OPTIMAL PROGRAMMING OF QUANTUM GATES

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INTRODUCTION:

QUANTUM PROGRAMMING

PROGRAMMING QUANTUM GATES

In quantum mechanics, unitary operators describe reversible evolutions.
In quantum computing, a computation is a sequence of unitary evolutions, also called ***unitary gates.***
A ***universal quantum computer*** is device that can be programmed to approximate any desired unitary gate.

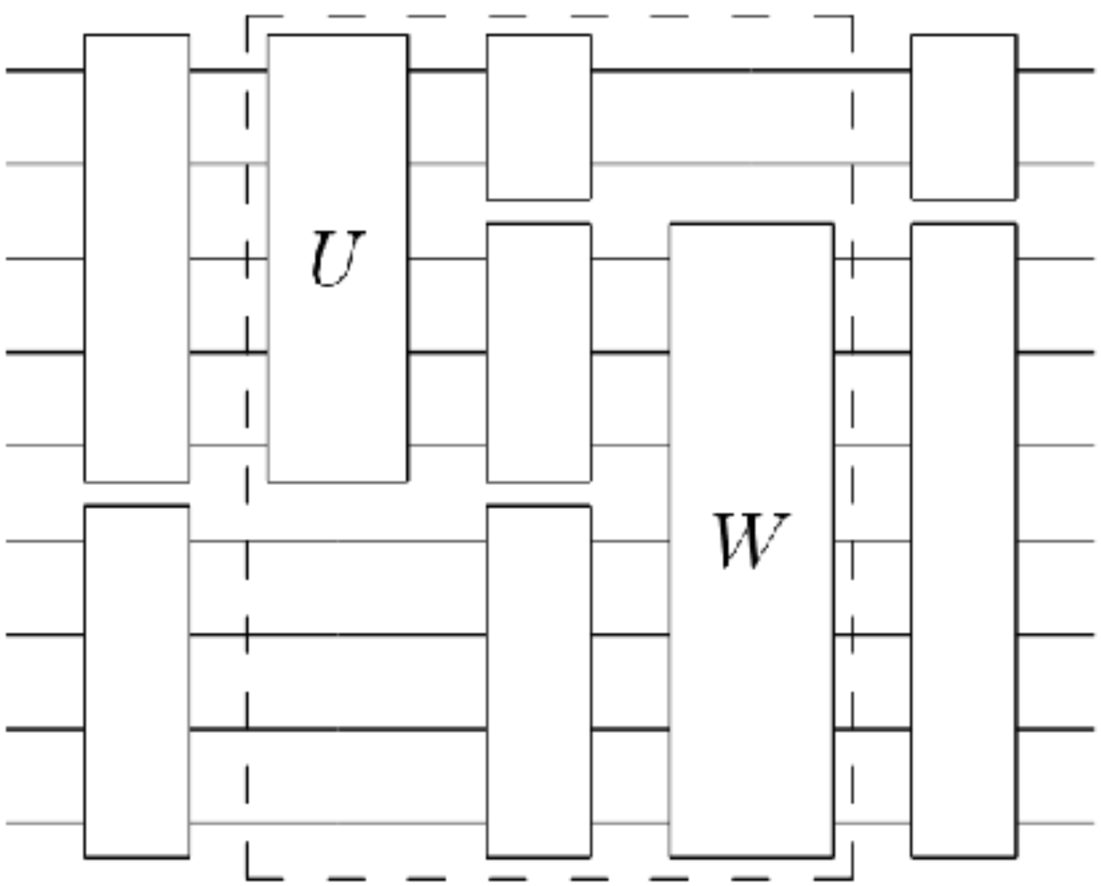


image from Wikipedia

Problem:
how to program an arbitrary unitary gate?

That is,
how to specify a unitary gate
in a set of instructions that
a quantum device can reliably follow?

CLASSICAL VS QUANTUM PROGRAMMING

Classical approach

- Fix a finite set of gates that can be implemented by the computer.
- Find approximate decomposition of the desired gate U.

e.g. decompose an element of SO(3) into rotations about Cartesian axes.

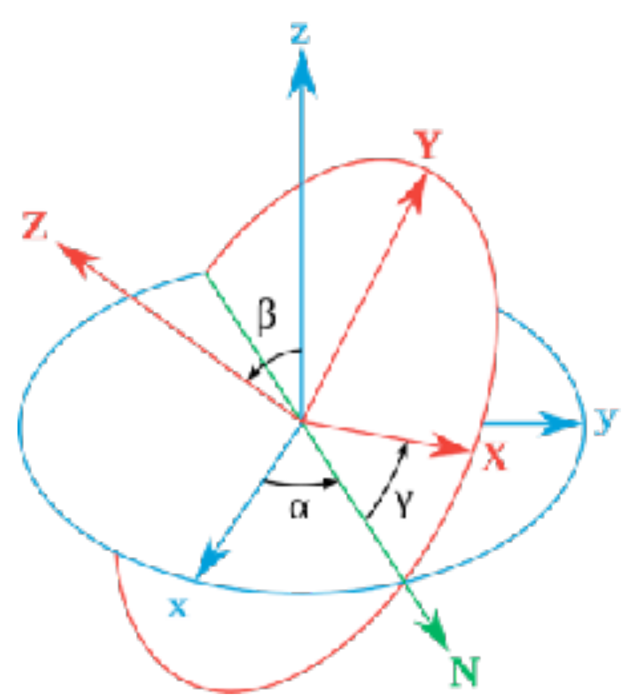
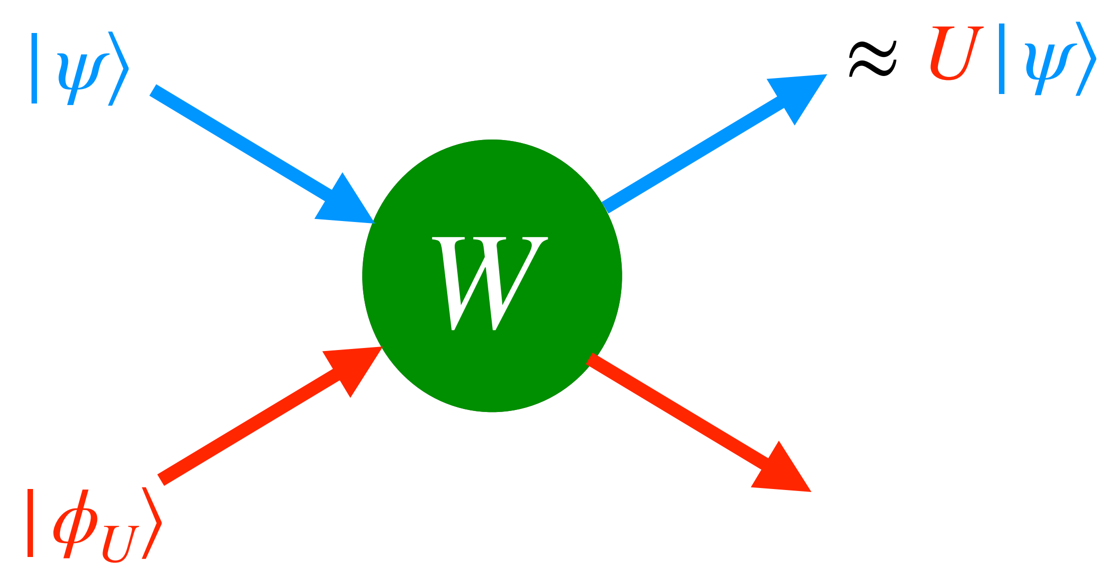


image from Wikipedia

more practical,
used in quantum computing

Quantum approach

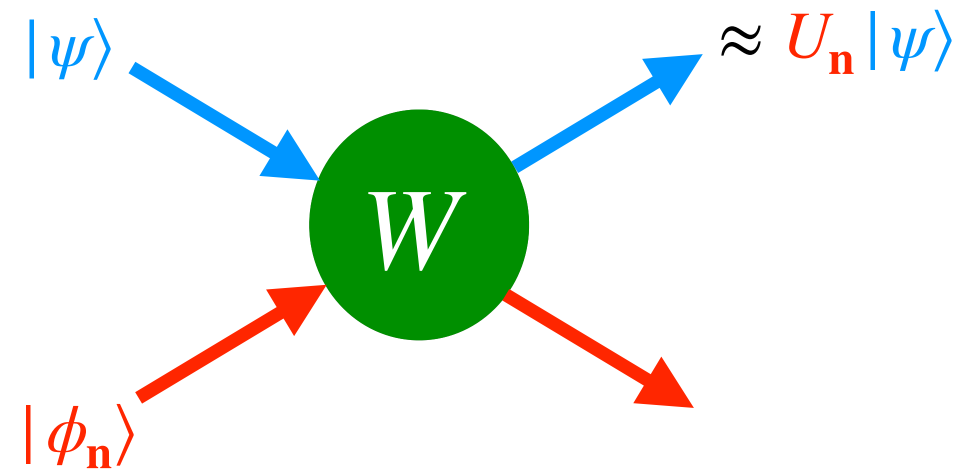
- Encode the gate U into a state $|\phi_U\rangle$ of a control system
- Let the target and the control interact through a fixed unitary gate W.



more fundamental, contains
classical programming as special case.

EXAMPLE OF QUANTUM PROGRAMMING

Problem: flip a spin about an axis $\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$



Possible solution:

- set up a Heisenberg interaction $\hat{J}_x^{\text{target}}\hat{J}_x^{\text{control}} + \hat{J}_y^{\text{target}}\hat{J}_y^{\text{control}} + \hat{J}_z^{\text{target}}\hat{J}_z^{\text{control}}$ between the target spin and a control spin
- put the control spin in the spin coherent state $|\phi_{\mathbf{n}}\rangle$ corresponding to the maximum eigenvalue of the operator $n_x\hat{J}_x^{\text{control}} + n_y\hat{J}_y^{\text{control}} + n_z\hat{J}_z^{\text{control}}$

With this scheme, the overlap with the desired state is $1 - O\left[\frac{\left(j^{\text{target}}\right)^2}{j^{\text{control}}}\right]$

Mo and Chiribella, New Journal of Physics, 21, 113003 (2019)
Marvian and Mann, Phys. Rev. A 78 022304 (2008)

ACCURACY-DIMENSION TRADEOFF

No Programming Theorem (Nielsen and Chuang, *Phys. Rev. Lett.* 79 321, 1997): exact programming of an infinite set of unitary gates require an infinite-dimensional control system.

Question: ***what is the minimum dimension of the control system required to achieve* a given level of accuracy?**

Open problem
for more than 2 decades.

Nielsen and Chuang, Phys. Rev. Lett. 79.321 (1997);
Kim, Cheong, Lee, and Lee, Phys. Rev. A 65, 012302 (2001);
Hillery, Bužek, and Ziman Phys. Rev. A 65 022301 (2002);
Vidal, Masanes, and Cirac, Phys. Rev. Lett. 88 047905 (2002);
Chiribella, D'Ariano, Perinotti, and Sacchi, Phys. Rev. Lett. 93, 180503 (2004);
Hayashi, Physics Letters A 354, 183 (2006);
Ishizaka and Hiroshima, Phys. Rev. Lett. 101, 240501 (2008);
Bartlett, Rudolph, Spekkens, and Turner, New J. Phys. 11 063013 (2009);
Bisio, Chiribella D'Ariano, Facchini, and Perinotti, Phys. Rev. A 81 032324 (2010);
Kubicki, Palazuelos, and Perez-Garcıa, Phys. Rev. Lett. 122, 080505 (2019);
Sedlak, Bisio, and Ziman, Phys. Rev. Lett. 122, 170502 (2019);
…

Important for quantifying resources in quantum information processing.

THIS TALK

Asymptotic solution of the open problem:

the minimum dimension of the control system for programming an arbitrary unitary gate in the special unitary group $\mathsf{SU}(d_{\text{target}})$ satisfies

$$\log_2 d_{\text{control}} = \frac{d_{\text{target}}^2 - 1}{2} \log_2 \frac{1}{\epsilon}$$

at the leading order in $1/\epsilon$.

Yang, Renner, and Chiribella, PRL 125, 210501 (2020)

COMPARISON WITH PREVIOUS RESULTS

	Upper bounds	Lower bounds
Prior works	$d^2 \log_2(1/\epsilon)$ (QIP'19) [Kubicki-Palazuelos-Pérez-García-2019-PRL] $(4d^2 \log_2 d)/\epsilon^2$ port-based teleportation	$\propto (1-\epsilon)d$ [Kubicki-Palazuelos-Pérez-García-2019-PRL] $\left(\frac{d-1}{2}\right) \log_2(1/\epsilon)$ [Pérez-García-2006-PRA]
This work	$\left(\frac{d^2-1}{2}\right) \log_2(1/\epsilon)$	$\left(\frac{d^2-1}{2} - \delta\right) \log_2(1/\epsilon) \quad \forall \delta > 0$

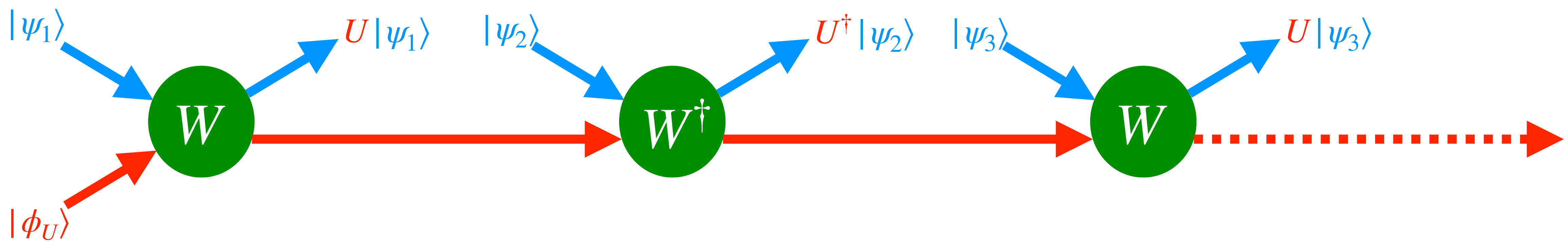
slide credit to Yuxiang Yang

METHODS
FOR
LOWER BOUND

ALTERNATIVE PROOF OF THE NO-PROGRAMMING THEOREM

Fact 1: If a gate U can be programmed *without error,* then the state of the control system can be used to implement *arbitrarily many repetitions of the gate* $U \otimes U^\dagger$.

Chiribella, D'Ariano, and Perinotti, Phys. Rev. A 81, 062348 (2010)



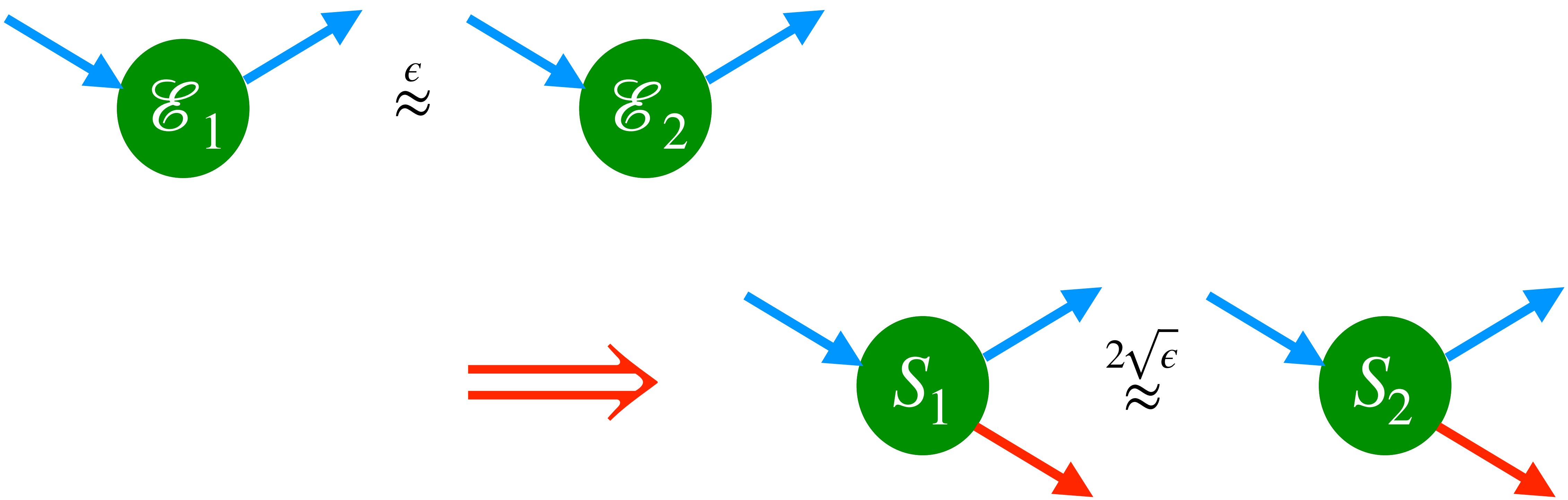
Consequence: the program states can be distinguished with arbitrary precision.

CONTINUITY OF STINESPRING'S DILATION

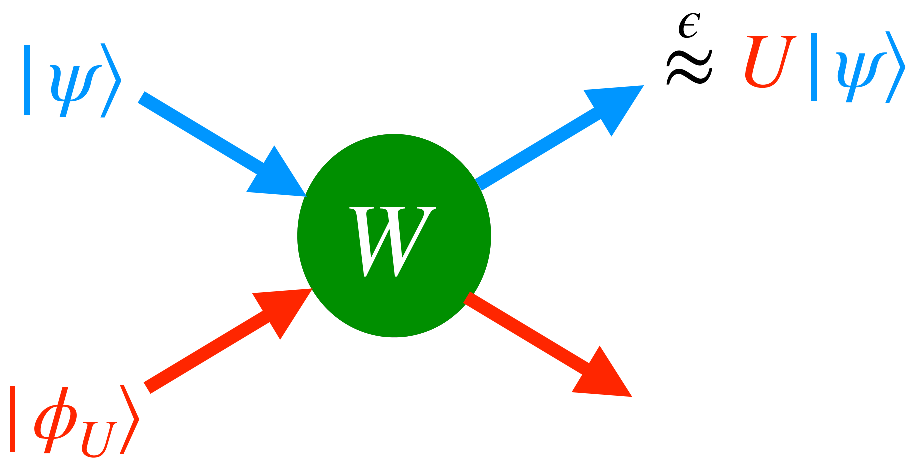
Fact 2

If two quantum evolutions (completely positive trace-preserving maps) are ϵ-close (with respect to the completely bounded trace-distance), then every Stinespring dilation of one evolution is $2\sqrt{\epsilon}$-close to some Stinespring dilation of the other.

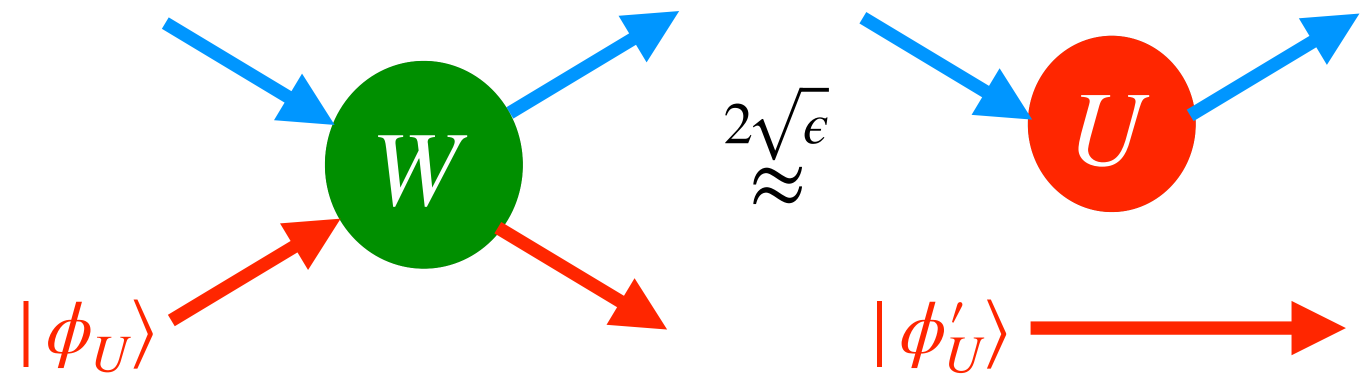
Kretschmann, Schlingemann, and Werner, Journal of Functional Analysis 255, 1889 (2008)



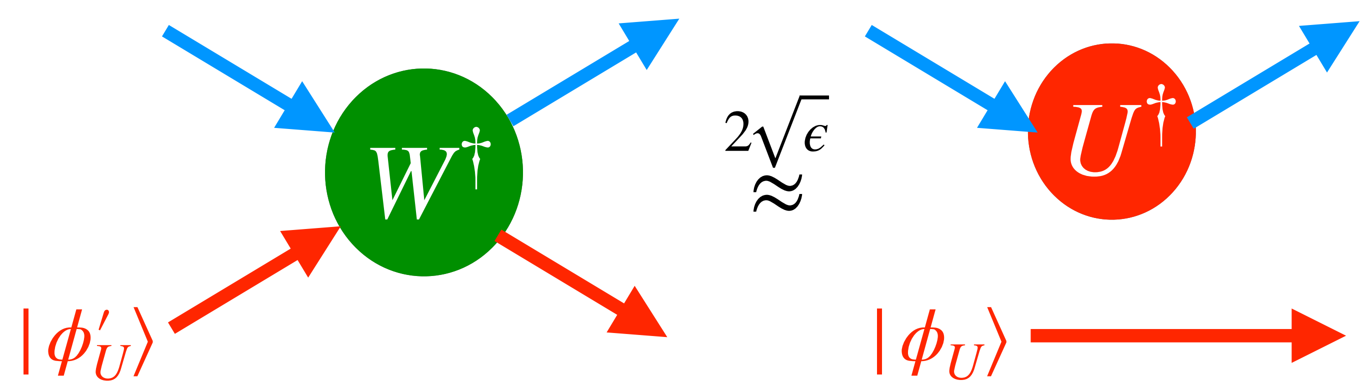
In the special case where one of the two evolution is the unitary gate U



implies



and

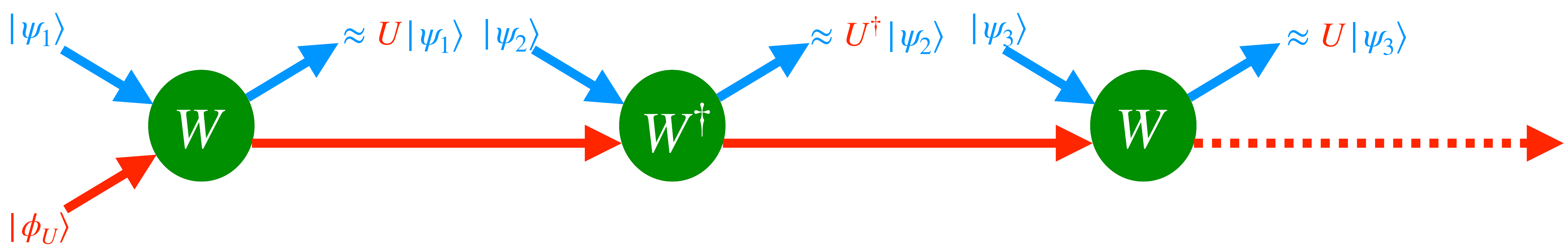


cf. Kretschmann, Kribbs, and Spekkens, Phys. Rev. A 78, 032330 (2008)

Similar technique used in many other works; recently, Tajima, Shiraishi, and Saito, Phys. Rev. Lett. 121, 110403 (2018).

APPROXIMATE RECYCLING

If a gate U can be programmed *with error* ϵ,
then the state of the control system can be used to
implement m *repetitions of the gate* $U \otimes U^\dagger$ *with error* $4m\sqrt{\epsilon}$.



Consequence: the information contained in the state $|\phi_U\rangle$ should be approximately equal to the information one can extract from the gate $(U \otimes U^\dagger)^{\otimes m}$

THE HOLEVO BOUND

Fact 3

For every set of states $\{|\phi_U\rangle\}$ in a Hilbert space of dimension d,
and for every probability distribution $p(\mathrm{d}U)$
one has the bound $\log_2 d \geq -\mathrm{Tr}[\rho \log_2 \rho]$ with $\rho := \int p(\mathrm{d}U)\, |\phi_U\rangle\langle\phi_U|$

A. S. Holevo, Problemy Peredachi Informatsii 9, 3 (1973).

Consequence: up to approximation errors, the log-dimension $\log_2 d_{\text{control}}$ should be larger than the maximum entropy of any set of states generated by the gate $(U \otimes U^\dagger)^{\otimes m}$

SCHUR-WEYL DUALITY

The maximum von Neumann entropy $-\mathsf{Tr}[\rho \log_2 \rho]$ generated by $(U \otimes U^\dagger)^{\otimes m}$ can be computed using the Schur-Weyl decomposition

$$(\mathbb{C}^d)^{\otimes m} = \bigoplus_\lambda \left(\mathscr{R}_\lambda \otimes \mathscr{M}_\lambda\right)$$

where λ labels the Young diagrams with m boxes and at most d rows.

The maximum entropy is $\log_2 \left(\sum_\lambda d^2_{\mathscr{R}_\lambda} \right)$

Chiribella, D'Ariano, Perinotti, Sacchi, Int. J. Quantum Inf. 4, 453 (2006)

The sum of squared dimensions is given by Schur's formula $\sum_\lambda d^2_{\mathscr{R}_\lambda} = \binom{m + d^2 - 1}{d^2 - 1}$

PUTTING EVERYTHING TOGETHER

Taking into account the approximation errors,
we obtain the asymptotic lower bound

$$\log_2 d_{\text{control}} \geq (1 - \delta - 4\sqrt{2\epsilon})\,\frac{d^2-1}{2}\left(\log_2\frac{1}{\epsilon} + \log_2\frac{\delta^2}{32(d^2-1)^2}\right) - 1$$

valid for every fixed $\delta > 0$

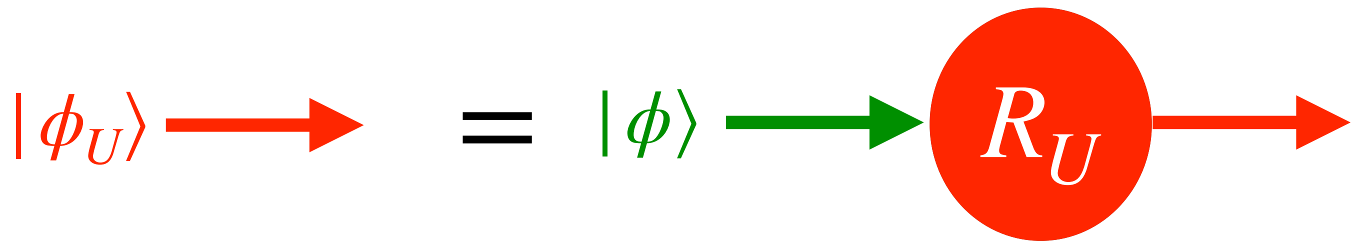
At leading order, $\log_2 d_{\text{control}} \geq (1-\delta)\left(\frac{d^2-1}{2}\right)\log_2\frac{1}{\epsilon}$

METHODS
FOR
UPPER BOUND

PROGRAM STATES FROM GROUP ACTIONS

Let G be a group of unitary gates
and let $R : U \mapsto R_U$ be a unitary representation of G
on some Hilbert space $\mathscr{H}_R$

A ***possible program state*** for the gate U is $|\phi_U\rangle := R_U|\phi\rangle$

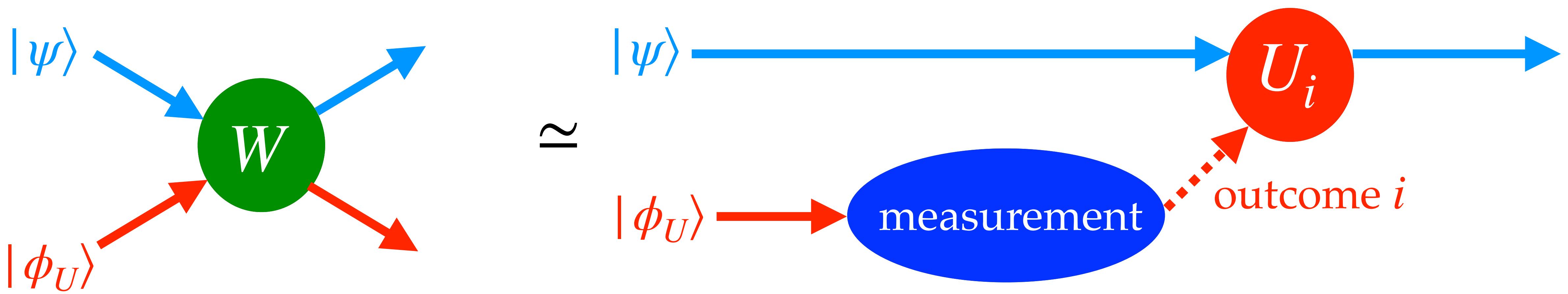


A priori, there is no guarantee that this choice will give the optimal dependence of $|\phi_U\rangle$ on U. Still, this is a reasonable *ansatz* one can make.

MEASURE-AND-OPERATE STRATEGIES

A *possible way to extract the gate U* from the program state $|\phi_U\rangle$ is to

- *perform a measurement*, and
- *perform a conditional operation* based on the measurement outcome.



We call this scheme a ***measure-and-operate (MO) strategy.***
Again, there is no *a priori* guarantee that MO strategies are optimal,
but one can use them as an *ansatz*.

BACK TO SCHUR-WEYL

In our work, $\mathsf{G} = \mathsf{SU}(d_{\text{target}})$
and we pick the truncated regular representation $R = \bigoplus_\lambda \left(R_\lambda \otimes I_\lambda\right)$

where λ runs over a subset of Young diagrams with n boxes in at most d_{target} rows.

For the input state, we use a state of the form $|\phi\rangle = \bigoplus_\lambda \sqrt{q_\lambda}\,|E_\lambda\rangle$

where (q_λ) is a probability distribution and each $|E_\lambda\rangle$ is a maximally entangled state.

States of this form are known to be optimal for the estimation of the gate U

Chiribella, D'Ariano, Perinotti, Phys. Rev. A 72 042338 (2005)

THE UPPER BOUND

We find that the estimation error is upper bounded as

$$\epsilon \leq 2\left(\frac{\pi(d_{\text{target}}-1)(3d_{\text{target}}-2)}{d_{\text{target}}\, n}\right)^2$$

and the dimension of the representation is upper bounded as $d_R \leq \left(\frac{9n}{3d_{\text{target}}-2}\right)^{d^2-1}$

The above error bound improves over the state of the art of $\mathsf{SU}(d)$ estimation
cf. Kahn, Physical Review A 75, 022326 (2007).

Expressing d_R in terms of ϵ, we obtain the bound

$$\log_2 d_R \leq \frac{d_{\text{target}}^2-1}{2}\left(\log_2\frac{1}{\epsilon}+\log_2\frac{162\pi^2(d_{\text{target}}^2-1)^4}{d_{\text{target}}^2}\right)$$

CONCLUSIONS

OUTLOOK

Asymptotic solution of the quantum programming problem:

$$\log_2 d_{\text{control}} = \frac{d_{\text{target}}^2 - 1}{2} \log_2 \frac{1}{\epsilon} \qquad \text{at leading order in } 1/\epsilon.$$

Yang, Renner, and Chiribella, PRL 125, 210501 (2020)

Conjecture: for an f-dimensional manifold of unitary gates

$$\log_2 d_{\text{control}} = \frac{f}{2} \log_2 \frac{1}{\epsilon} \qquad \text{at leading order in } 1/\epsilon.$$

- **From unitary gates programming to quantum error correction:** see talk by Mischa Wood later in this session.