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Incompatibility of quantum instruments

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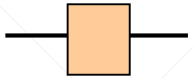
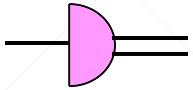
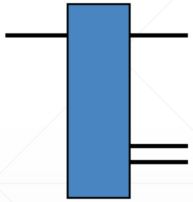
Incompatibility of quantum devices

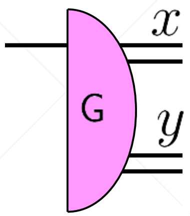
	In this talk quantum devices are:	Can be seen as:	Notation of the set:
	**Channel** = CPTP map	one outcome instrument	$\mathrm{Ch}(\mathcal{H},\mathcal{K})$
	**POVM** = Positive operator valued measure - describes only outcome probabilities	instrument with 1-dim out	$\mathcal{O}(\Omega,\mathcal{H})$
	**Instrument** = collection of CPTD maps summing up to a channel - describes also post measurement state		$\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$

Two quantum devices A and B are **compatible** if there exist a joint quantum device C, which in each single experimental run produces (complete) output for each of the devices A,B. Devices A,B are **incompatible** otherwise.

Compatibility conditions in different scenarios

Compatibility of POVMs $\mathsf{A} \in \mathcal{O}(\Omega, \mathcal{H})$ $\mathsf{B} \in \mathcal{O}(\Lambda, \mathcal{H})$

$\mathsf{A} \,\text{co}\, \mathsf{B} \Leftrightarrow \exists\, \mathsf{G} \in \mathcal{O}(\Omega \times \Lambda, \mathcal{H})$

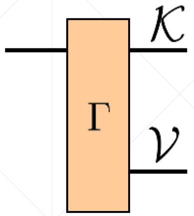


$$\sum_{y \in \Lambda} \mathsf{G}(x, y) = \mathsf{A}(x) \quad \forall x \in \Omega$$

$$\sum_{x \in \Omega} \mathsf{G}(x, y) = \mathsf{B}(y) \quad \forall y \in \Lambda$$

Compatibility of channels $\Phi \in \mathrm{Ch}(\mathcal{H}, \mathcal{K})$ $\Psi \in \mathrm{Ch}(\mathcal{H}, \mathcal{V})$

$\Phi \,\text{co}\, \Psi \Leftrightarrow \exists\, \Gamma \in \mathrm{Ch}(\mathcal{H}, \mathcal{K} \otimes \mathcal{V})$



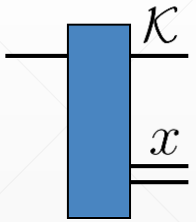
$$\mathrm{tr}_{\mathcal{V}} [\Gamma(\varrho)] = \Phi(\varrho)$$

$$\mathrm{tr}_{\mathcal{K}} [\Gamma(\varrho)] = \Psi(\varrho)$$

Compatibility of channel and POVM

$\Phi \in \mathrm{Ch}(\mathcal{H}, \mathcal{K})$ $\mathsf{A} \in \mathcal{O}(\Omega, \mathcal{H})$

$\Phi \,\text{co}\, \mathsf{A} \Leftrightarrow \exists\, \mathcal{G} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$



$$\sum_x \mathcal{I}_x(\rho) = \Phi(\rho) \qquad \Phi^{\mathcal{G}} = \Phi$$

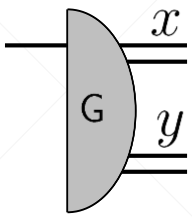
$$Tr(\mathcal{I}_x(\rho)) = Tr(\mathsf{A}(x)\rho) \qquad \Leftrightarrow \qquad \mathsf{A}^{\mathcal{G}} = \mathsf{A}$$

$\Phi^{\mathcal{G}}$ **induced channel of instrument $\mathcal{G}$** $\mathsf{A}^{\mathcal{G}}$ **induced POVM of instrument $\mathcal{G}$**

Compatibility conditions in different scenarios

Compatibility of POVMs $\mathsf{A} \in \mathcal{O}(\Omega, \mathcal{H})$ $\mathsf{B} \in \mathcal{O}(\Lambda, \mathcal{H})$

$$\mathsf{A} \circledcirc \mathsf{B} \Leftrightarrow \exists\, \mathsf{G} \in \mathcal{O}(\Omega \times \Lambda, \mathcal{H})$$

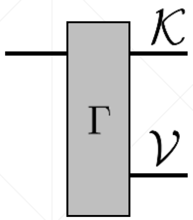


$$\sum_{y \in \Lambda} \mathsf{G}(x, y) = \mathsf{A}(x) \quad \forall x \in \Omega$$

$$\sum_{x \in \Omega} \mathsf{G}(x, y) = \mathsf{B}(y) \quad \forall y \in \Lambda$$

Compatibility of channels $\Phi \in \mathrm{Ch}(\mathcal{H}, \mathcal{K})$ $\Psi \in \mathrm{Ch}(\mathcal{H}, \mathcal{V})$

$$\Phi \circledcirc \Psi \Leftrightarrow \exists\, \Gamma \in \mathrm{Ch}(\mathcal{H}, \mathcal{K} \otimes \mathcal{V})$$



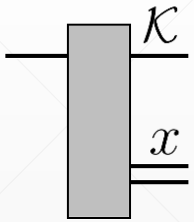
$$\mathrm{tr}_{\mathcal{V}}\left[\Gamma(\varrho)\right] = \Phi(\varrho)$$

$$\mathrm{tr}_{\mathcal{K}}\left[\Gamma(\varrho)\right] = \Psi(\varrho)$$

Compatibility of channel and POVM

$\Phi \in \mathrm{Ch}(\mathcal{H}, \mathcal{K})$ $\mathsf{A} \in \mathcal{O}(\Omega, \mathcal{H})$

$$\Phi \circledcirc \mathsf{A} \Leftrightarrow \exists\, \mathcal{G} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$$



$$\sum_x \mathcal{I}_x(\rho) = \Phi(\rho) \qquad \Phi^{\mathcal{G}} = \Phi$$

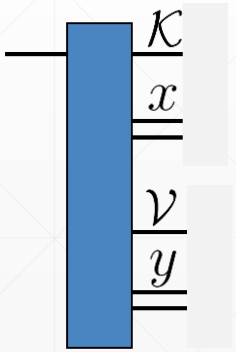
$$Tr(\mathcal{I}_x(\rho)) = Tr(\mathsf{A}(x)\rho) \quad \Leftrightarrow \quad \mathsf{A}^{\mathcal{G}} = \mathsf{A}$$

Compatibility of instruments

$\mathcal{I} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$

$\mathcal{J} \in \mathrm{Ins}(\Lambda, \mathcal{H}, \mathcal{V})$

$$\mathcal{I} \circledcirc \mathcal{J} \Leftrightarrow \exists\, \mathcal{G} \in \mathrm{Ins}(\Omega \times \Lambda, \mathcal{H}, \mathcal{K} \otimes \mathcal{V})$$



$$\sum_{y \in \Lambda} \mathrm{tr}_{\mathcal{V}}\left[\mathcal{G}_{(x,y)}(\varrho)\right] = \mathcal{I}_x(\varrho) \quad \forall x \in \Omega$$

$$\sum_{x \in \Omega} \mathrm{tr}_{\mathcal{K}}\left[\mathcal{G}_{(x,y)}(\varrho)\right] = \mathcal{J}_y(\varrho) \quad \forall y \in \Lambda$$

Parallel compatibility defined in: A. Mitra, M. Farkas, On the compatibility of quantum instruments, Phys. Rev. A 105, 052202 (2022)

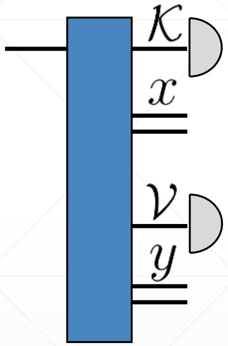
Compatibility of instruments – definition consequences

$\mathcal{I} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$
$\mathcal{J} \in \mathrm{Ins}(\Lambda, \mathcal{H}, \mathcal{V})$

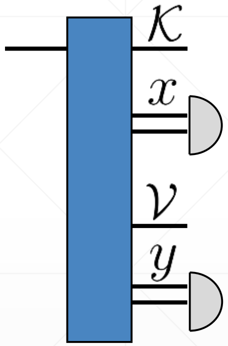
$\mathcal{I} \circ\!\!\circ \mathcal{J} \Leftrightarrow \exists\, \mathcal{G} \in \mathrm{Ins}(\Omega \times \Lambda, \mathcal{H}, \mathcal{K} \otimes \mathcal{V})$ **such that**

$$\sum_{y \in \Lambda} tr_{\mathcal{V}} \circ \mathcal{G}_{(x,y)} = \mathcal{I}_x \quad \forall x \in \Omega$$

$$\sum_{x \in \Omega} tr_{\mathcal{K}} \circ \mathcal{G}_{(x,y)} = \mathcal{J}_y \quad \forall y \in \Lambda$$



$$\mathcal{I} \circ\!\!\circ \mathcal{J} \Rightarrow \mathrm{A}^{\mathcal{I}} \circ\!\!\circ \mathrm{A}^{\mathcal{J}}$$



$$\mathcal{I} \circ\!\!\circ \mathcal{J} \Rightarrow \Phi^{\mathcal{I}} \circ\!\!\circ \Phi^{\mathcal{J}}$$

$\mathrm{A}^{\mathcal{G}}$ **induced POVM of instrument** $\mathcal{G}$

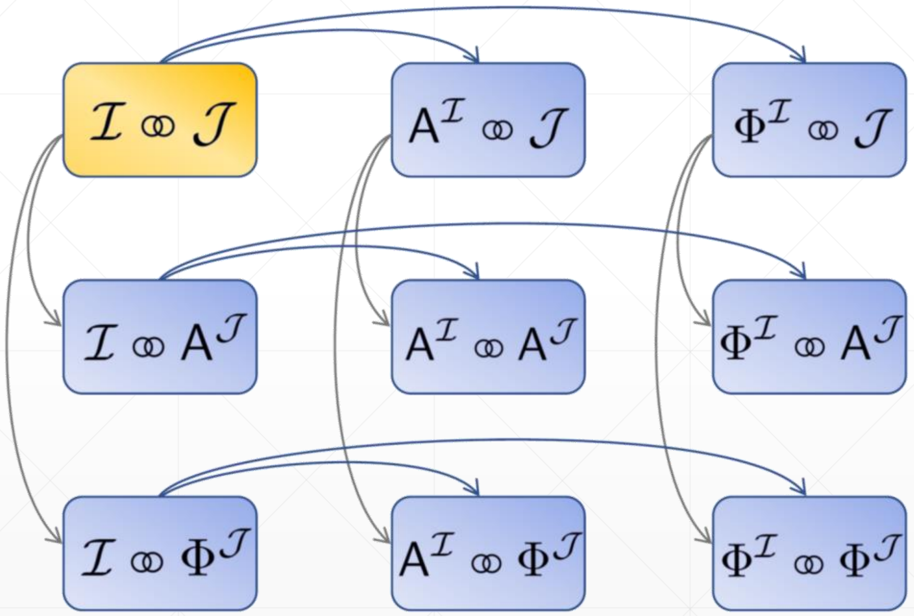
$\Phi^{\mathcal{G}}$ **induced channel of instrument** $\mathcal{G}$

Compatibility of instruments – definition consequences

Proposition 1:

If a quantum device is compatible with an instrument then it is compatible also with its induced channel and POVM

Compatibility of two instruments implies:



$A^{\mathcal{G}}$ induced POVM of instrument $\mathcal{G}$

$\Phi^{\mathcal{G}}$ induced channel of instrument $\mathcal{G}$

Compatibility of instruments – Attempted reduction

Hypotheses:

$$\mathsf{A}^{\mathcal{I}} \mathrel{\circ\!\circ} \mathsf{A}^{\mathcal{J}} \quad \text{and} \quad \Phi^{\mathcal{I}} \mathrel{\circ\!\circ} \Phi^{\mathcal{J}} \overset{?}{\cancel{\Rightarrow}} \mathcal{I} \mathrel{\circ\!\circ} \mathcal{J}$$

Counterexample:

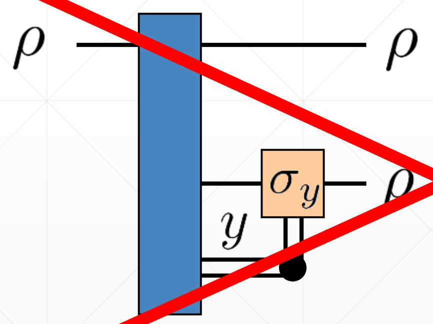
$$\mathcal{I}(\rho) = \rho$$

$$\mathcal{J}_y(\rho) = \frac{1}{4}\sigma_y \rho \sigma_y \qquad y = \{0, \ldots, 3\}$$

$$\mathsf{A}^{\mathcal{I}} = \{I\} \qquad \mathsf{A}^{\mathcal{J}} = \{\frac{1}{4} I\}_{y=0}^{3}$$

$$\Phi^{\mathcal{I}}(\rho) = \rho \qquad \Phi^{\mathcal{J}}(\rho) = \mathrm{tr}\rho\, \xi$$

But if the joint instrument exist then:



we could build universal cloner

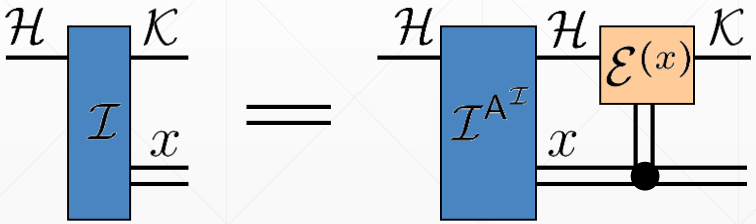
Compatibility of instruments – Attempted reduction 2

Lüders instrument for POVM $\mathsf{A} \in \mathcal{O}(\Omega, \mathcal{H})$

$$\mathcal{I}^{\mathsf{A}}_x(\varrho) = \sqrt{\mathsf{A}(x)}\,\varrho\,\sqrt{\mathsf{A}(x)}$$

Known fact:

Every instrument $\mathcal{I}$ can be realized as a Lüders instrument of its induced POVM $\mathsf{A}^{\mathcal{I}}$ concatenated with conditional channels that depend on the instrument outcome.

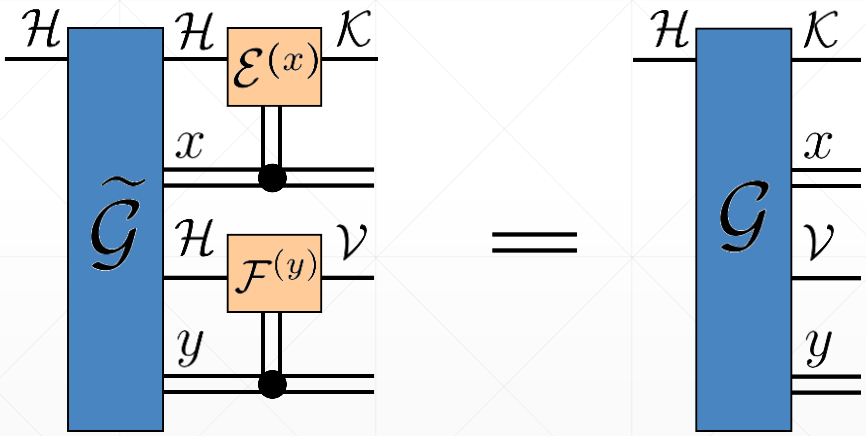


Hypotheses:

$$\mathcal{I} \circ\!\circ \mathcal{J} \overset{?}{\Leftrightarrow} \mathcal{I}^{\mathsf{A}^{\mathcal{I}}} \circ\!\circ \mathcal{J}^{\mathsf{A}^{\mathcal{J}}}$$

Proof for $\Leftarrow$

$\widetilde{\mathcal{G}}$ joint instrument for $\mathcal{I}^{\mathsf{A}^{\mathcal{I}}}$ and $\mathcal{J}^{\mathsf{A}^{\mathcal{J}}}$

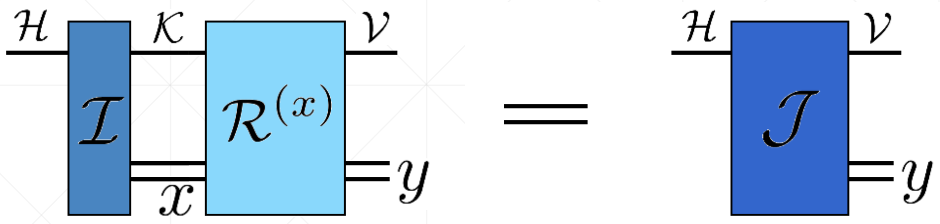


Counterexample to $\mathcal{I} \circ\!\circ \mathcal{J} \Rightarrow \mathcal{I}^{\mathsf{A}^{\mathcal{I}}} \circ\!\circ \mathcal{J}^{\mathsf{A}^{\mathcal{J}}}$

$$\mathcal{I}(\varrho) = \varrho \qquad \mathcal{J}(\varrho) = \mathrm{tr}[\varrho]\,\xi$$

Lüders inst. for both is identity channel, so contradicts no cloning

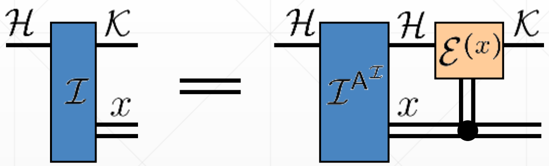
Postprocessing of instruments



$$\mathcal{J}_y(\varrho) = \sum_{x\in\Omega} \mathcal{R}_y^{(x)}(\mathcal{I}_x(\varrho))$$

Suppose that for $\mathcal{I}, \mathcal{J}$ instruments $\mathcal{R}^{(x)}$ exist so that the above Eq. hold, then **we denote it** $\boxed{\mathcal{J} \leq \mathcal{I}}$

Known fact
from previous slide

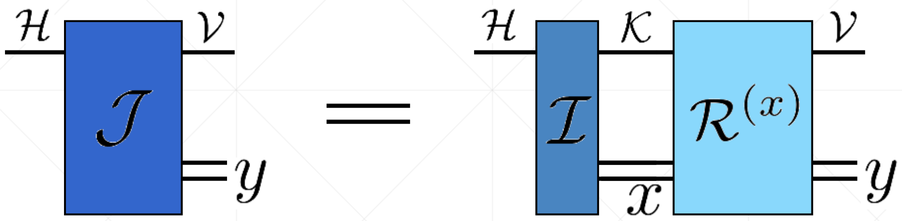


$\leftrightarrow$ $\mathcal{I} \leq \mathcal{I}^{\mathsf{A}^{\mathcal{I}}}$

Leevi Leppäjärvi, Michal Sedlák, Physical Review A 103, 022615 (2021)

Incompatibility vs. postprocessing of instruments

$\mathcal{J} \leq \mathcal{I}$ $\leftrightarrow$



$$\mathcal{J}_y(\varrho) = \sum_{x\in\Omega} \mathcal{R}_y^{(x)}(\mathcal{I}_x(\varrho))$$

Proposition 2:

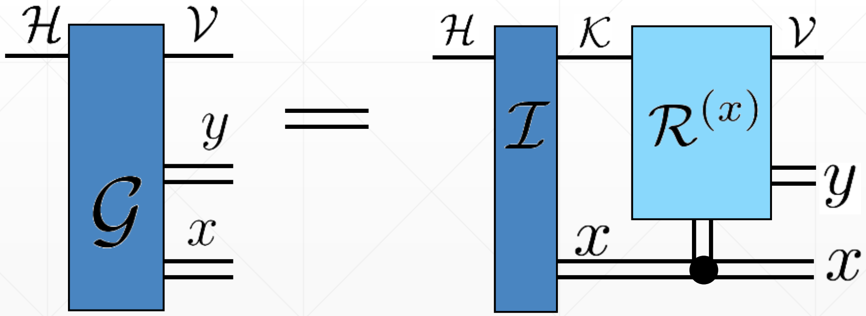
If $\mathcal{J} \leq \mathcal{I}$ then $\mathcal{J} \circ\!\!\circ\, \mathrm{A}^{\mathcal{I}}$ and by Prop. 1 also $\mathrm{A}^{\mathcal{J}} \circ\!\!\circ\, \mathrm{A}^{\mathcal{I}}$ and $\Phi^{\mathcal{J}} \circ\!\!\circ\, \mathrm{A}^{\mathcal{I}}$

Proof:

Set

$$\mathcal{G}_{(x,y)} = \mathcal{R}_y^{(x)} \circ \mathcal{I}_x$$

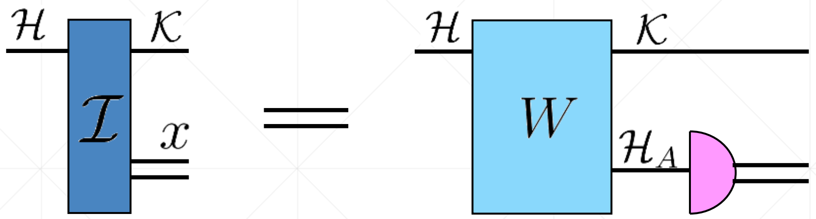
and verify



Leevi Leppäjärvi, Michal Sedlák, **Post-processing of quantum instruments**, Physical Review A 103, 022615 (2021)

Dilation of an instrument

Dilation of instrument $\mathcal{I} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$ **is a triple** $(\mathcal{H}_A, W, \mathsf{E})$**:**



Hilbert space $\mathcal{H}_A$

Isometry $W : \mathcal{H} \to \mathcal{H}_A \otimes \mathcal{K}$

POVM $\mathsf{E} \in \mathcal{O}(\Omega, \mathcal{H}_A)$

$$\mathcal{I}_x(\varrho) = \mathrm{tr}_{\mathcal{H}_A}\left[W \varrho W^* \left(\mathsf{E}(x) \otimes I_{\mathcal{K}}\right)\right]$$

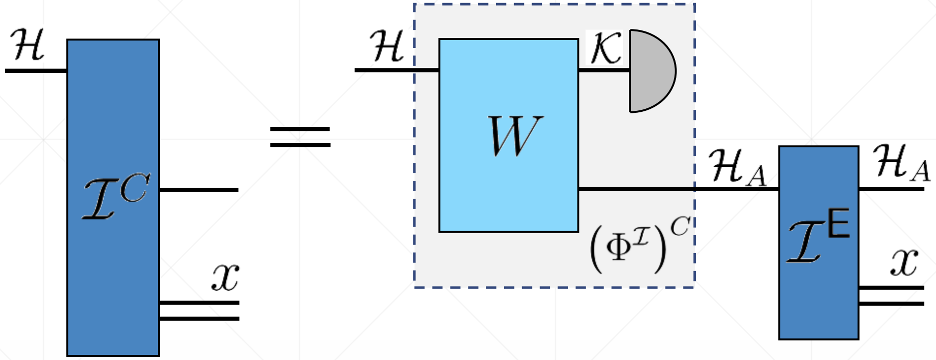
Remarks:

$(\mathcal{H}_A, W)$ must be a dilation of channel $\Phi^{\mathcal{I}}$

Dilation $(\mathcal{H}_A, W, \mathsf{E})$ is minimal if channel dilation $(\mathcal{H}_A, W)$ is minimal

Complementary instrument

We define $\mathcal{I}^C \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{H}_A)$ **relative to a dilation** $(\mathcal{H}_A, W, \mathsf{E})$ **of instrument** $\mathcal{I} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$ **as**



$$\mathcal{I}^C_x(\varrho) = \mathrm{tr}_{\mathcal{K}}\left[\left(\sqrt{\mathsf{E}(x)} \otimes I_{\mathcal{K}}\right) W \varrho W^* \left(\sqrt{\mathsf{E}(x)} \otimes I_{\mathcal{K}}\right)\right]$$

or equivalently

$$\mathcal{I}^C_x = \mathcal{I}^{\mathsf{E}}_x \circ \left(\Phi^{\mathcal{I}}\right)^C$$

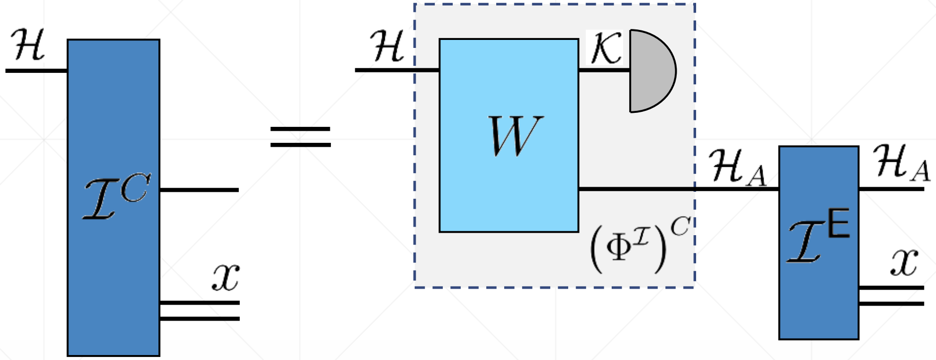
Remarks:

For minimal dilation $(\mathcal{H}_A, W, \mathsf{E})$ POVM $\mathsf{E} \in \mathcal{O}(\Omega, \mathcal{H}_A)$ is unique

Clearly, $\mathcal{I} \circ\!\!\circ\, \mathcal{I}^C$ by definition

Complementary instrument

We define $\mathcal{I}^C \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{H}_A)$ **relative to a dilation** $(\mathcal{H}_A, W, \mathsf{E})$ **of instrument** $\mathcal{I} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$ **as**



$$\mathcal{I}^C_x(\varrho) = \mathrm{tr}_{\mathcal{K}}\left[\left(\sqrt{\mathsf{E}(x)} \otimes I_{\mathcal{K}}\right) W \varrho W^* \left(\sqrt{\mathsf{E}(x)} \otimes I_{\mathcal{K}}\right)\right]$$

or equivalently

$$\mathcal{I}^C_x = \mathcal{I}^{\mathsf{E}}_x \circ \left(\Phi^{\mathcal{I}}\right)^C$$

Proposition 3:

All complementary instruments of instrument $\mathcal{I} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$ are postprocessing equivalent, i.e. let $(\mathcal{H}_A, W, \mathsf{E}), (\mathcal{H}'_A, W', \mathsf{E}')$ be two dilations of $\mathcal{I}$, then $\mathcal{I}^C \preceq \mathcal{I}^{C'}$ and $\mathcal{I}^{C'} \preceq \mathcal{I}^C$

Incompatibility of instruments – main result

Theorem 1:

Following three statements for instruments $\mathcal{I} \in \mathrm{Ins}(\Omega, \mathcal{H}, \mathcal{K})$ and $\mathcal{J} \in \mathrm{Ins}(\Lambda, \mathcal{H}, \mathcal{V})$ are equivalent:

i) $\mathcal{I} \,\mathrm{co}\, \mathcal{J}$

ii) $\mathcal{I} \leq \mathcal{J}^C$ for any complementary instrument $\mathcal{J}^C$ of $\mathcal{J}$

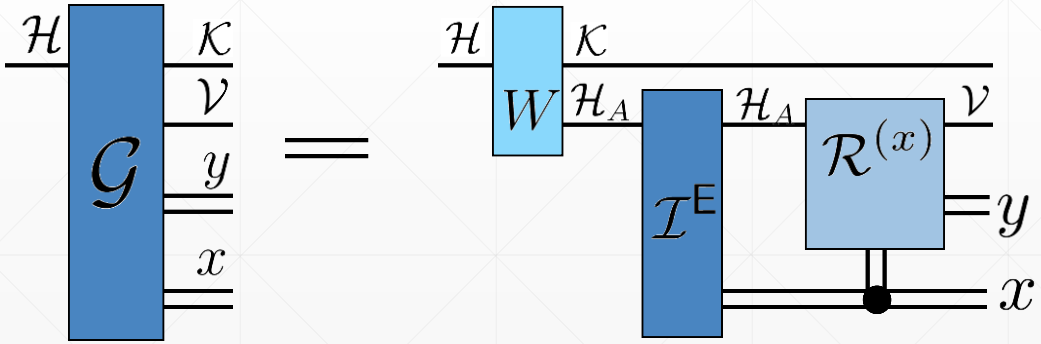
iii) $\mathcal{J} \leq \mathcal{I}^C$ for any complementary instrument $\mathcal{I}^C$ of $\mathcal{I}$

Proof: $\mathcal{J} \leq \mathcal{I}^C \Rightarrow \mathcal{I} \,\mathrm{co}\, \mathcal{J}$

Set

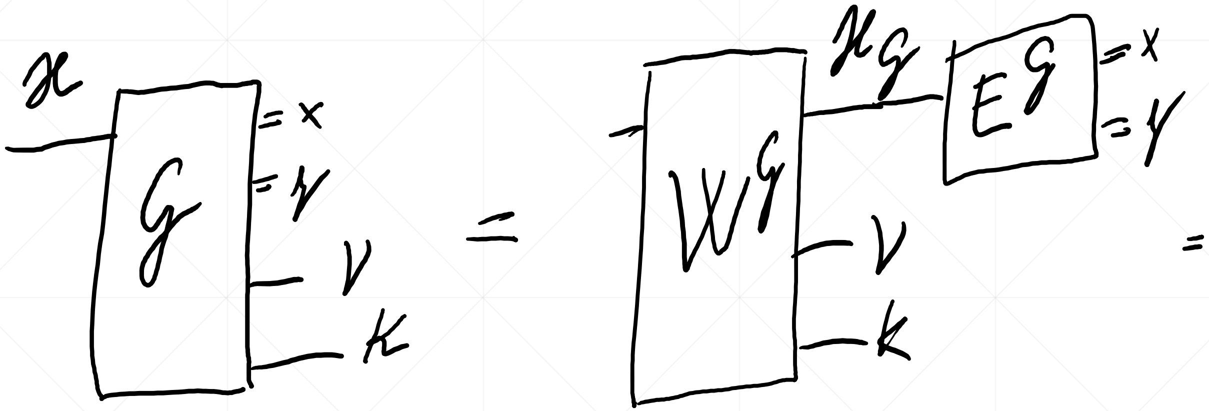
$$\mathcal{G}_{(x,y)}(\varrho) := (\mathcal{R}_y^{(x)} \otimes id_{\mathcal{K}})\left(\left(\sqrt{\mathsf{E}(x)} \otimes I_{\mathcal{K}}\right) W \varrho W^* \left(\sqrt{\mathsf{E}(x)} \otimes I_{\mathcal{K}}\right)\right)$$

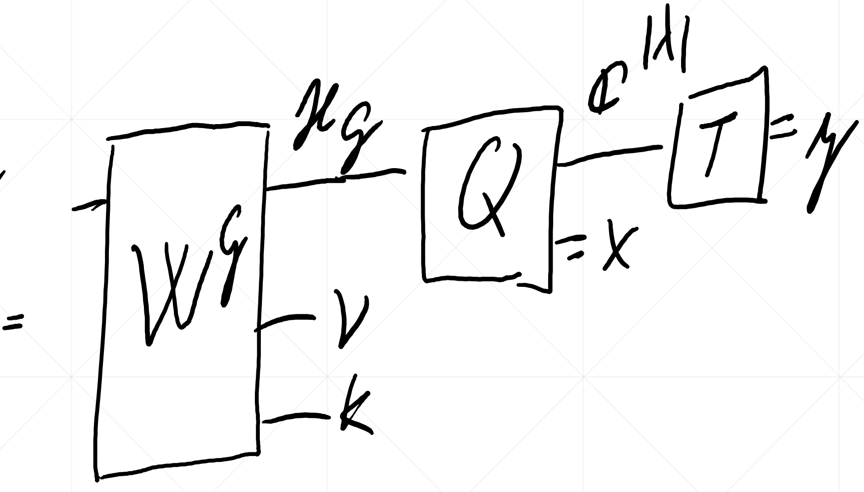
and verify

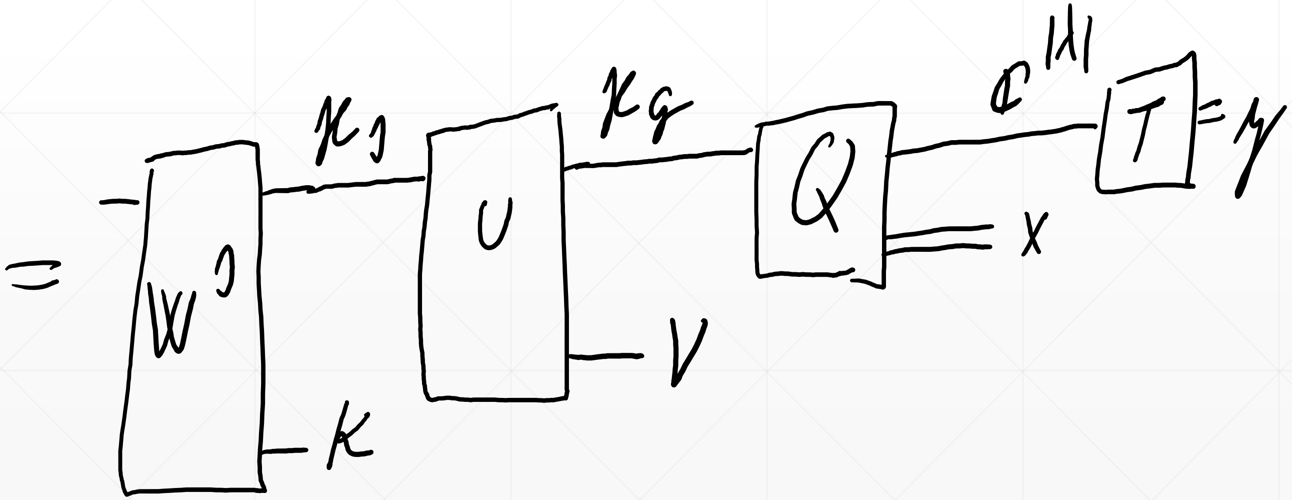


Incompatibility of instruments – main result

Sketch of proof: $\mathcal{I} \infty \mathcal{J} \Rightarrow \mathcal{J} \leq \mathcal{I}^C$

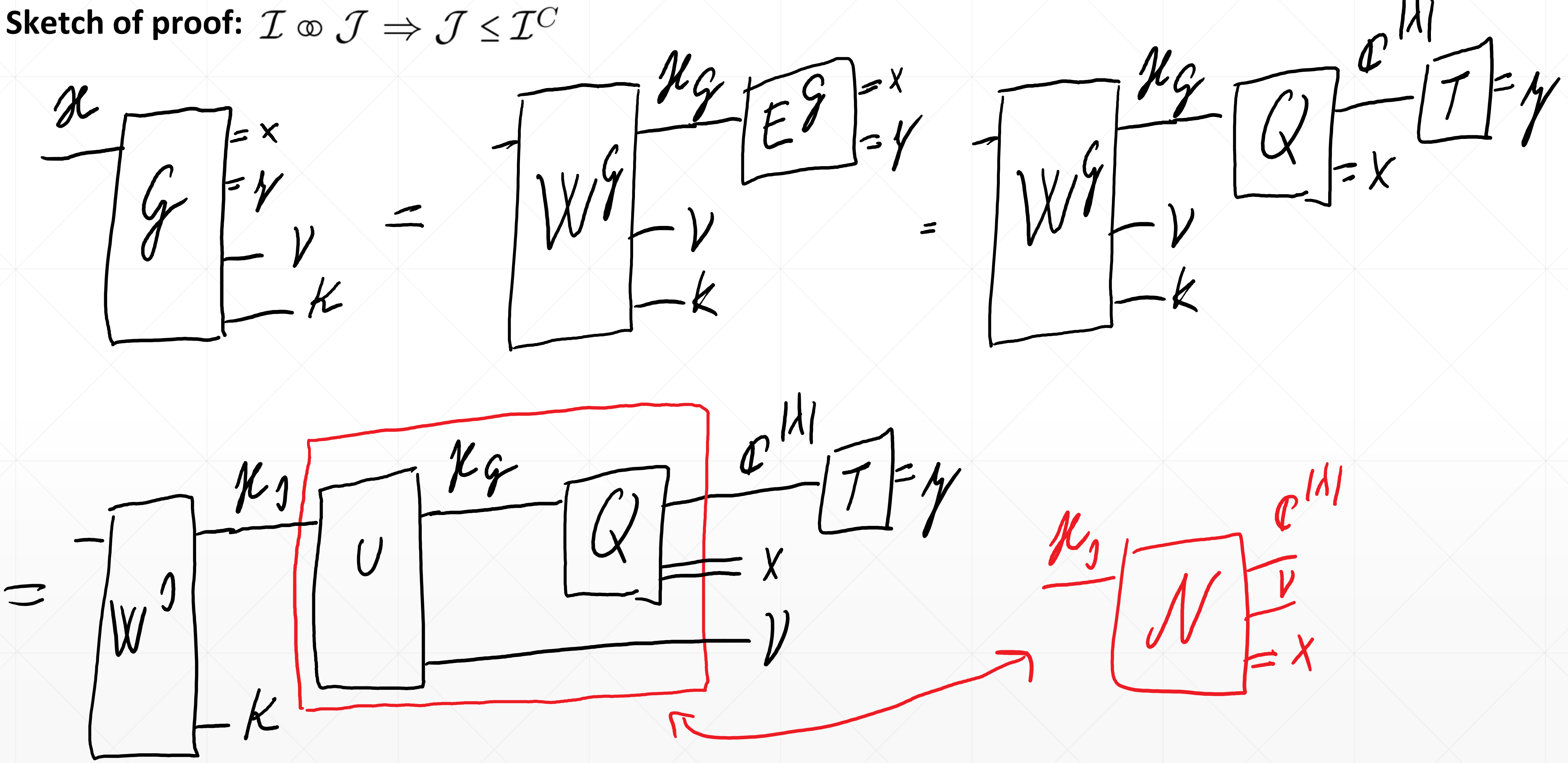






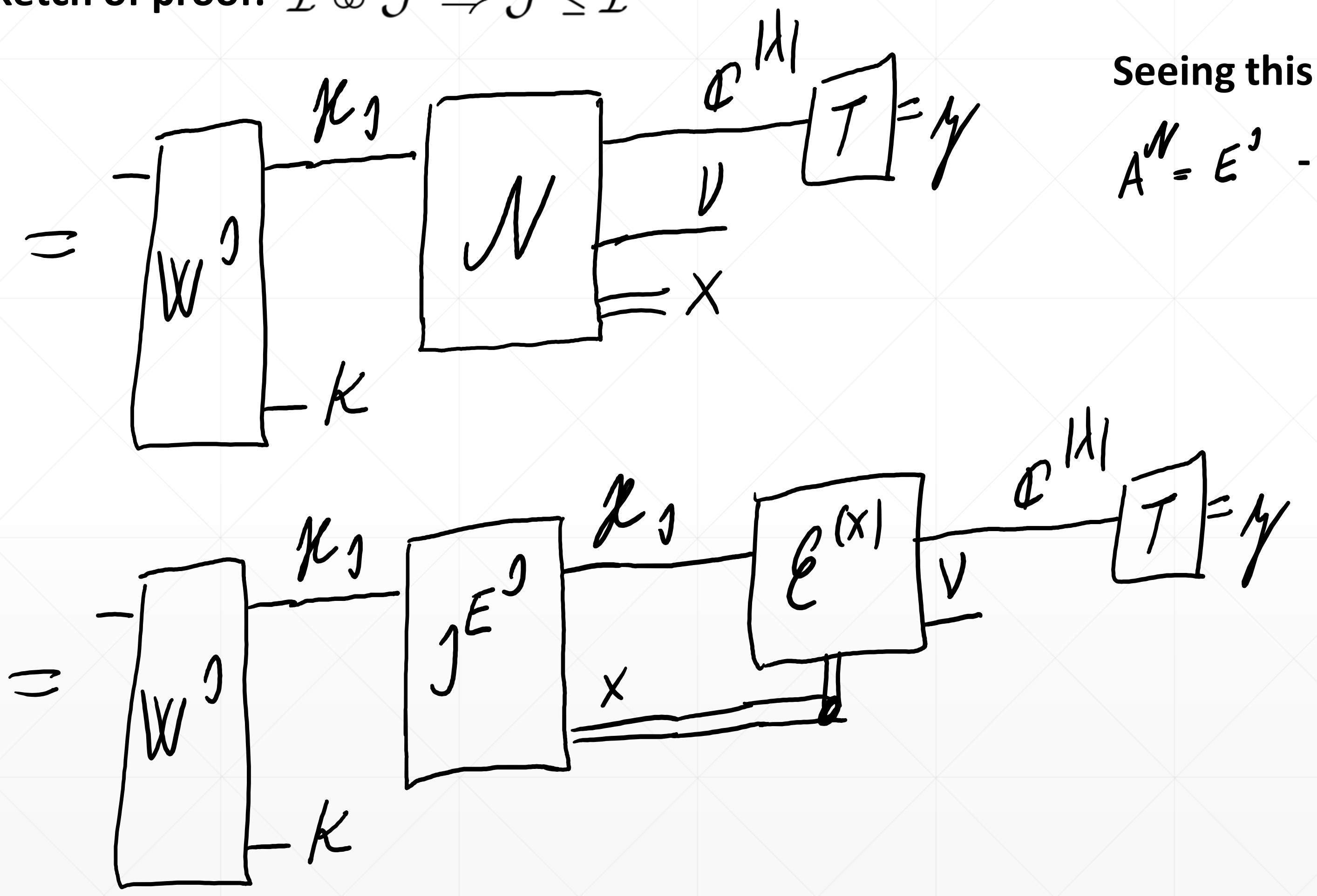
Incompatibility of instruments – main result

Sketch of proof: $\mathcal{I} \text{ co } \mathcal{J} \Rightarrow \mathcal{J} \leq \mathcal{I}^C$



Incompatibility of instruments – main result

Sketch of proof: $\mathcal{I} \circledcirc \mathcal{J} \Rightarrow \mathcal{J} \leq \mathcal{I}^C$

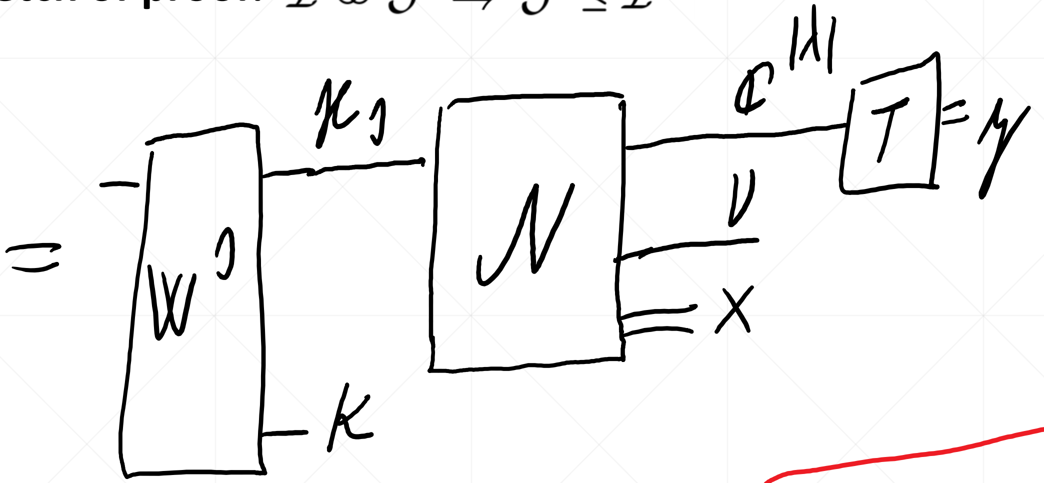


Seeing this as dilation of $\mathcal{I}$ **implies**

$A^{\mathcal{N}} = E^{\mathcal{J}}$ **- unique POVM from minimal dilation** $(\mathcal{H}_{\mathcal{I}}, W^{\mathcal{I}}, \mathsf{E}^{\mathcal{I}})$

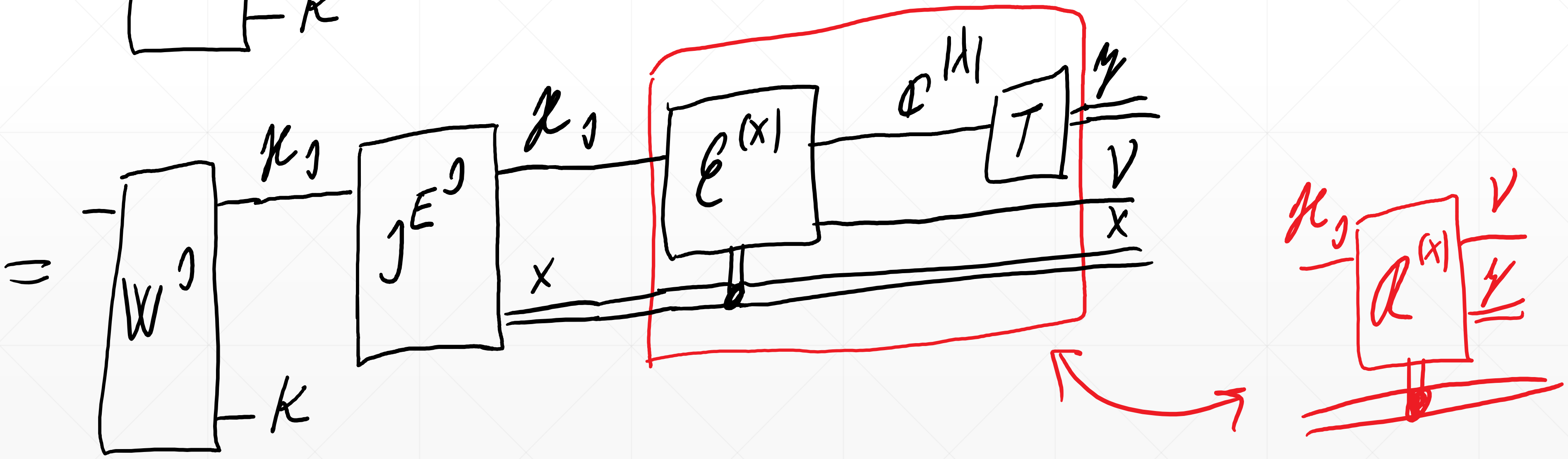
Incompatibility of instruments – main result

Sketch of proof: $\mathcal{I} \,\text{∞}\, \mathcal{J} \Rightarrow \mathcal{J} \leq \mathcal{I}^C$



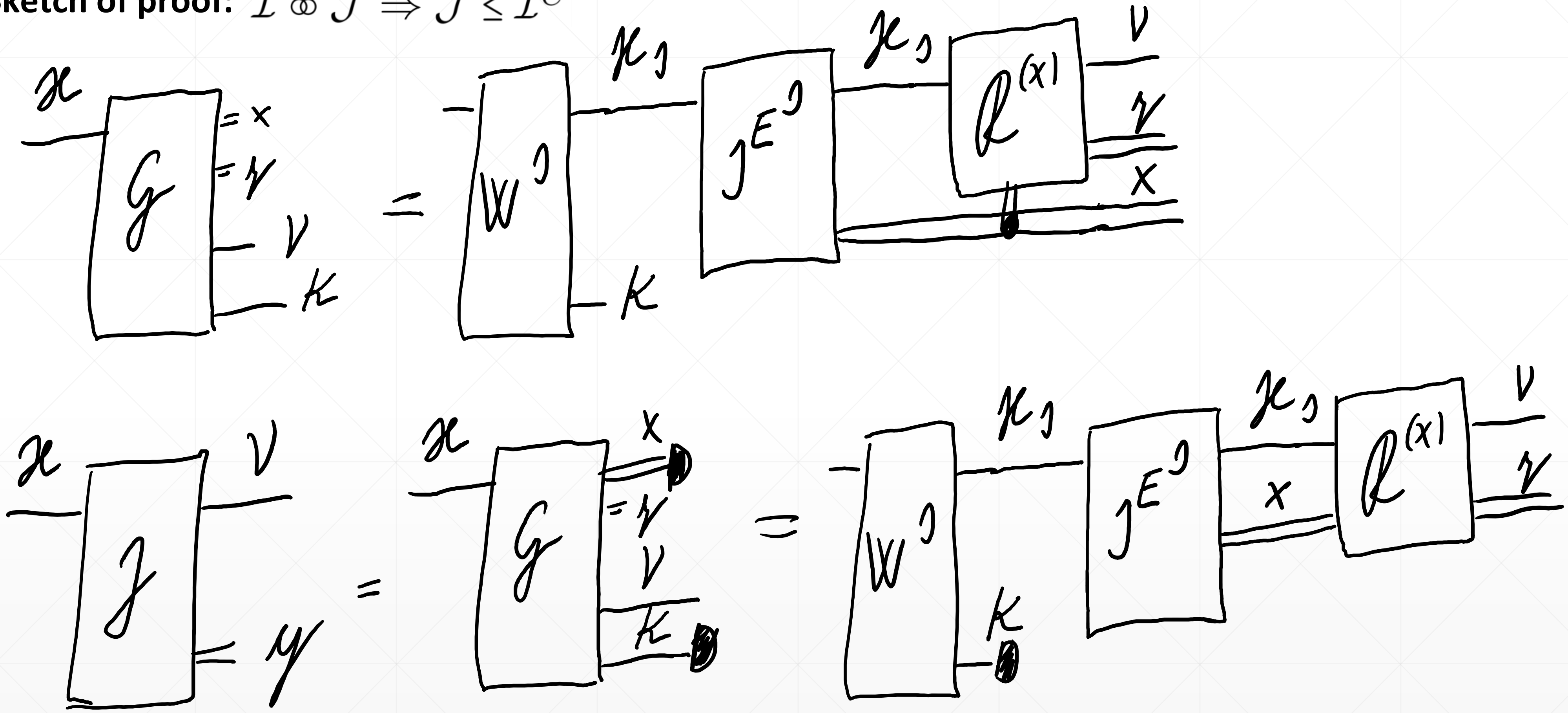
Seeing this as dilation of $\mathcal{I}$ implies

$A^{\mathcal{N}} = E^{\mathcal{J}}$ - unique POVM from minimal dilation $(\mathcal{H}_\mathcal{I}, W^\mathcal{I}, \mathrm{E}^\mathcal{I})$



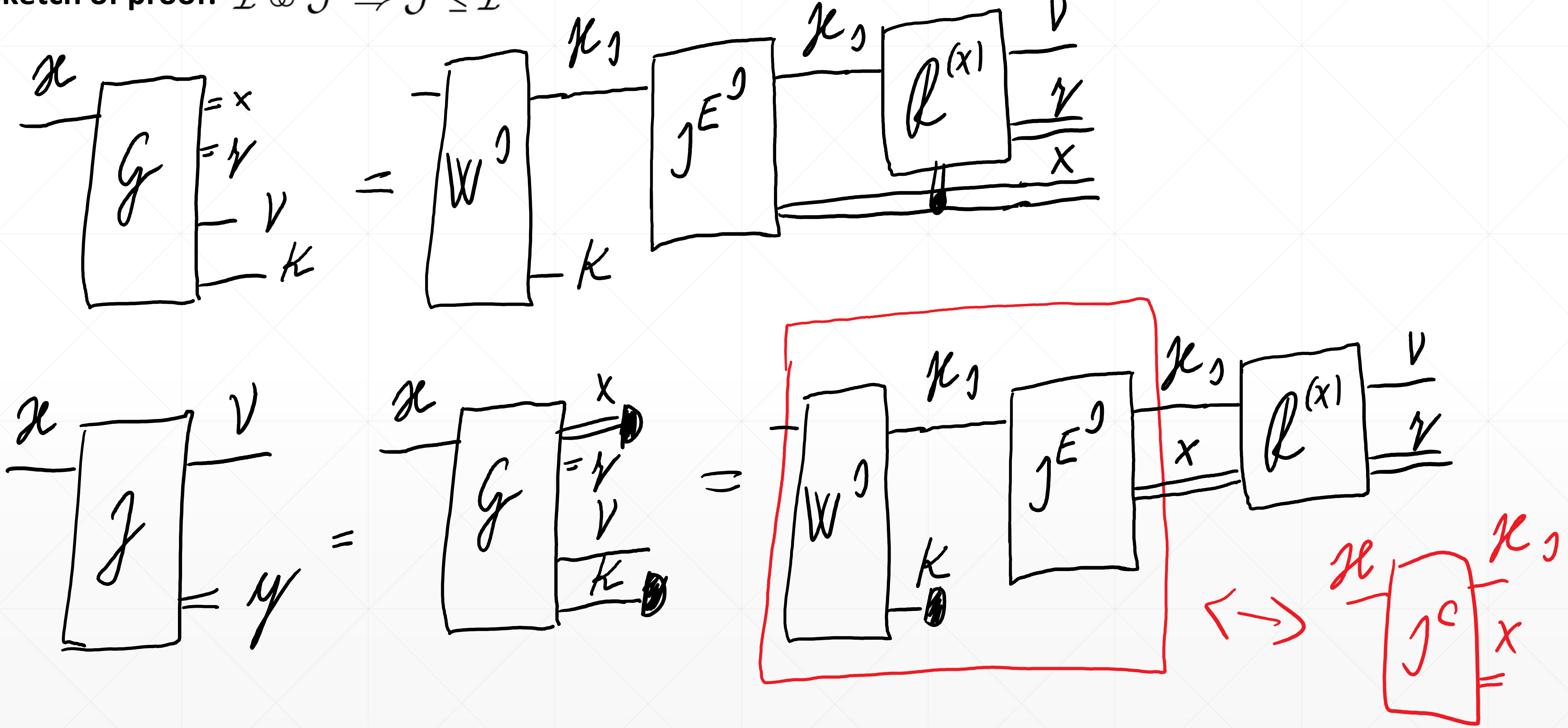
Incompatibility of instruments – main result

Sketch of proof: $\mathcal{I} \infty \mathcal{J} \Rightarrow \mathcal{J} \leq \mathcal{I}^C$



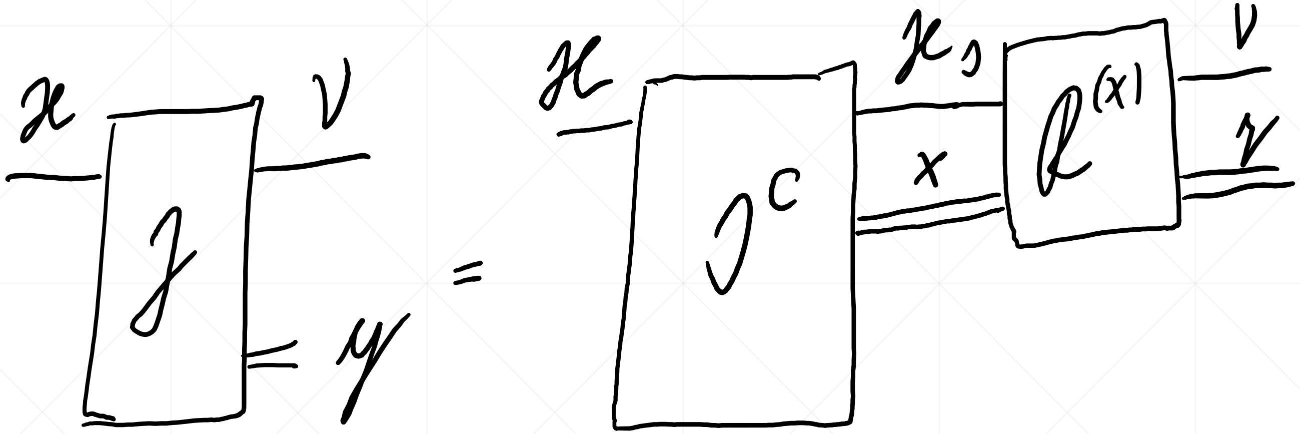
Incompatibility of instruments – main result

Sketch of proof: $\mathcal{I} \infty \mathcal{J} \Rightarrow \mathcal{J} \leq \mathcal{I}^C$



Incompatibility of instruments – main result

Sketch of proof: $\mathcal{I} \not\!\!\circ\, \mathcal{J} \Rightarrow \mathcal{J} \leq \mathcal{I}^C$



which concludes the proof!

Incompatibility of instruments

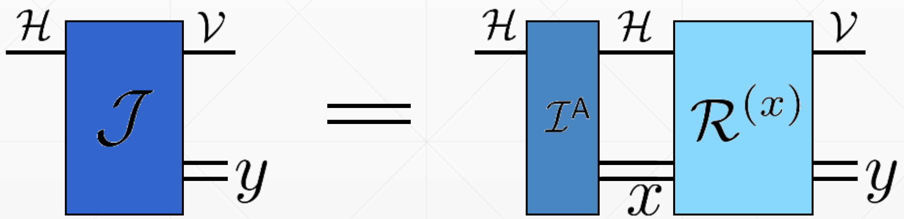
If a POVM $\mathsf{A} \in \mathcal{O}(\Omega, \mathcal{H})$ is considered as an instrument with one dimensional output space then its complementary instrument is Lüders instrument $\mathcal{I}^{\mathsf{A}} \in \mathrm{Ins}(\Omega, \mathcal{H})$

Corrolary of Theorem 1:

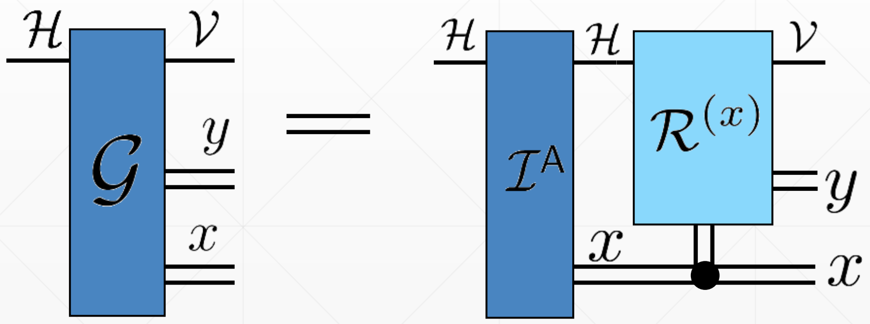
Let $\mathsf{A} \in \mathcal{O}(\Omega, \mathcal{H})$ and $\mathcal{J} \in \mathrm{Ins}(\Lambda, \mathcal{H}, \mathcal{V})$ then

$$\mathsf{A} \circ\!\!\circ\, \mathcal{J} \Leftrightarrow \mathcal{J} \leq \mathcal{I}^{A}$$

Thus,



and



Summary & Outlook

- **Presented results = to appear on ArXiv soon**
- **Basic relations for incompatibility of instruments found**
- **Complementary instruments introduced and their equivalences characterized**
- **Compatibility of instruments** shown to be equivalent to **postprocessing of complementary instruments**
- **further results: Non-disturbance for instruments, particular classes of incompatible instruments**

Open questions & future work:

- Compatibility of binary qubit instruments
- Characterize Complementary instruments to indecomposable instrument

Thanks for your attention.