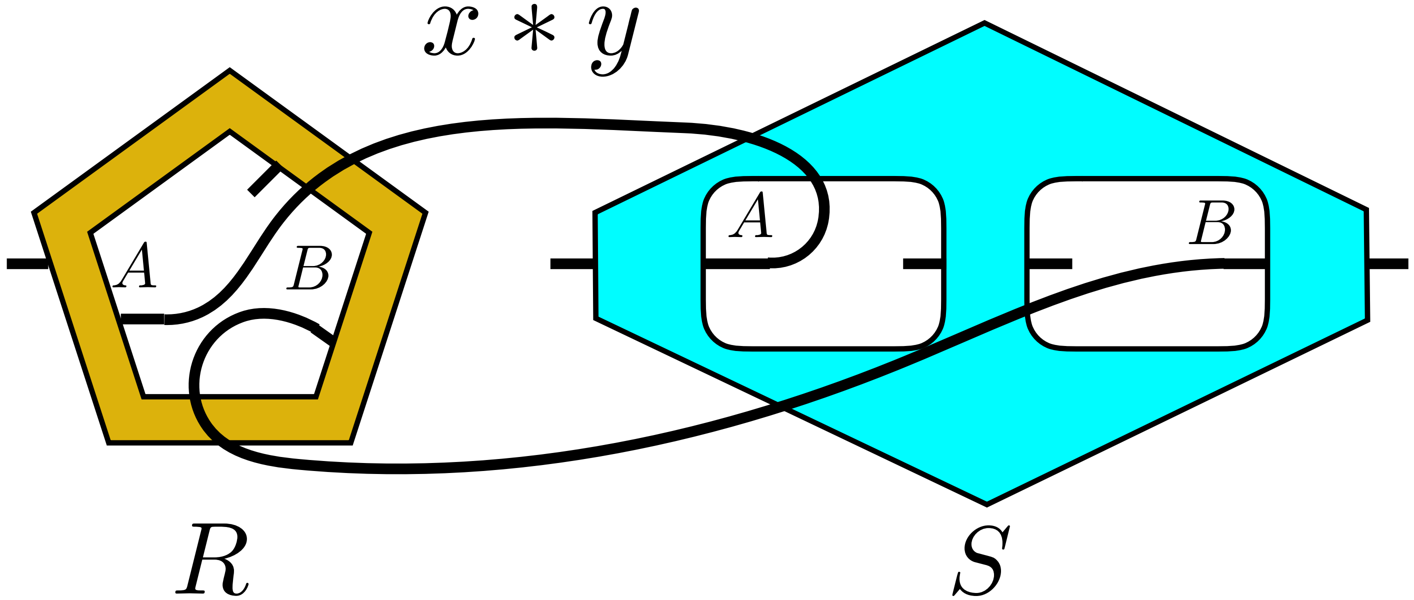
$R * S$ is a higher order map for any $R \in x$ and $S \in y$



$(A \to B) * (B \to C)$

Compositional structure

Connecting wires



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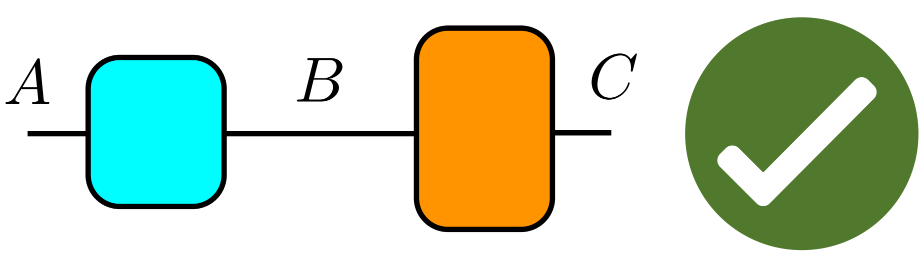

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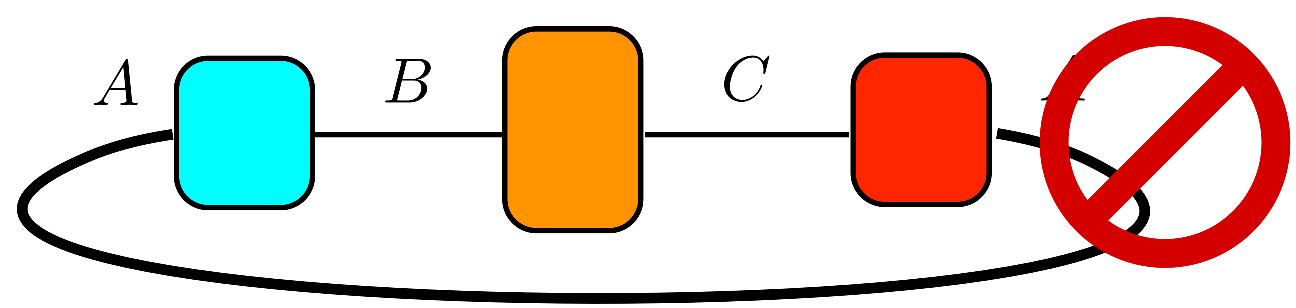
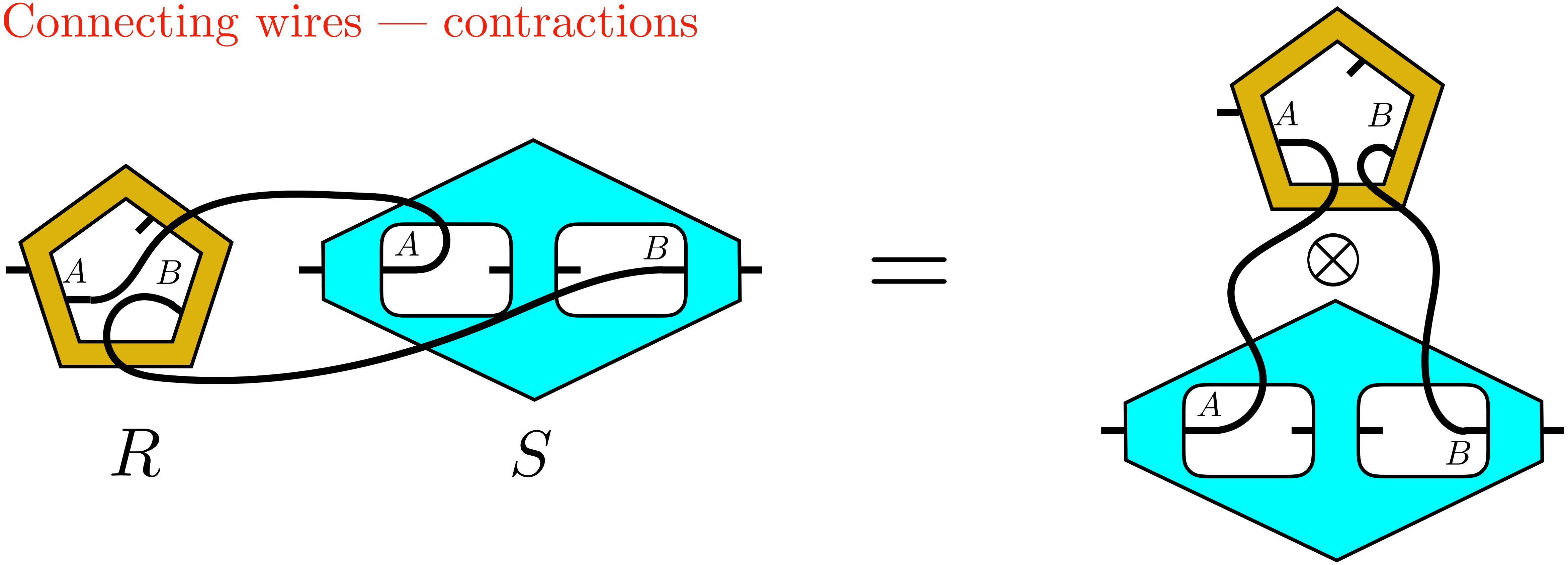


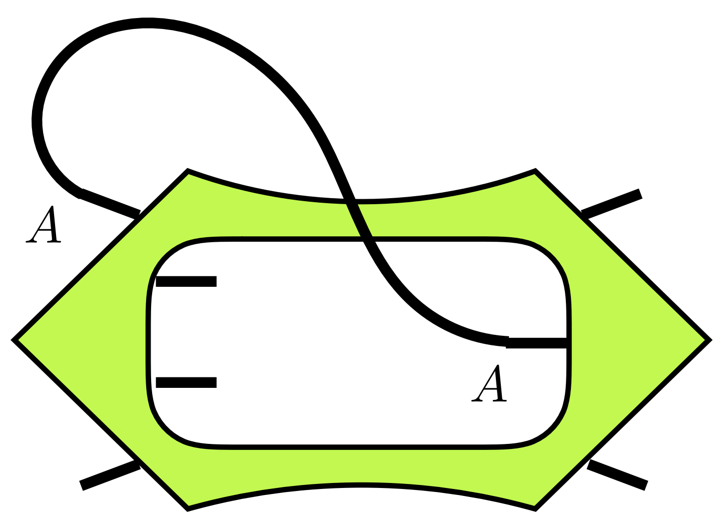
$(A \to B) * (B \to C) * (C \to A)$

Compositional structure

Connecting wires — contractions



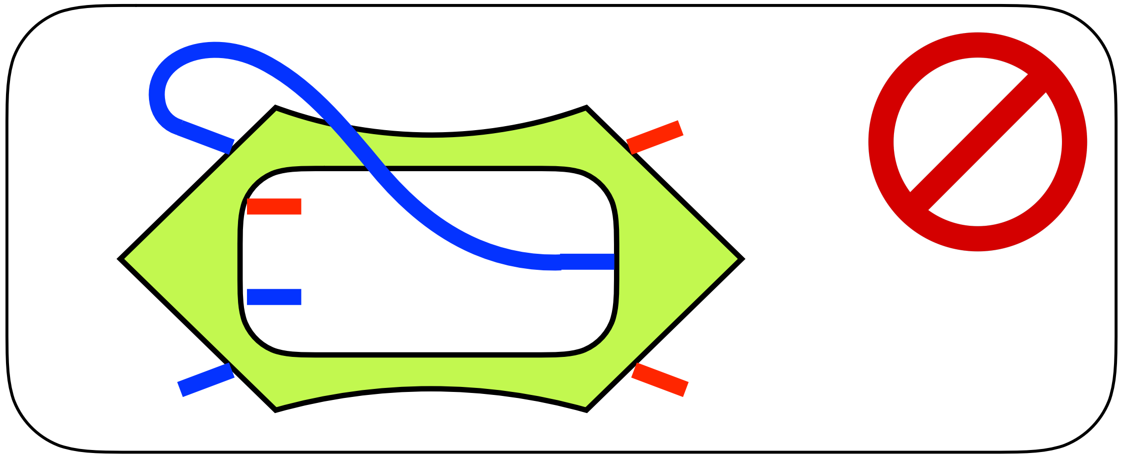
$$\mathcal{C}_{AA}(R) =$$

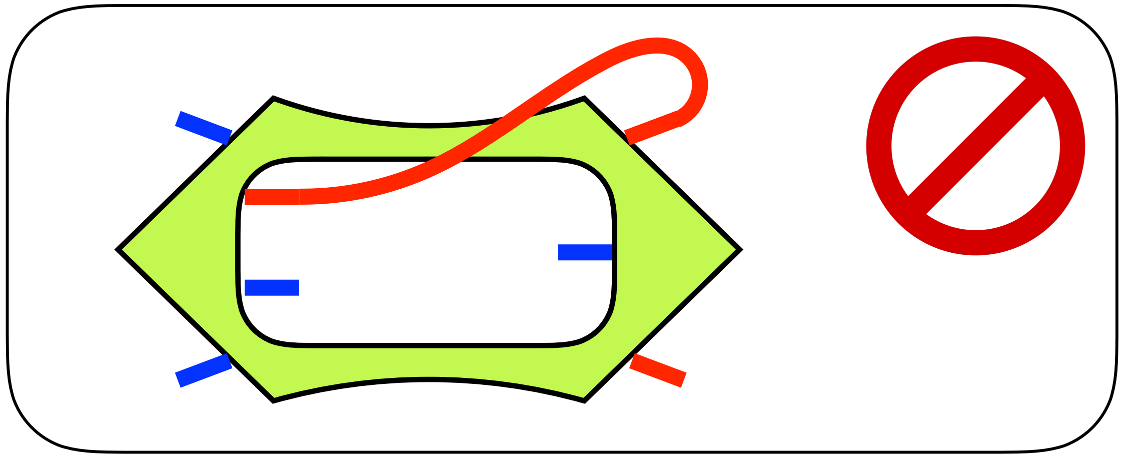


$$R * S = \mathcal{C}_{AA}(\mathcal{C}_{BB}(R \otimes S))$$

Compositional structure

Admissible contractions





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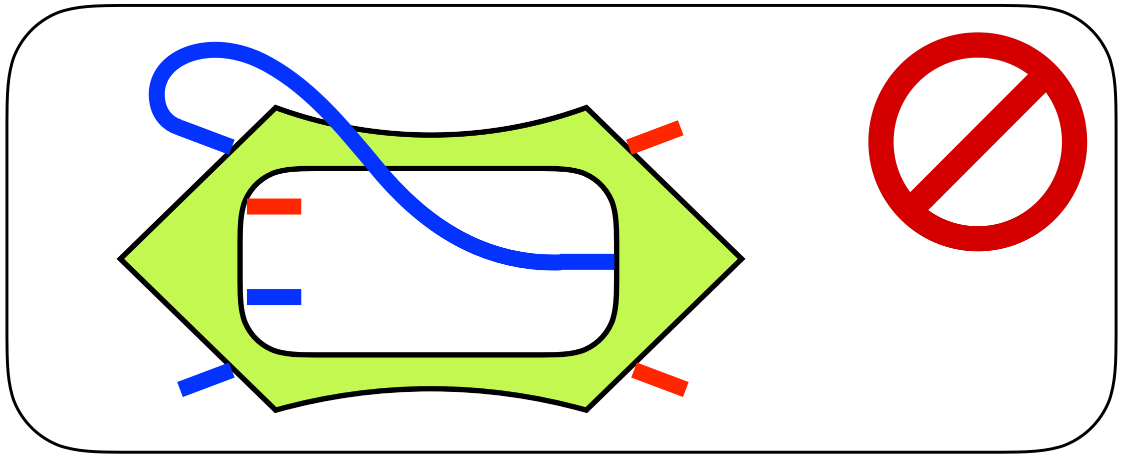

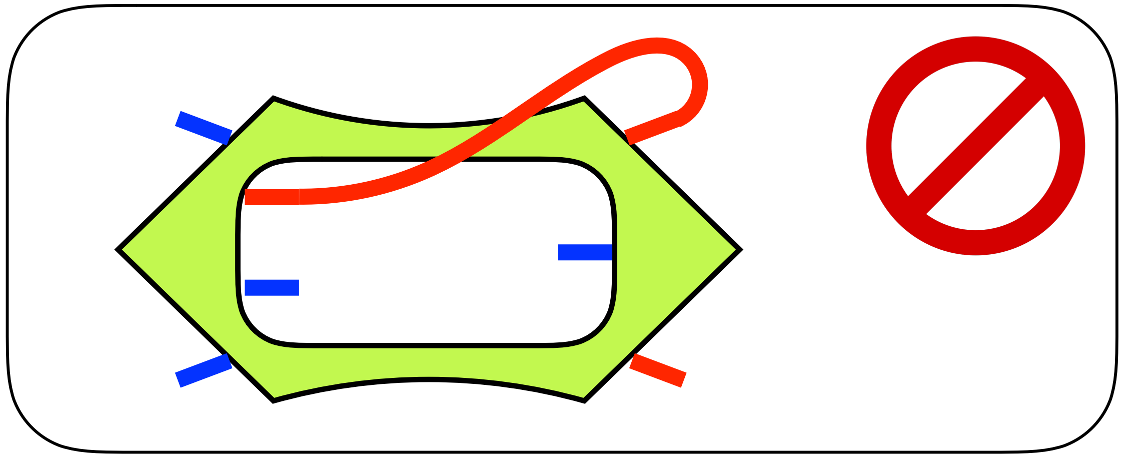

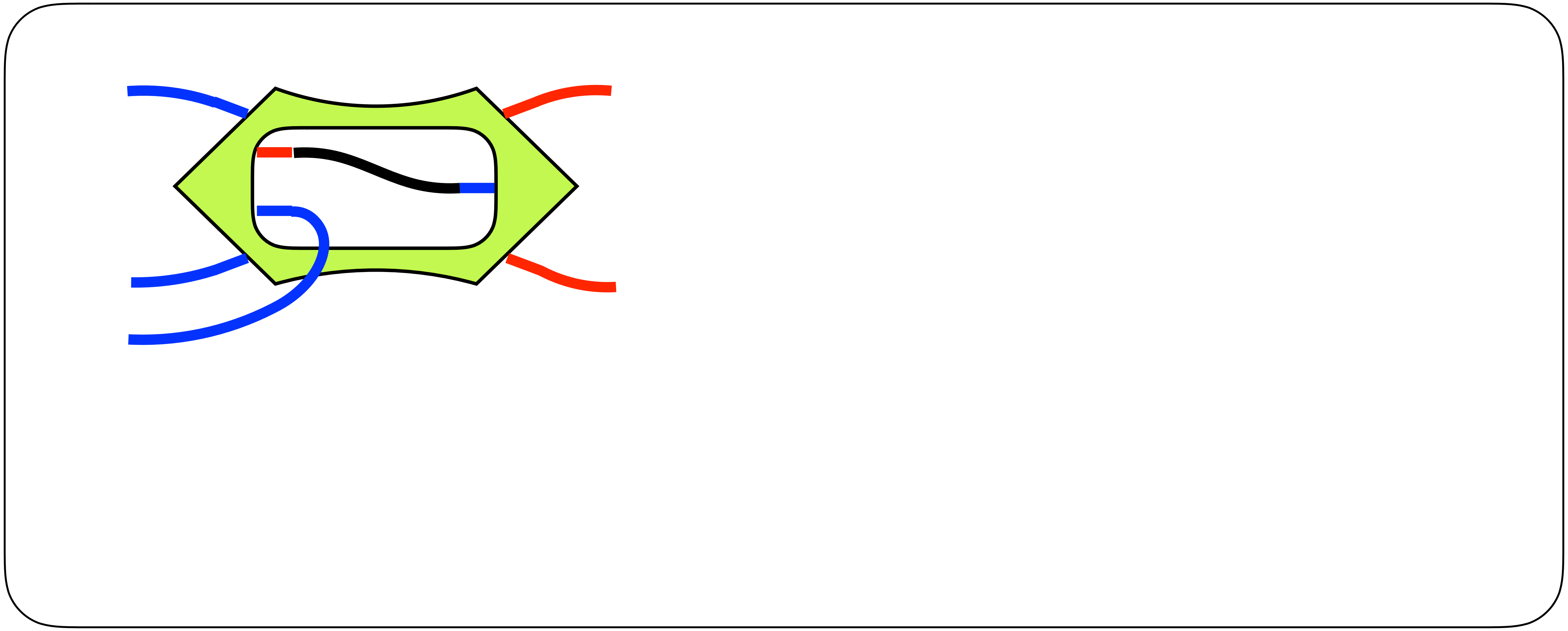

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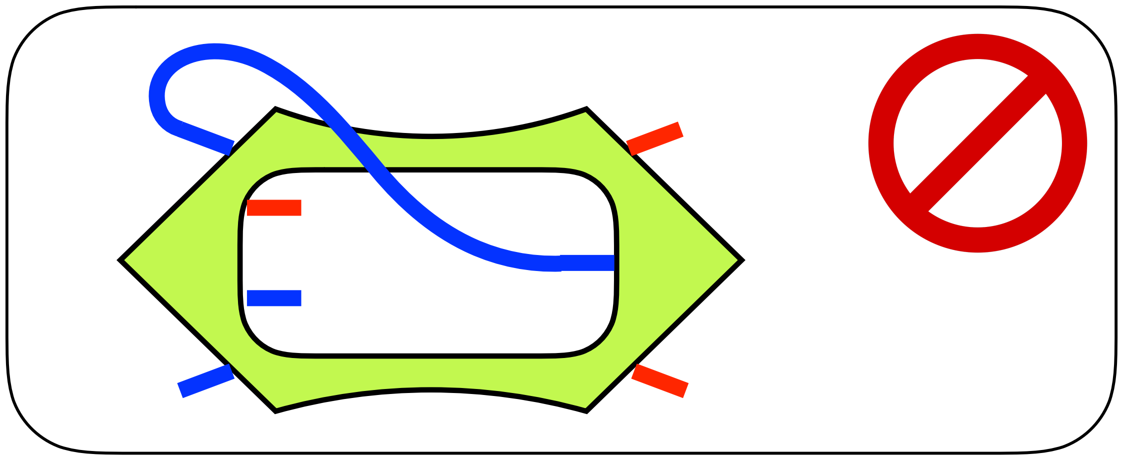
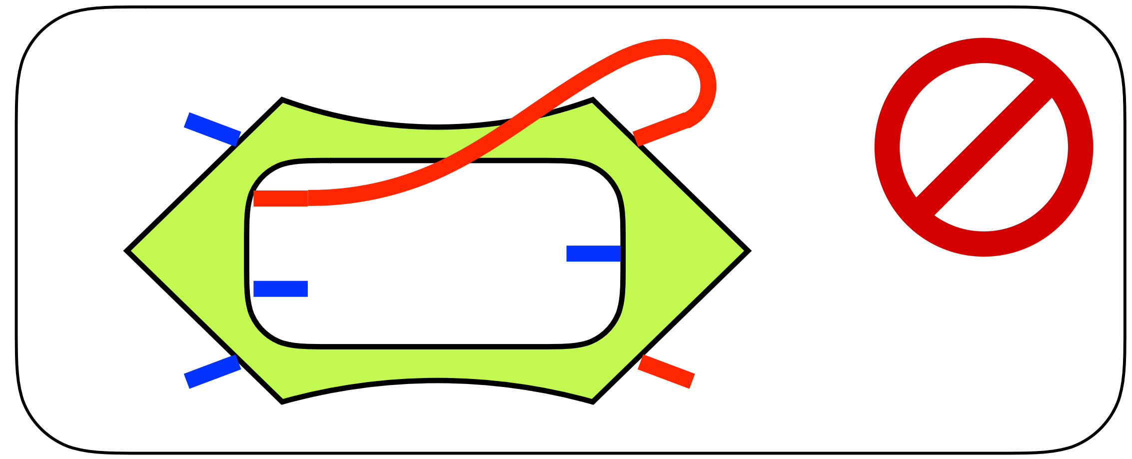
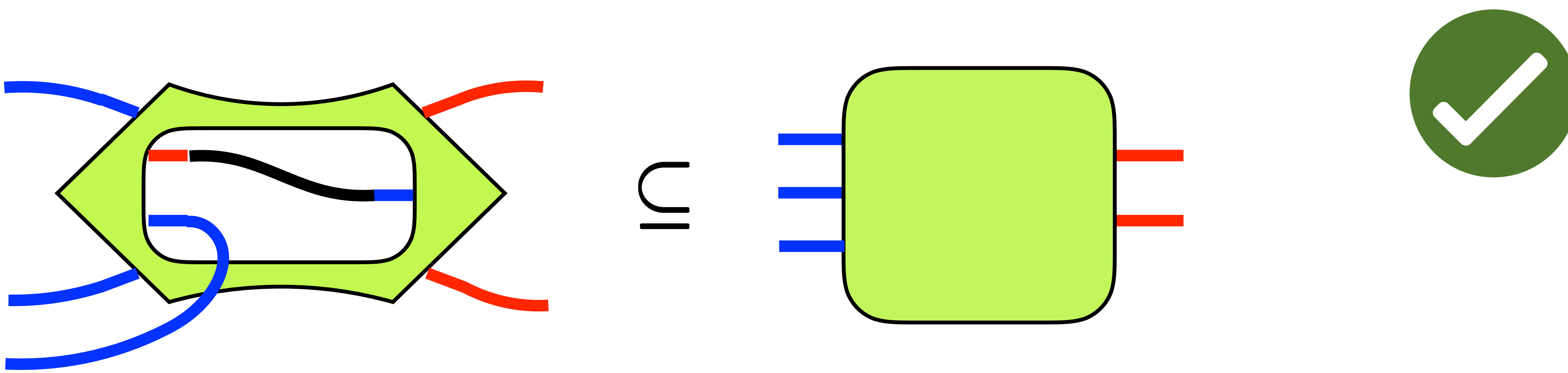


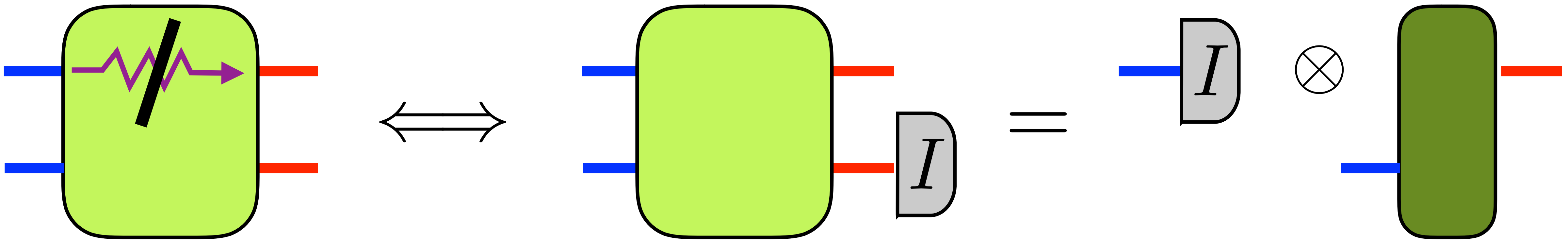


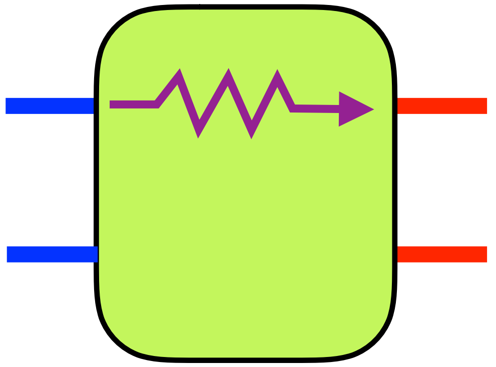


$$\mathcal{C}_{AA}(R) \subseteq \Big(\mathsf{in} \setminus A\Big) \to \Big(\mathsf{out} \setminus A\Big)$$

Signalling and type contraction

No-signalling condition

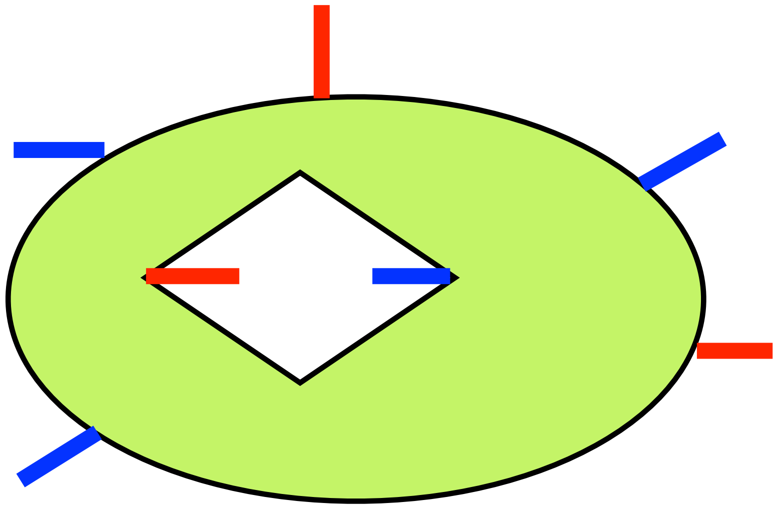




Otherwise

Signalling and type contraction

Signalling constraints admissible contractions



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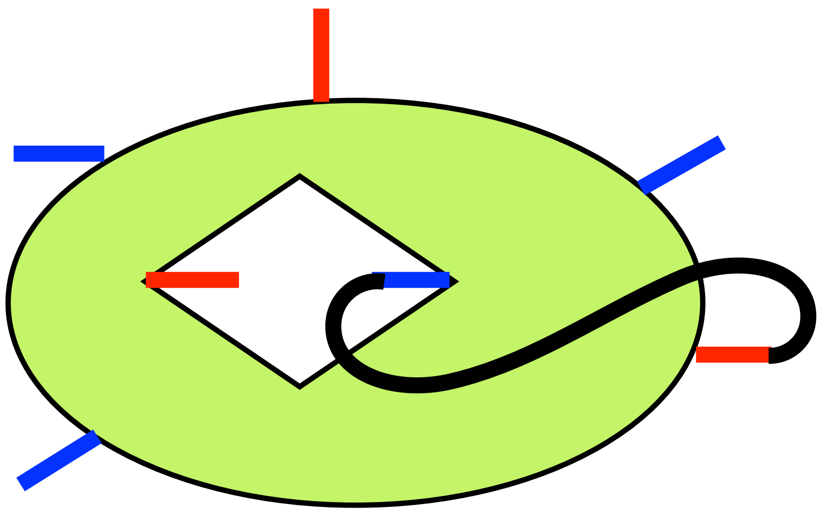

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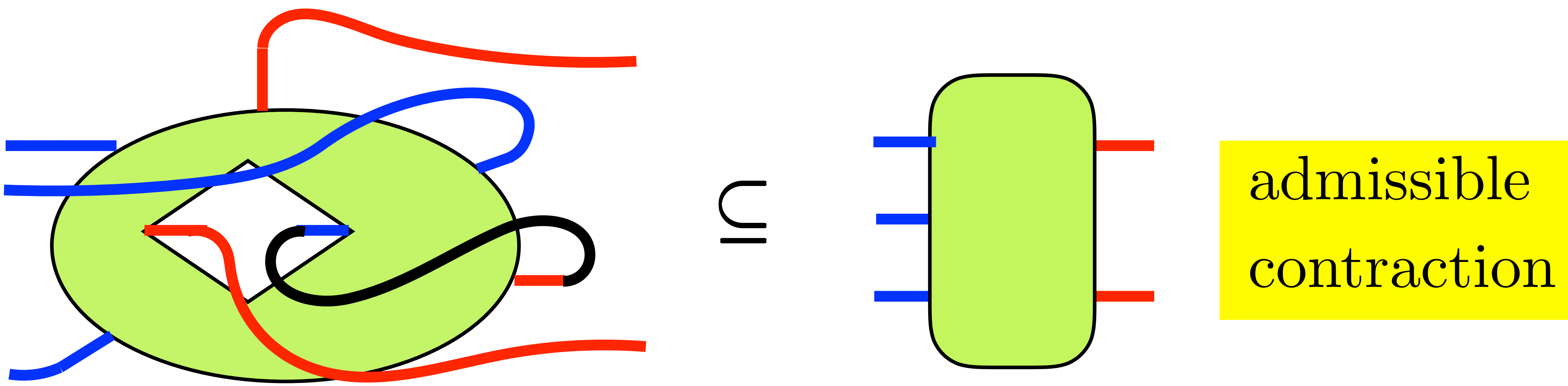

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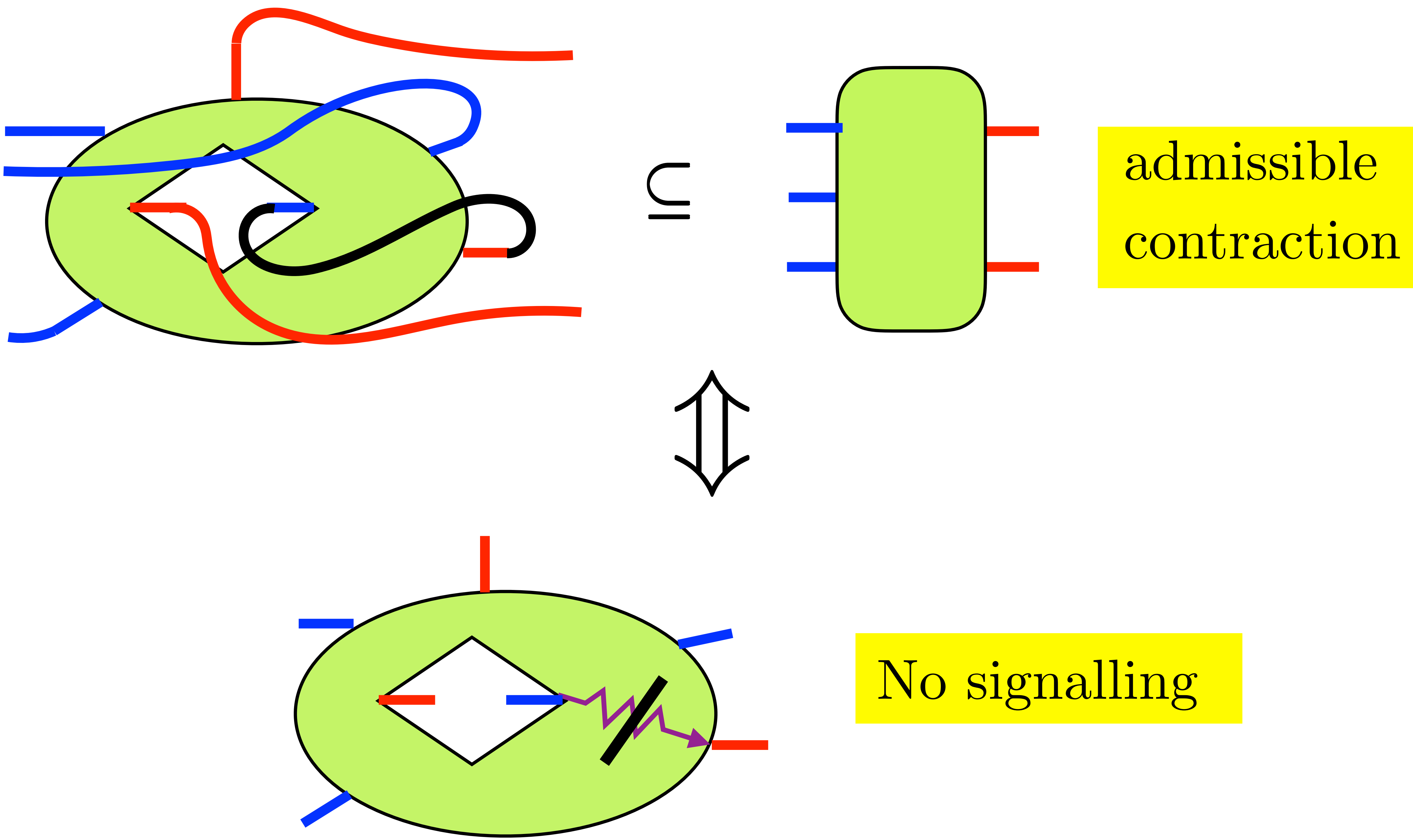

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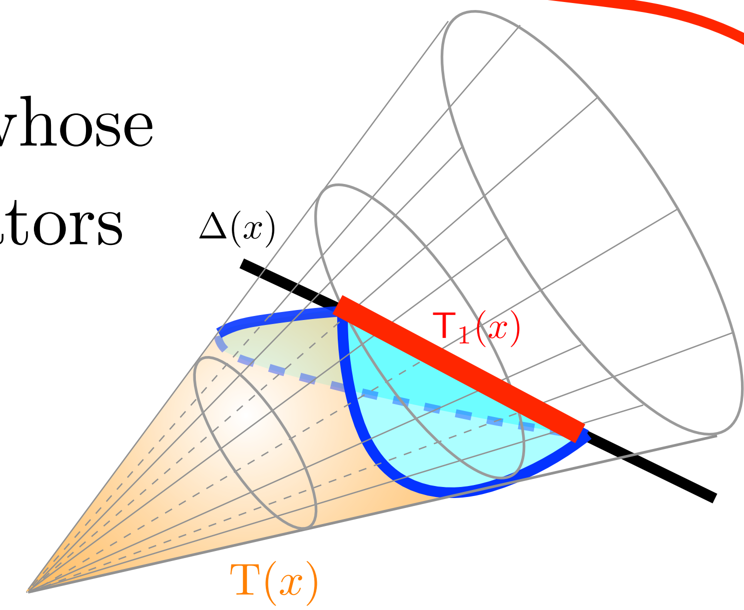

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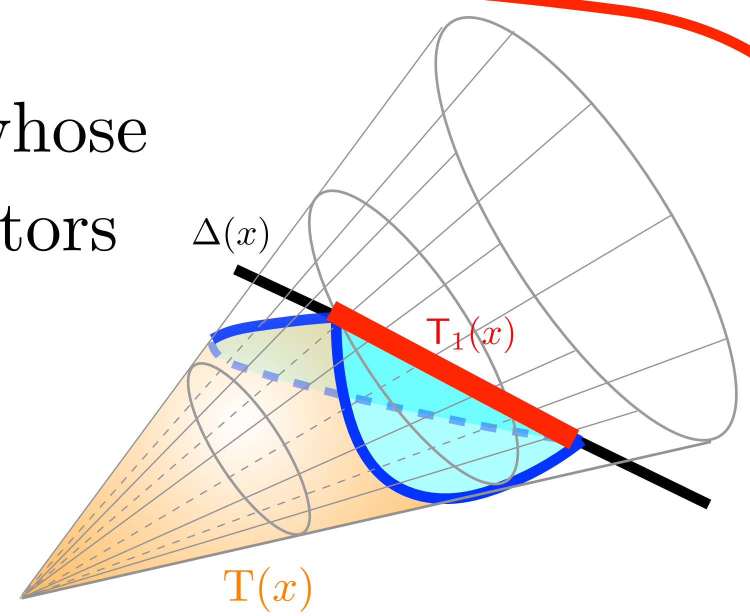

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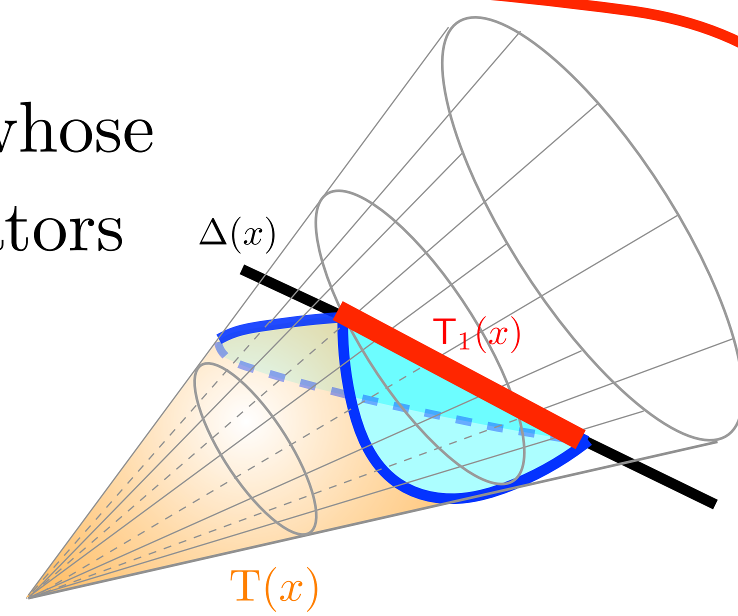

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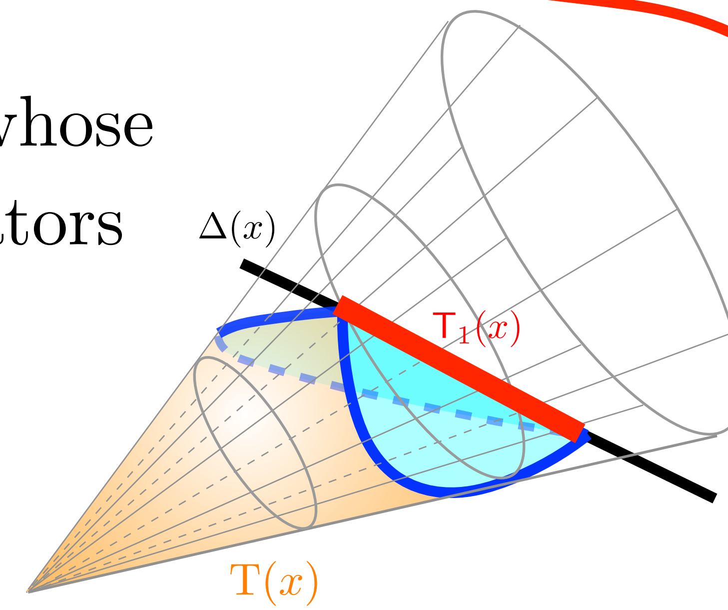

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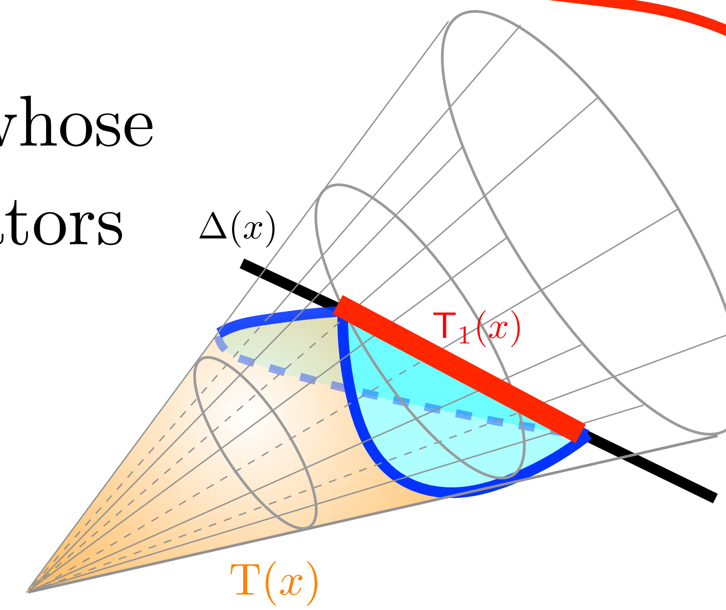

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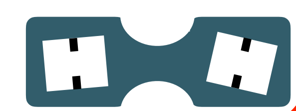

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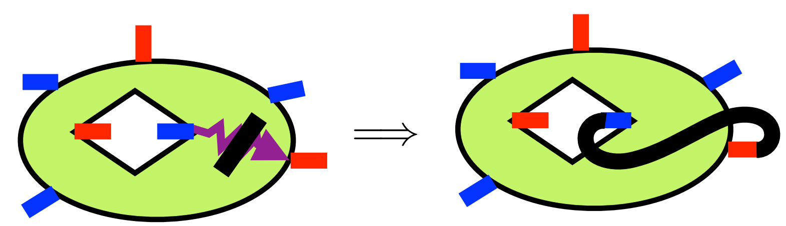
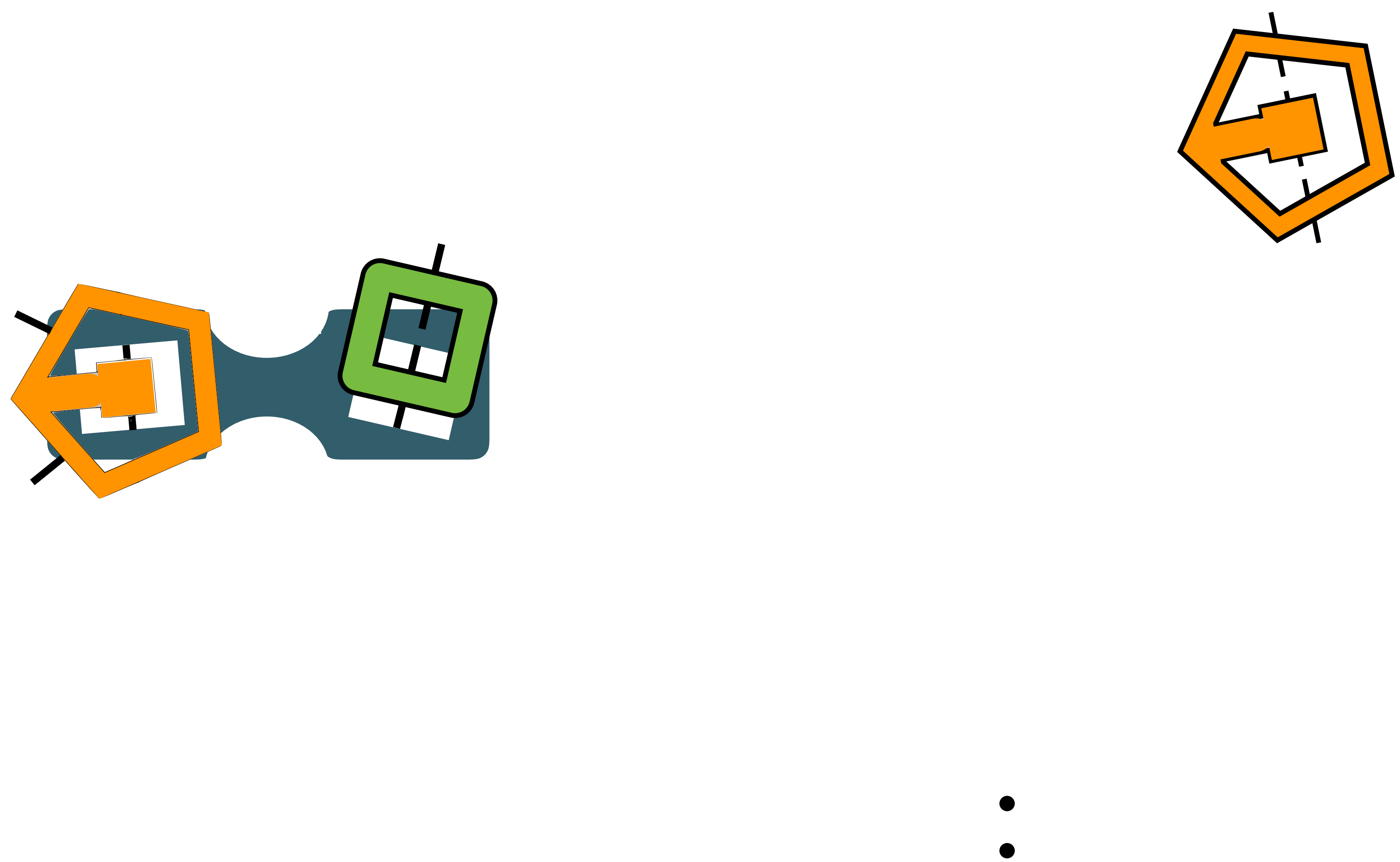
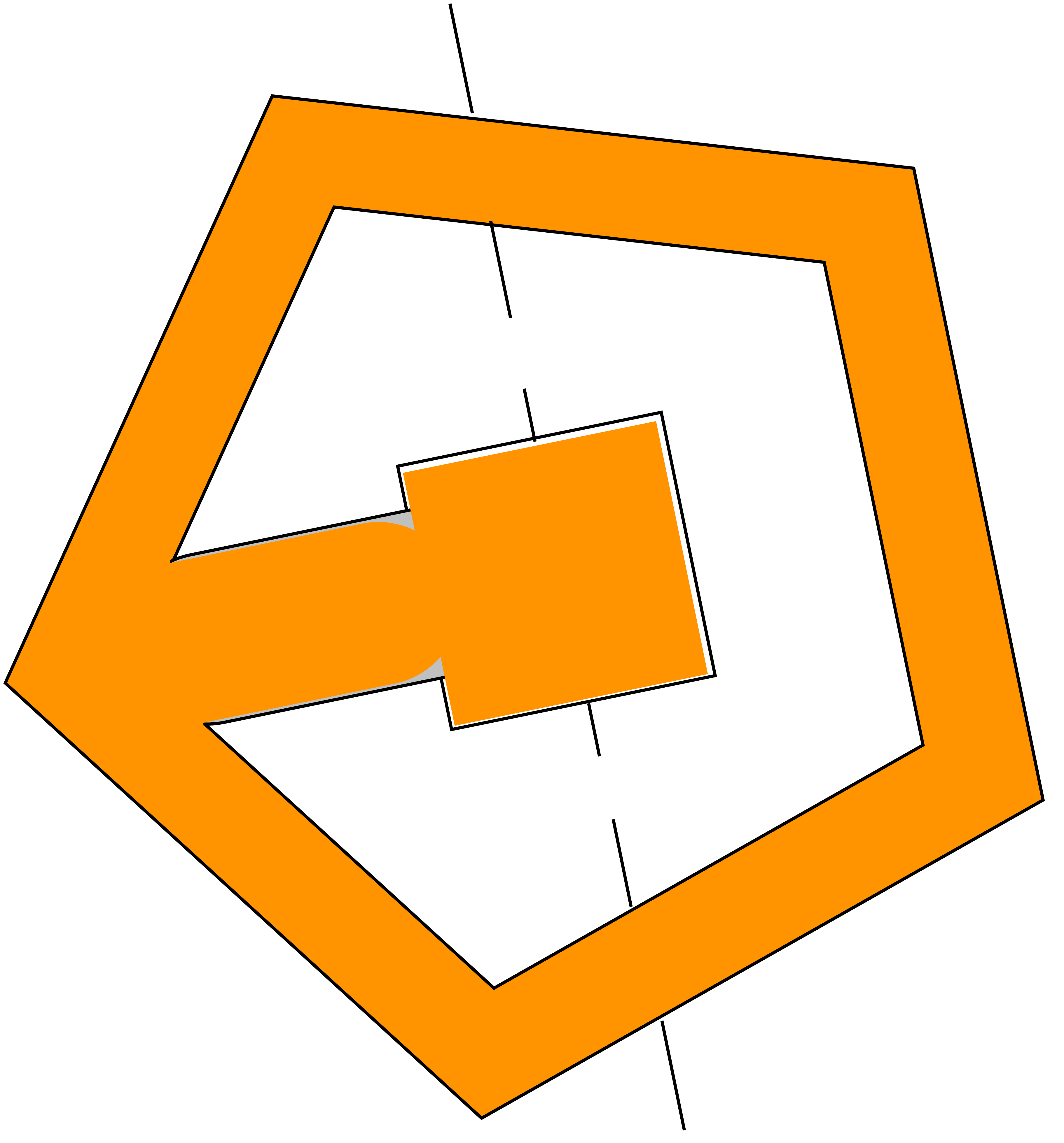
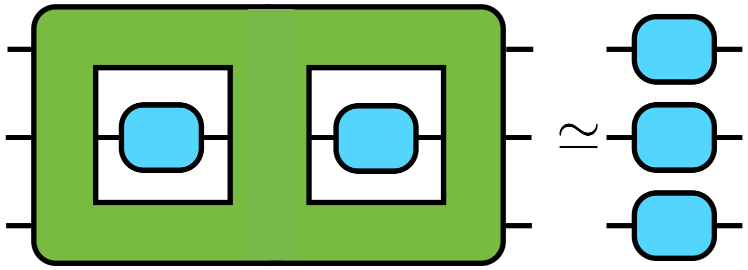






Higher order quantum computation

- Higher order maps are a convenient framework for Quantum Information Processing when the carriers of information are quantum objects more general than states (transformations, quantum network...)



- The framework of higher order maps encompasses quantum networks and quantum processes with indefinite causal order.

- Quantum processes with indefinite causal order may outperform circuital strategies:

M. Araújo, F Costa, C. Brukner Phys. Rev. Lett. 113 250402 (2014)	computational speedup
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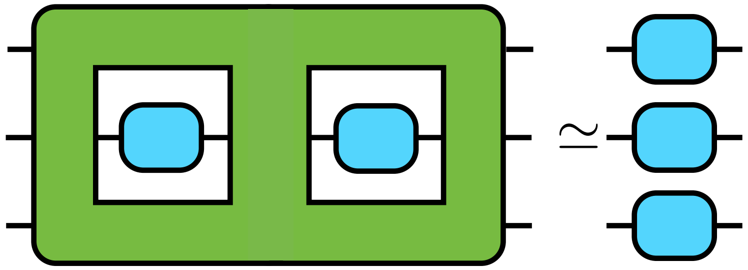

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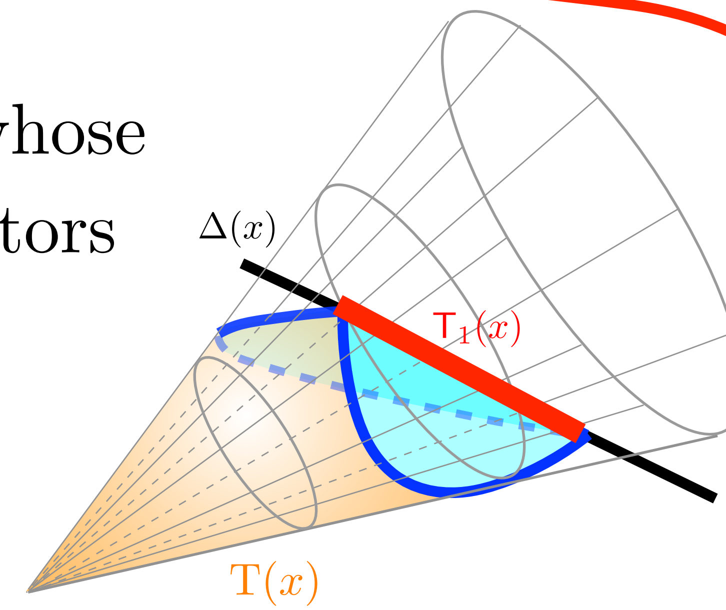

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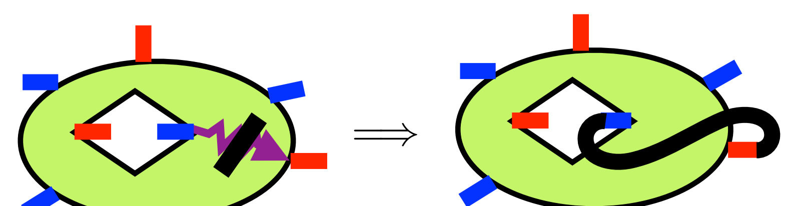

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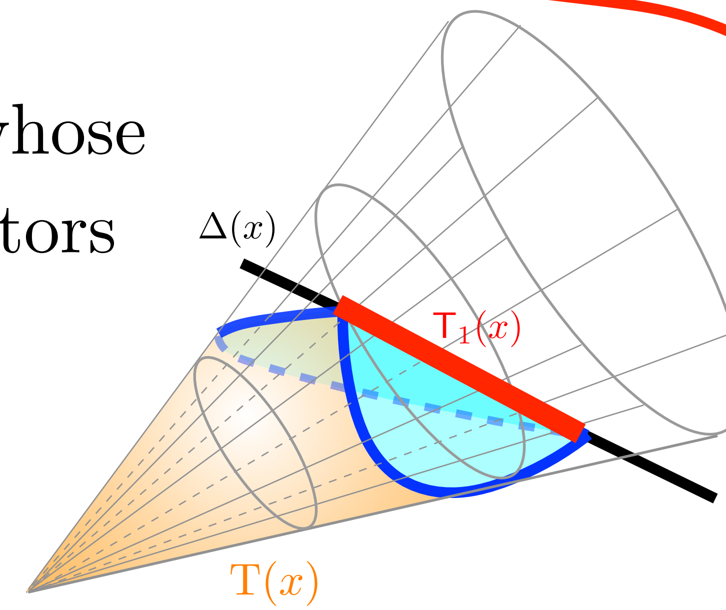

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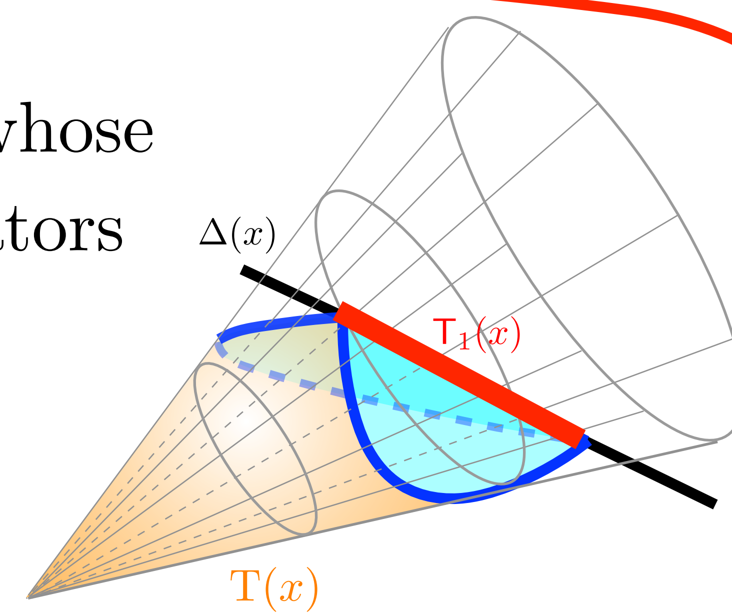

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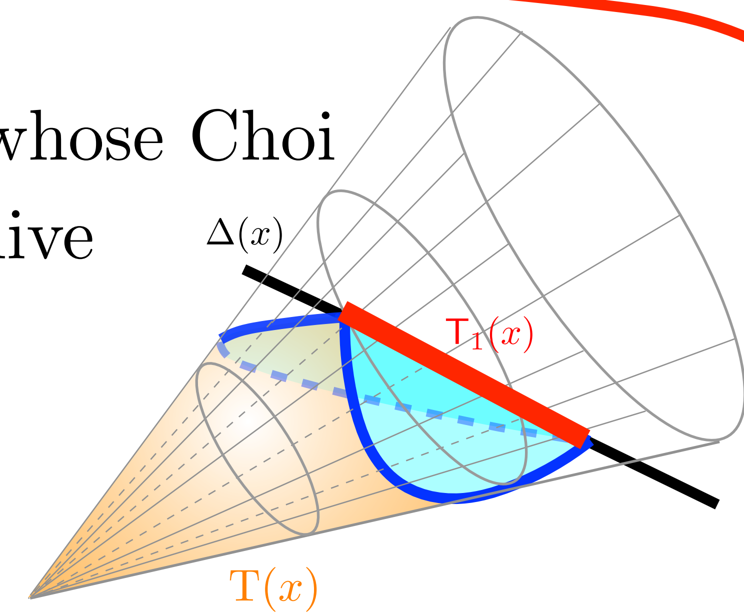

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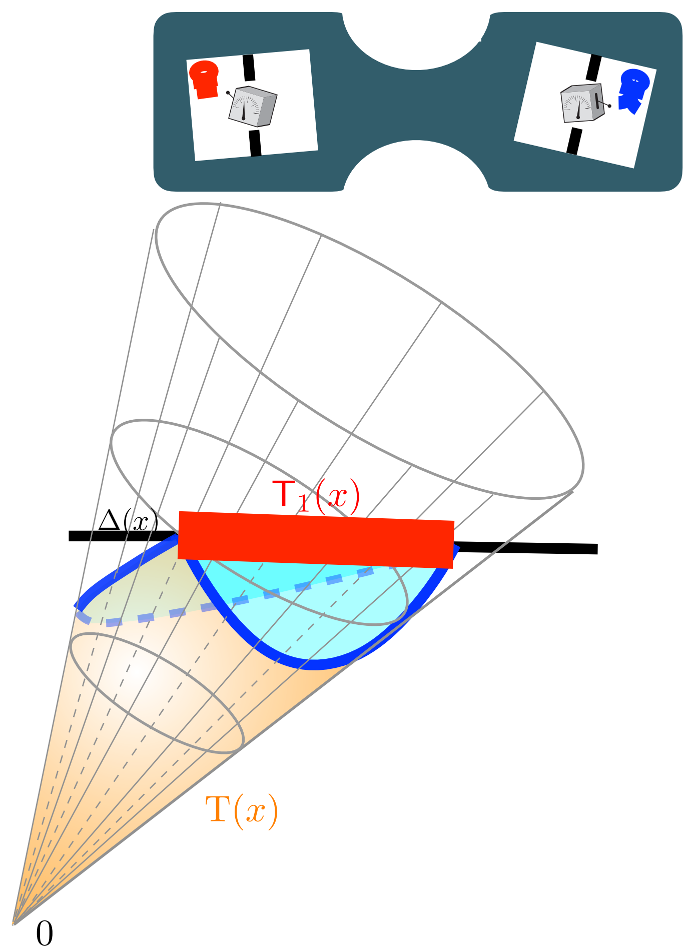

$\Delta(x)$
$\mathsf{T}_I(x)$
$\mathrm{T}(x)$
0