Quantum Betting Tasks

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28th July 2022

PRX Quantum (2022)

University of BRISTOL

THE ROYAL SOCIETY

CIFAR

Outline

1. Quantum state discrimination & exclusion
2. Expected utility theory & risk averse gamblers
3. Quantum state betting with risk averse gamblers
4. Summary & Conclusions

Part I

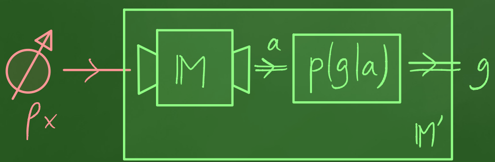
Quantum State Discrimination & Exclusion

Quantum state discrimination & exclusion

Discrimination

- Ensemble $\mathcal{E} = \{ p(x), \rho_x \}$
- Goal: Correctly identify state
- Resource: Fixed measurement $\mathbb{M} = \{ M_a \}$

  POVM: $M_a \geq 0 \quad \sum_a M_a = \mathbb{1}$
- Strategy:



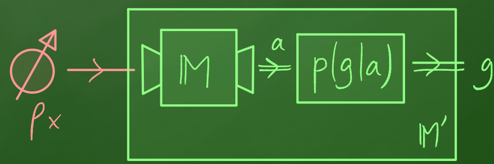
$g = x$?

- F.o.M. $P_{succ}(\mathcal{E}, \mathbb{M}) = \max\limits_{\mathbb{M}' < \mathbb{M}} \sum\limits_x p(x) \, \mathrm{tr}\left[M'_{g=x} \rho_x \right]$

Exclusion

- Ensemble $\mathcal{E} = \{ p(x), \rho_x \}$
- Goal: Correctly exclude state
- Resource: Fixed measurement $\mathbb{M} = \{ M_a \}$

  POVM: $M_a \geq 0 \quad \sum_a M_a = \mathbb{1}$
- Strategy:



$g \neq x$?

- F.o.M. $P_{err}(\mathcal{E}, \mathbb{M}) = \min\limits_{\mathbb{M}' < \mathbb{M}} \sum\limits_x p(x) \, \mathrm{tr}\left[M'_{g=x} \rho_x \right]$

Quantification of usefulness in discrimination & exclusion

Q: How useful is a given measurement for state discrimination & exclusion?

Discrimination

$$\max_{\mathcal{E}} \frac{P_{guess}(\mathcal{E}, \mathbb{M})}{\max_x p(x)} = 1 + R(\mathbb{M})$$

- $R(\mathbb{M})$ – Generalised Robustness of Measurement Informativeness

$$R(\mathbb{M}) = \min r$$
$$\text{s.t. } \frac{M_a + r N_a}{1+r} = q(a)\mathbb{1} \quad \forall a$$
$$N_a \geq 0, \quad \sum_a N_a = \mathbb{1}$$

Exclusion

$$\min_{\mathcal{E}} \frac{P_{err}(\mathcal{E}, \mathbb{M})}{\min_x p(x)} = 1 - W(\mathbb{M})$$

- $W(\mathbb{M})$ = Weight of Measurement Informativeness

$$W(\mathbb{M}) = \min s$$
$$\text{s.t. } s N_a + (1-s) q(a)\mathbb{1} = M_a \quad \forall a$$
$$N_a \geq 0, \quad \sum_a N_a = \mathbb{1}$$

Connection to Renyi entropies?

- Above two results can be related to extremes of Renyi entropies

$$H_\alpha(X) = \frac{1}{1-\alpha} \log \sum_x p(x)^\alpha$$

$$H_\infty(X) = -\log \max_x p(x)$$

$$H_{-\infty}(X) = -\log \min_x p(x)$$

$$H_\alpha(X|G) = \frac{\alpha}{1-\alpha} \log \sum_g p(g) \left(\sum_x p(x|g)^\alpha \right)^{1/\alpha}$$

$$H_\infty(X|G) = -\log \sum_g p(g) \max_x p(x|g)$$

$$H_{-\infty}(X|G) = -\log \sum_g p(g) \min_x p(x|g)$$

- $$\log \max_{\mathcal{E}} \frac{P_{succ}(\mathcal{E}, \mathbb{M})}{\max_x p(x)} = \max_{\mathcal{E}} \underbrace{H_\infty(X) - H_\infty(X|G)}_{I_\infty(X:G)}$$

w/ $p(x,g) = p(x) \operatorname{tr}[M_g \rho_x]$

- $$\log \min_{\mathcal{E}} \frac{P_{err}(\mathcal{E}, \mathbb{M})}{\min_x p(x)} = \min_{\mathcal{E}} \underbrace{H_{-\infty}(X|G) - H_{-\infty}(X)}_{I_{-\infty}(X:G)}$$

Q: Is there a continuous family of tasks that

(i) connect discrimination & exclusion ?

(ii) where the usefulness is quantified by Renyi-information ?

Yes

Expected Utility Theory
& Risk Averse Gamblers

(Quantum state) Betting

- Consider betting on an ensemble of quantum states $\mathcal{E} = \{p(x), \rho_x\}$
  - Bookmaker offers odds $o(x)$-for-1 on state ρ_x
    - Pays out £$o(x)$ on £1 bet if ρ_x is state.
  - Gambler will bet proportion $b(x)$ of their wealth on state ρ_x
  - Expected wealth at end of bet is $\mathbb{E}[W] = \sum_x p(x)\, b(x)\, o(x)$
- Want to take into account risk aversion of gamblers

Risk aversion

- Consider state betting on two states: $p(0) = \frac{1}{2}, \quad p_0$

  $p(1) = \frac{1}{2}, \quad p_1$

- Bookmaker offer odds: $o(0) = £100$

  $o(1) = £0$

- Gambler bets all money on p_0, $b(0) = 1 \qquad b(1) = 0$

- Expected wealth after bet $\mathbb{E}[W] = \frac{1}{2} \times 1 \times £100 = £50$

Question: If offered £40 would you walk away from bet? £30? £20?

- Rational to have a preference for certain wealth over uncertain wealth

- The smallest amount of money a gambler would accept to walk away is a measure of their risk aversion

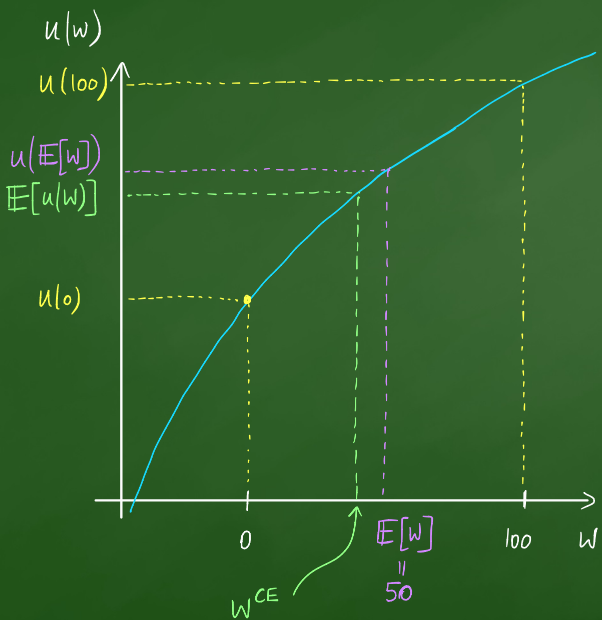
lower figure = more risk averse

higher figure = less risk averse

Expected Utility Theory

- Model behaviour of gamblers by introducing concept of utility = happiness / satisfaction
  - Renormalise value of wealth to account for risk tendencies

- $u(w)$ – utility function



- For risk averse gamblers, utility grows slower than wealth.

- Certain-equivalent wealth W^{CE} is the amount of wealth that has the same utility / happiness / satisfaction as expected utility of wealth.

$$\boxed{W^{CE} = u^{-1}\left(\mathbb{E}[u(W)]\right)}$$

- In expected utility theory agents aim to maximise expected utility (rather than expected wealth)
  - this is equivalent to maximising certainty-equivalent wealth.

Risk averse gamblers

- Curvature of utility curve determines level of risk aversion
  - Coefficient of Relative Risk Aversion $R(w) = -w \dfrac{\frac{d^2u}{dw^2}}{\frac{du}{dw}}$
    - invariant under $u(w) \to \alpha u(w) + \beta$
    - dimensionless

- A gambler that has constant relative risk aversion R has utility function satisfying $R = -w \dfrac{\frac{d^2u}{dw^2}}{\frac{du}{dw}}$

Solution: isoelastic utility function

$$\boxed{u_R^I(w) = \begin{cases} \dfrac{w^{1-R} - 1}{1-R} & R \neq 1 \\ \ln w & R = 1 \end{cases}}$$

- $R = 0$: $u_0^I(w) = w$ risk neutral
- $R > 0$: $u_R^I(w)$ grows slower than w

Quantum state betting with risk averse gamblers

(Quantum state) betting with risk-averse gamblers

- Consider betting on an ensemble of quantum states $\mathcal{E} = \{p(x), \rho_x\}$
  - Bookmaker offers odds $o(x)$-for-1 on state ρ_x
  - Gambler will bet proportion $b(x)$ of their wealth on state ρ_x
  - Expected utility of gambler w/ constant relative risk aversion

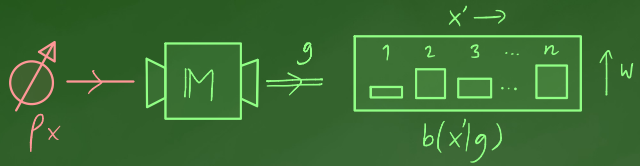
$$\mathbb{E}[u_R^I(w)] = \begin{cases} \sum_x p(x) \left(\frac{[b(x)\,o(x)]^{1-R} - 1}{1-R} \right) & R \neq 1 \\ \sum_x p(x) \ln[b(x)\,o(x)] & R = 1 \end{cases}$$

  - Certainty-equivalent wealth

$$w_R^{ICE}(\mathcal{E}, o(x), b(x)) = (u_R^I)^{-1}\left(\mathbb{E}[u_R^I(w)]\right) = \begin{cases} \left(\sum_x p(x)[b(x)\,o(x)]^{1-R}\right)^{1/1-R} & R \neq 1 \\ e^{\sum_x p(x) \ln[b(x)\,o(x)]} & R = 1 \end{cases}$$

Quantum state betting with risk averse gamblers

- Ensemble: $\mathcal{E} = \{ p(x), \rho_x \}$
- Odds: $o(x)$ - for - 1
- Resource: Measurement $\mathbb{M} = \{ M_g \}$
- Strategy:



– place conditional bet $b(x'|g)$ depending on side information g generated by measurement

- Figure of Merit: maximised certainty equivalent wealth

$$
w_R^{ICE}(\mathcal{E}, o(x), \mathbb{M}) = \max_{b(x|g)} \left(\sum_{xg} p(x) \operatorname{tr}\left[M_g \rho_x \right] \left(b(x|g)\, o(x) \right)^{1-R} \right)^{\frac{1}{1-R}}
$$

Recall: this is the amount of money risk averse gambler would accept to walk away from bet.

Quantification of usefulness in quantum state betting

Q: How useful is a given measurement in quantum state betting?

→ How much can a measurement increase certainty-equivalent wealth?

$r(x) \propto \frac{1}{o(x)}$ (normalised prob. distribution)

- $$\boxed{\log \frac{W_R^{ICE}(\mathcal{E}, o(x), \mathbb{M})}{\max\limits_{b(x)} W_R^{ICE}(\mathcal{E}, o(x), b(x))} = \underbrace{D_{1/R}\left(p(x|g) \,\|\, r(x) \,|\, p(g)\right)} - \underbrace{D_{1/R}\left(p(x) \,\|\, r(x)\right)}}$$

Conditional Renyi divergence of order $\alpha = 1/R$

$$= \frac{\frac{1}{R}}{\frac{1}{R}-1} \log \sum_g p(g) \left[\sum_x p(x|g)^{\frac{1}{R}} r(x)^{1-\frac{1}{R}} \right]^R$$

– measure of how 'far' the bookmakers odds are from the updated probabilities given the side information.

Renyi divergence of order $\alpha = 1/R$

$$= \frac{1}{\frac{1}{R}-1} \log \sum_x p(x)^{\frac{1}{R}} r(x)^{1-\frac{1}{R}}$$

– measure of how 'far' the bookmakers odds are from the true probabilities

- Renyi parameter $\alpha = \frac{1}{R}$ determines risk tendency of gambler

Constant odds

- In case of constant odds things simplify
  - Bookmaker offers $o(x) = c =$ constant odds
    ↳ $r(x) = \frac{1}{n}$ uniform distribution

rewarded uniformly for guessing state correctly.

$$D_{1/R}\left(p(x|g) \,\|\, \tfrac{1}{n} \,\middle|\, p(g)\right) = \frac{\frac{1}{R}}{\frac{1}{R}-1} \log \sum_g p(g) \left[\sum_x p(x|g)^{\frac{1}{R}}\right]^R + \log n$$

$$= H_{1/R}(X|G) + \log n$$

$$D_{1/R}\left(p(x) \,\|\, \tfrac{1}{n}\right) = \frac{1}{\frac{1}{R}-1} \log \sum_x p(x)^{\frac{1}{R}} + \log n$$

$$= H_{1/R}(X) + \log n$$

$$\boxed{\log \frac{W_R^{ICE}(\mathcal{E}, c, M)}{\max_{b(x)} W_R^{ICE}(\mathcal{E}, c, b(x))} = H_{1/R}(X|G) - H_{1/R}(X) = I_{1/R}(X:G)}$$

Renyi - Arimoto α-mutual information

Risk-neutral gamblers

- $R = 0$ corresponds to risk neutral gambler

$$W_0^{ICE}(\mathcal{E}, o(x), \mathbb{M}) = \max_{b(x|g)} \sum_{xg} p(x)\, \mathrm{tr}[M_g \rho_x]\, b(x|g)\, o(x)$$

$$W_0^{ICE}(\mathcal{E}, o(x), b(x)) = \sum_x p(x)\, b(x)\, o(x)$$

- with constant odds $o(x) = c = \text{constant}$

$$W_0^{ICE}(\mathcal{E}, c, \mathbb{M}) = c\, p_{succ}(\mathcal{E}, \mathbb{M})$$

Success prob. in q. state discrimination using $\mathbb{M}$

$$\max_{b(x)} W_0^{ICE}(\mathcal{E}, c, b(x)) = c \max_x p(x)$$

Best classical guess in q. state discrimination

$$\longrightarrow \quad \frac{W_0^{ICE}(\mathcal{E}, c, \mathbb{M})}{\max_{b(x)} W_0^{ICE}(\mathcal{E}, c, b(x))} = \frac{p_{succ}(\mathcal{E}, \mathbb{M})}{\max_x p(x)}$$

Recovers discrimination in limit of risk neutral players with constant odds

unit-risk gamblers

- $R = 1$ is a special gambler $u_1^I(w) = \ln w$
  - Equivalent to situation where gambler wants to maximise growth rate of wealth.

$$W_1^{ICE}(\mathcal{E}, o(x), \mathbb{M}) = \exp\left[\sum_{xg} p(x)\, \mathrm{tr}\left[M_g\, \rho_x\right] \ln\left(b(x|g)\, o(x)\right)\right]$$

$$W_1^{ICE}(\mathcal{E}, o(x), b(x)) = \exp\left[\sum_{x} p(x) \ln\left(b(x)\, o(x)\right)\right]$$

$$\log \frac{W_1^{ICE}(\mathcal{E}, o(x), \mathbb{M})}{\max_{b(x)} W_1^{ICE}(\mathcal{E}, o(x), b(x))} = D\left(p(x,g) \,\|\, r(x)\, p(g)\right) - D\left(p(x) \,\|\, r(x)\right)$$

$$= I(X : G)$$

Called 'Golden formula'

- Recovers result that increase in growth rate of wealth equals mutual information w/ side information.

Negative Renyi parameters & 'loss games'

- So far have only considered $\alpha \geq 0$ because $R \geq 0$
- Can extend to negative α by considering loss games

  - As before ensemble of quantum states $\mathcal{E} = \{p(x), \rho_x\}$
  - odds now represent losses : $o(x)$ - for - 1 with $o(x) < 0$

    Gambler must pay out $\pm o(x)$ when unit stake is placed on state ρ_x

  - Gambler will bet proportion $b(x)$ of their wealth on state ρ_x
  - Risk averse gambler will accept fixed loss $w^{CE} < \mathbb{E}[W] < 0$ to walk away from bet

$$\longrightarrow \quad u_R^I(w) = \begin{cases} -\dfrac{|w|^{1-R} - 1}{1-R} & R \neq 1 \\ -\ln|w| & R = 1 \end{cases} \qquad w < 0$$

$$\longrightarrow \quad w_R^{ICE}(\mathcal{E}, o(x), \mathbb{M}) = \max_{b(x|g)} \left(\sum_{xg} p(x) \operatorname{tr}[M_g \rho_x] \left(b(x|g)\, o(x) \right)^{1-R} \right)^{\frac{1}{1-R}} < 0$$

$\longrightarrow$ Risk averse now corresponds to $R < 0$

Same as for 'gain' games

Quantification of usefulness in quantum state betting

- when $o(x) < 0$ amount by which gambler can minimise certainty equivalent loss

$$\boxed{\bullet \quad \log \frac{W_R^{ICE}(\mathcal{E}, o(x), \mathbb{M})}{\max_{b(x)} W_R^{ICE}(\mathcal{E}, o(x), b(x))} = D_{1/R}\big(p(x) \,\|\, r(x)\big) - D_{1/R}\big(p(x|g) \,\|\, r(x) \,|\, p(g)\big)}$$

$$r(x) \propto \frac{1}{|o(x)|} \geq 0$$

- $R \to 0$ from below limit of risk neutral gambler if $o(x) = c < 0$

$$\frac{W_0^{ICE}(\mathcal{E}, c, \mathbb{M})}{\max_{b(x)} W_0^{ICE}(\mathcal{E}, c, b(x))} = \frac{P_{err}(\mathcal{E}, \mathbb{M})}{\min_x p(x)}$$

Summary & Conclusions

Summary & Conclusions

- Introduced quantum state betting with risk averse gamblers
- Shown that usefulness of a measurement in this task is quantified by Renyi-esque quantities
  - Renyi parameter interpreted as risk aversion of gambler.
- Generalises previous results on state discrimination & state exclusion
- (Didn't show you): Result hold for other betting tasks – channel & subchannel betting
  - Results are ultimately about usefulness of (classical) side information

Future work

- Explore more general gamblers – i.e. alternative utility functions
- Fully quantum betting tasks?
- More general investigation of utility theory & risk aversion in quantum information.

Thank you!