Minimally-committal quantum measurements

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Third Kyoto Workshop on Quantum Information, Computation, and Foundations, October 2022

Range of quantum measurements

States:

$$\rho \geq 0 \text{ s.t. } \mathrm{Tr}[\rho] = 1$$

Measurements:

$$\boldsymbol{\pi} : \mathbb{S} \to \mathbb{P} \qquad \text{s. t. } \pi_j \geq 0 \; \forall j \text{ and } \sum_j \pi_j = \mathbb{1}$$

$$\rho \mapsto \left(\mathrm{Tr}[\rho \pi_j]\right)_j$$

Measurement range:

$$\mathrm{rng}(\boldsymbol{\pi}) := \left\{ \mathbf{p} \;\middle|\; \exists\, \rho \text{ s.t. } p_j = \mathrm{Tr}[\rho \pi_j],\ \forall j \right\}.$$

Properties of the inference of quantum measurements

Data-driven: the only input is a set $\mathcal{P}$ of probability distributions

Consistent:

$$\mathcal{P} \subseteq \operatorname{rng}(\boldsymbol{\pi})$$

Minimally committal: the (partial) order induced by the committal degree:

- is invariant under linear transformations:

$$F(\boldsymbol{\pi}_0) \leq F(\boldsymbol{\pi}_1) \leftrightarrow F(\mathcal{L}(\boldsymbol{\pi}_0)) \leq F(\mathcal{L}(\boldsymbol{\pi}_1))$$

- preserves the (partial) order induced by range inclusion

$$\operatorname{rng}(\boldsymbol{\pi}_0) \subseteq \operatorname{rng}(\boldsymbol{\pi}_1) \Leftrightarrow F(\boldsymbol{\pi}_0) \leq F(\boldsymbol{\pi}_1)$$

Data-driven inference of quantum measurements

Definition (Data-driven inference)

For any spanning set $\mathcal{P}$ of d^2-outcome probability distributions

$$\mathtt{ddi}_{\mathbb{S}}\left(\mathcal{P}\right) := \operatorname*{argmin}_{\substack{\boldsymbol{\pi} \\ \sum_j \pi_j = \mathbb{1} \\ \mathcal{P} \subseteq \mathrm{rng}(\boldsymbol{\pi})}} \mathrm{vol}\,\mathrm{rng}\left(\boldsymbol{\pi}\right).$$

Caveat: as for linear inversion, the positivity constraint is relaxed

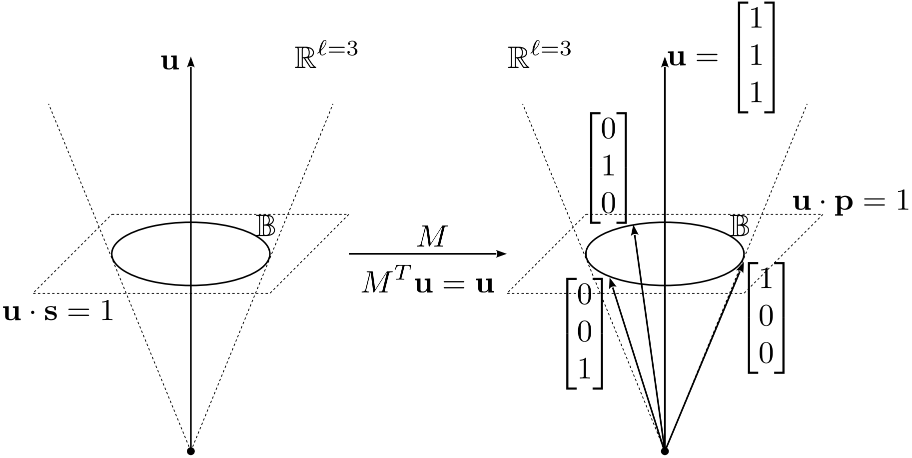
Qubit case solved in M. D., F. Buscemi, A. Bisio, A. Tosini, PRA 102 (2020)

A geometrical reformulation

	Hilbert formalism	Geometrical formalism
Linear space	Hermitian $d \times d$	$\mathbb{R}^{\ell = d^2}$
Inner prod.	Hilbert-Schmidt	Dot product
Born rule	$p_j = \mathrm{Tr}[\rho \pi_j]$	$\mathbf{p} = M\mathbf{s}$
Unit effect	$\mathbb{1}$	$\mathbf{u} := (1, \dots, 1)^T$
States	$\mathrm{Tr}[\rho] = 1$	$\mathbf{u} \cdot \mathbf{s} = 1$
Measurements	$\sum_j \pi_j = \mathbb{1}$	$M^T\mathbf{u} = \mathbf{u}$ (see [1])
Purity	$\mathrm{Tr}[\rho^2]$	$\lvert\mathbf{s}\rvert^2$

[1] Follows from the fact that for any state $\mathbf{s}$ one has $\mathbf{u}^T M\mathbf{s} = 1$.

A pictorial representation



Data-driven inference of quantum measurements

Definition (Data-driven inference)

For any spanning set $\mathcal{P}$ of ℓ-outcome probability distributions

$$\mathtt{ddi}_{\mathbb{S}}\left(\mathcal{P}\right) := \operatorname*{argmin}_{\substack{M \in \mathbb{R}^{\ell \times \ell} \\ M^T \mathbf{u} = \mathbf{u} \\ \mathcal{P} \subseteq M\mathbb{S}}} \det M^T M.$$

Pure states:

$$|\mathbf{s}|^2 = 1$$

"Purity" cone (not to be confused with the "positivity" cone):

$$f(\mathbf{s}) := |\mathbf{s}|^2 - (\mathbf{u} \cdot \mathbf{s})^2 = \mathrm{Tr}\left[\mathbf{s} \otimes \mathbf{s}\right] - \mathbf{u}^T (\mathbf{s} \otimes \mathbf{s})\, \mathbf{u} \leq 0$$

"Purity" ball $\mathbb{B}$:

$$\mathbf{u} \cdot \mathbf{s} = 1 \quad \cap \quad f(\mathbf{s}) \leq 0$$

Spherical 2-**design** (not to be confused with the quantum 2-design):

$$\sum_{\mathbf{s} \in \mathcal{S}} p(\mathbf{s})\, \mathbf{s} \otimes \mathbf{s} = \int (O \oplus \hat{\mathbf{u}} \otimes \hat{\mathbf{u}})(\mathbf{s} \otimes \mathbf{s})(O \oplus \hat{\mathbf{u}} \otimes \hat{\mathbf{u}})^T \, \mathrm{d}\, O = \lambda \mathbb{1} + \mu \hat{\mathbf{u}} \otimes \hat{\mathbf{u}}$$

Main result

Theorem

For any spanning set $\mathcal{P}$ of ℓ-outcome quasi-probability distributions

$$\mathcal{S} \textit{ supports a spherical } 2\textit{-design} \Rightarrow M \in \mathtt{ddi}_{\mathbb{S}}\left(\mathcal{P}\right),$$

where M is uniquely given by $\mathcal{P} = M\mathcal{S}$.

SICs as standard measurements

SIC measurements (regular simplices): proportional to $\mathbb{1}$

Lemma

For any spanning set $\mathcal{P}$ of ℓ-outcome probability distributions

$$M \in \mathtt{ddi}_{\mathbb{S}}\left(\mathcal{P}\right) \Leftrightarrow \mathbb{1} \in \mathtt{ddi}_{\mathbb{S}}\left(M^{-1}\mathcal{P}\right).$$

Proof given in M. D., F. Buscemi, A. Bisio, A. Tosini, PRA 102 (2020)

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{M^{-1}} & M^{-1}\mathcal{P} \\
\mathtt{ddi}_{\mathbb{S}} \big\downarrow & & \mathtt{ddi}_{\mathbb{S}} \big\downarrow \\
\mathtt{ddi}_{\mathbb{S}}\left(\mathcal{P}\right) & \xleftarrow{M} & \mathtt{ddi}_{\mathbb{S}}\left(M^{-1}\mathcal{P}\right)
\end{array}$$

Outer approximation of the state space

Lemma

For any spanning set $\mathcal{S}$ of ℓ-dimensional states

$$\mathbb{1} \in \mathtt{ddi}_{\mathbb{B}}\left(\mathcal{S}\right) \Rightarrow \mathbb{1} \in \mathtt{ddi}_{\mathbb{S}}\left(\mathcal{S}\right).$$

The role of spherical 2-designs

Lemma

For any set $\mathcal{S}$ of ℓ-dimensional states that supports a spherical 2-design

$$\mathcal{S} \textit{ supports a sph. 2-design} \Rightarrow \mathbb{1} \in \mathtt{ddi}_{\mathbb{B}}\left(\mathcal{S}\right).$$

Sketch of the proof

A variation of a proof technique by F. John (1948)

$$\begin{aligned}
&\sum_{\mathbf{s}\in\mathcal{S}} p(\mathbf{s}) f\left(M^{-1}\mathbf{s}\right)\\
=&\operatorname{Tr}\left[M^{-1}\left(\lambda \mathbb{1} + \mu \hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\right) M^{-T}\right] - \mathbf{u}^T M^{-1}\left(\lambda \mathbb{1} + \mu \hat{\mathbf{u}}\otimes\hat{\mathbf{u}}\right) M^{-T}\mathbf{u}\\
=&\lambda \operatorname{Tr}\left[M^{-1}M^{-T}\right] + \mu\left|M^{-1}\hat{\mathbf{u}}\right|^2 - \lambda\left|M^{-T}\mathbf{u}\right|^2 - \mu\left(\hat{\mathbf{u}}M^{-T}\mathbf{u}\right)^2\\
=&\lambda \operatorname{Tr}\left[M^{-1}M^{-T}\right] + \mu\left|M^{-1}\hat{\mathbf{u}}\right|^2 - |\mathbf{u}|^2(\lambda+\mu)\\
\geq&\lambda \operatorname{Tr}\left[M^{-1}M^{-T}\right] + \mu - |\mathbf{u}|^2(\lambda+\mu)\\
=&\operatorname{Tr}\left[M^{-1}M^{-T}\right] - \ell\\
=&\operatorname{Tr}\left[M^{-1}M^{-T} - \mathbb{1}\right]\\
\geq&\ln\det M^{-1}M^{-T}
\end{aligned}$$

Conclusion

Data-driven: the only input is a set $\mathcal{P}$ of probability distributions

Consistent:

$$\mathcal{P} \subseteq M\mathbb{S}$$

Minimally committal: committal degree:

$$F(M) := \det M^T M$$

Characterization of a subset of $\mathtt{ddi}_{\mathbb{S}}(\mathcal{P})$ for arbitrary quantum dimension