Correlation in Catalysts Enables Arbitrary Manipulation of Quantum Coherence

Ryuji Takagi

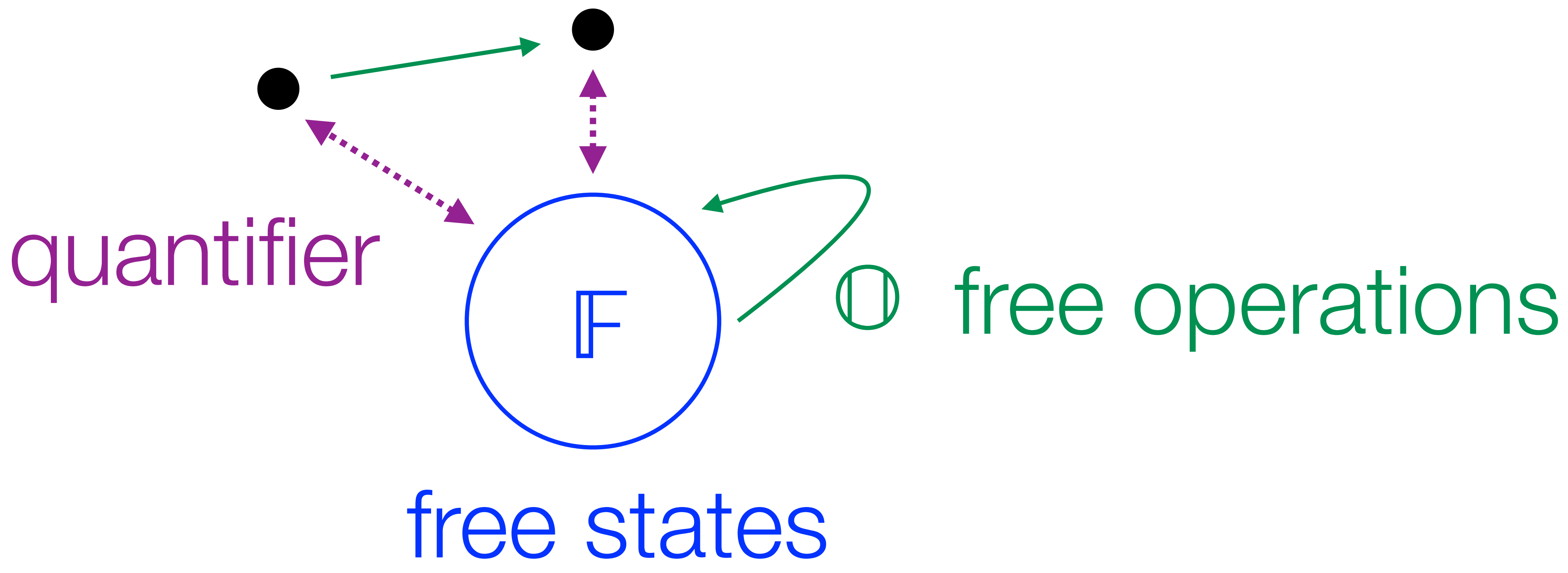
NTU Singapore

Joint work with Naoto Shiraishi (U. Tokyo)

Phys. Rev. Lett. 128, 240501 (2022)

Quantum Resource Theories

Framework to deal with **quantification** and **manipulation** of physical quantities that are considered “precious” under a given setting.



[Chitambar, Gour, RMP ’19]

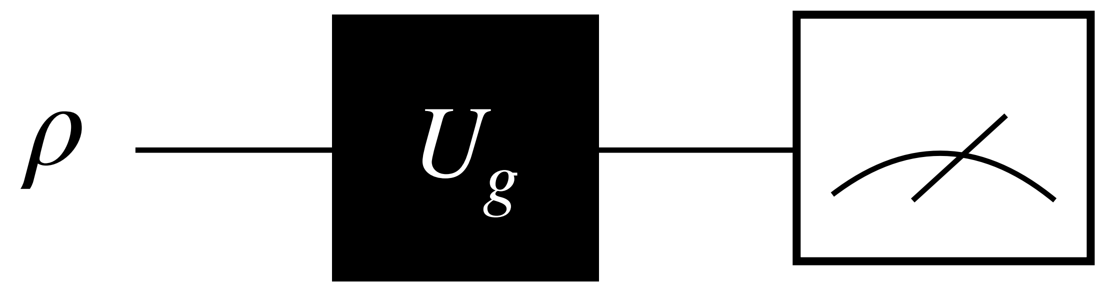
- Quantum entanglement [Horodecki et al. ’09]
- Fault-tolerant quantum computation [Veitch et al., ’14] [Howard, Campbell, ’17]
- Quantum non-Gaussianity [Genoni et al., ’08] [Takagi, Zhuang, ‘18]

Resource Theory of Asymmetry

Symmetry group G with representation U_g for $g \in G$.

A state ρ is *symmetric* if $U_g \rho U_g^\dagger = \rho, \forall g \in G$. Otherwise, ρ is *asymmetric*.

Operational resource for metrology



- Quantify the amount of symmetry violation
- Characterize the possible state transformation with operations that respect symmetry

Covariant operations: $U_g \mathscr{E}(\rho) U_g^\dagger = \mathscr{E}(U_g \rho U_g^\dagger), \forall g \in G, \forall \rho$

Resource theory of asymmetry $\mathbb{F}$: symmetric states and $\mathbb{O}$: Covariant operations

Quantum Coherence as Asymmetry

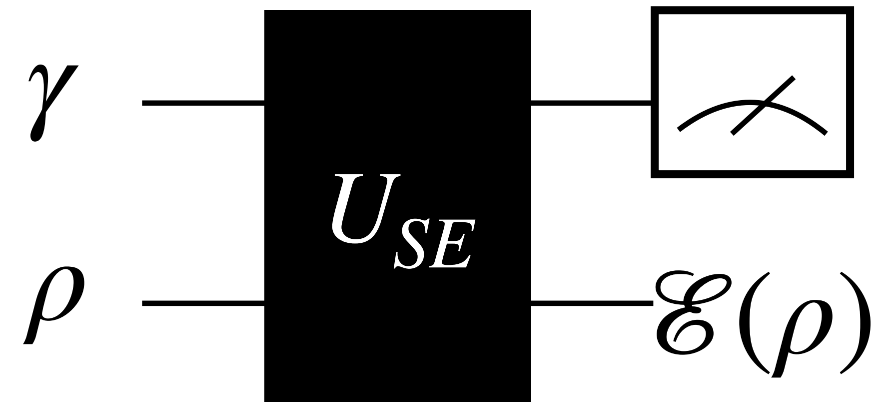
Consider a specific group $G = \mathrm{U}(1)$ with representation $U_t = e^{iHt}$. $H = \sum_i E_i |i\rangle\langle i|$

Asymmetric state has energetic coherence. e.g., $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- Resource for quantum clock
- "Quantum part" of quantum thermodynamics

Thermal Operations

$$\mathscr{E}(\rho) = \mathrm{Tr}_E\left(U_{SE}(\rho \otimes \gamma)\, U_{SE}^\dagger\right) \quad [U_{SE}, H_S + H_E] = 0 \quad \gamma = e^{-\beta H_E}/\mathrm{Tr}(e^{-\beta H})$$



- Thermal Operations are subclass of Covariant Operations with energy constraint
- Free energy can be decomposed into classical and quantum parts.

$$\beta[F(\rho) - F(\gamma)] = S(\rho\|\gamma) = \underbrace{S(\Pi(\rho)\|\gamma)}_{\text{classical}} + \underbrace{C(\rho)}_{\text{quantum}}$$

$\Pi(\rho)$: rephasing with energy eigenabsis

$C(\rho)$: relative entropy of coherence

Resource Manipulation with Catalysts

Ultimate transformation capability: One should consider using the help of *catalysts.*

Product catalysts

For a resource theory with free operations $\mathbb{O}$, $\rho \otimes \tau_C \underset{\mathbb{O}}{\rightarrow} \rho' \otimes \tau_C$ $\quad \tau_C$: catalyst

e.g.) Entanglement

There exists states ρ and ρ' such that $\rho \underset{\text{LOCC}}{\not\longrightarrow} \rho'$ but $\rho \otimes \tau_C \underset{\text{LOCC}}{\longrightarrow} \rho' \otimes \tau_C$

[Jonathan, Plenio, PRL ’99]

e.g.) Quantum thermodynamics (ρ, ρ': block-diagonal)

$\rho \underset{\text{Thermal}}{\longrightarrow} \rho'$: thermo-majorization [Horodecki, Oppeheim, Nat. Comm. ’13]

$\rho \otimes \tau_C \underset{\text{Thermal}}{\longrightarrow} \rho' \otimes \tau_C$: “second laws”. $F_\alpha(\rho) \geq F_\alpha(\rho'), \forall \alpha$ [Brandao et al., PNAS, ‘15]

Correlated Catalysts

$\rho \otimes \tau_C \to \rho'_{SC}$ such that $\mathrm{Tr}_C\, \rho'_{SC} = \rho'$ and $\mathrm{Tr}_S\, \rho'_{SC} = \tau_C$

Correlation between system and catalyst

e.g.) Quantum thermodynamics with correlated catalysts

$F(\rho) \geq F(\rho')$ with (1) thermal operations for block-diagonal ρ, ρ' [Müller, PRX, '18]

(2) Gibbs-preserving operations [Shiraishi, Sagawa, PRL, '21]

e.g.) Entanglement

Pure-to-pure transformation $\psi \xrightarrow[\mathrm{LOCC}]{} \phi$ with correlated catalysts.

$S_E(\psi) \geq S_E(\phi)$

$S_E(\psi) := -\mathrm{Tr}(\rho_A \log \rho_A),\ \rho_A := \mathrm{Tr}_B\, \psi$

[Kondra et al., PRL, '21]
[Lipka-Bartosik, Skrzypczyk, PRL, '21]

Coherence Manipulation with Catalysts

Covariant operations: $U_g \mathscr{E}(\rho) U_g^\dagger = \mathscr{E}(U_g \rho U_g^\dagger), \forall g \in G, \forall \rho$

Product catalysts

Pure catalysts do not help coherence transformation

If $\rho \otimes \psi_C \underset{\text{cov.}}{\longrightarrow} \rho' \otimes \psi_C$, then $\rho \underset{\text{cov.}}{\longrightarrow} \rho'$ [Marvian, Spekkens, NJP ’13] [Ding, Hu, Fan, PRA ’21]

Correlated catalysts

Coherence no-broadcasting: If ρ is incoherent, ρ' is also incoherent. [Marvian, Spekkens, PRL ’19] [Lostaglio, Muller, PRL ’19]

Other catalytic setting?

Marginal Catalysts

$$\rho \otimes \tau_{C^{(0)}} \cdots \otimes \tau_{C^{(K-1)}} \to \rho' \otimes \tau_{C^{(0)}\dots C^{(K-1)}}$$

$$\mathrm{Tr}_{\overline{C^{(j)}}}\left(\tau_{C^{(0)}\dots C^{(K-1)}}\right) = \tau_{C^{(j)}},\ \forall j$$

Correlation between multiple catalysts

Previously considered in quantum thermodynamics

For states ρ, ρ' that are block diagonal, ρ can be transformed to σ with a marginal-catalytic thermal operations with an arbitrary small error iff $F(\rho) \geq F(\rho')$

[Lostaglio et al., PRL ‘15]

Coherence

No general characterization except for qubit case. [Ding, Fu, Fan, PRA, ’21]

Arbitrary Manipulation is Possible

$$\rho \otimes \tau_{C^{(0)}} \cdots \otimes \tau_{C^{(K-1)}} \to \rho' \otimes \tau_{C^{(0)} \ldots C^{(K-1)}}$$

$$\mathrm{Tr}_{\overline{C^{(j)}}} \left(\tau_{C^{(0)} \ldots C^{(K-1)}} \right) = \tau_{C^{(j)}}, \ \forall j$$

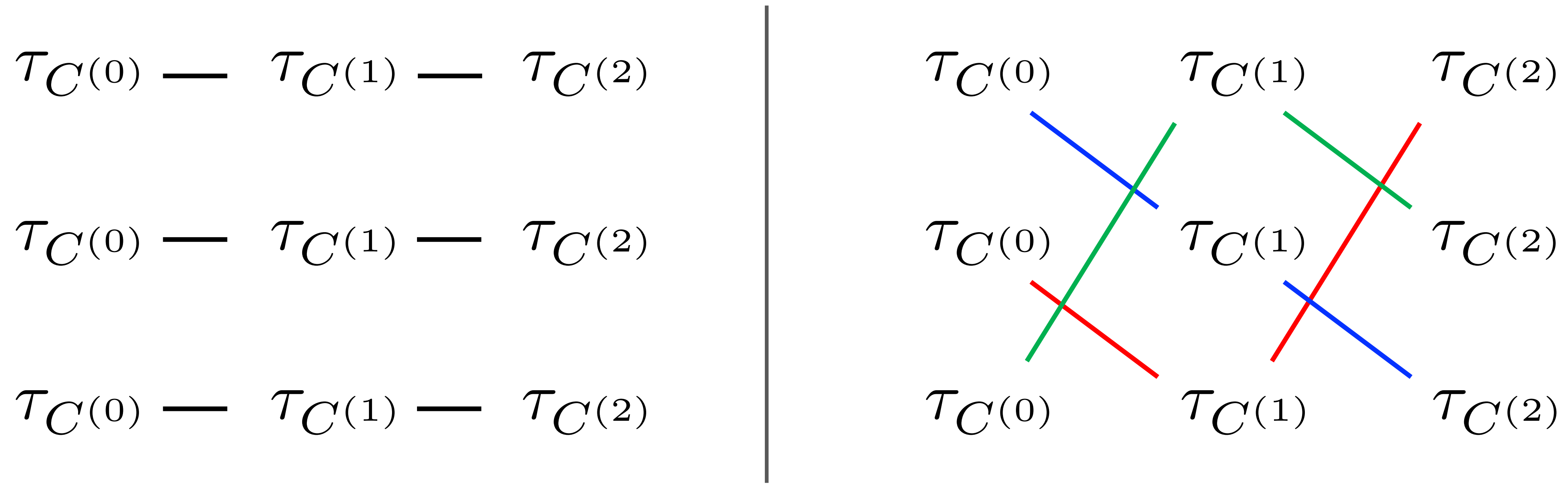
Main result

For arbitrary states ρ and ρ', ρ can be transformed to ρ' by a marginal-catalytic covariant transformation with arbitrarily small error.

- ρ can even be an incoherent state.
- Valid for arbitrary dimension and arbitrary Hamiltonian.
- Unlike the case of thermodynamics, coherence transformation has no restriction.
- Can be regarded as a new type of embezzlement, but with a very different mechanism.

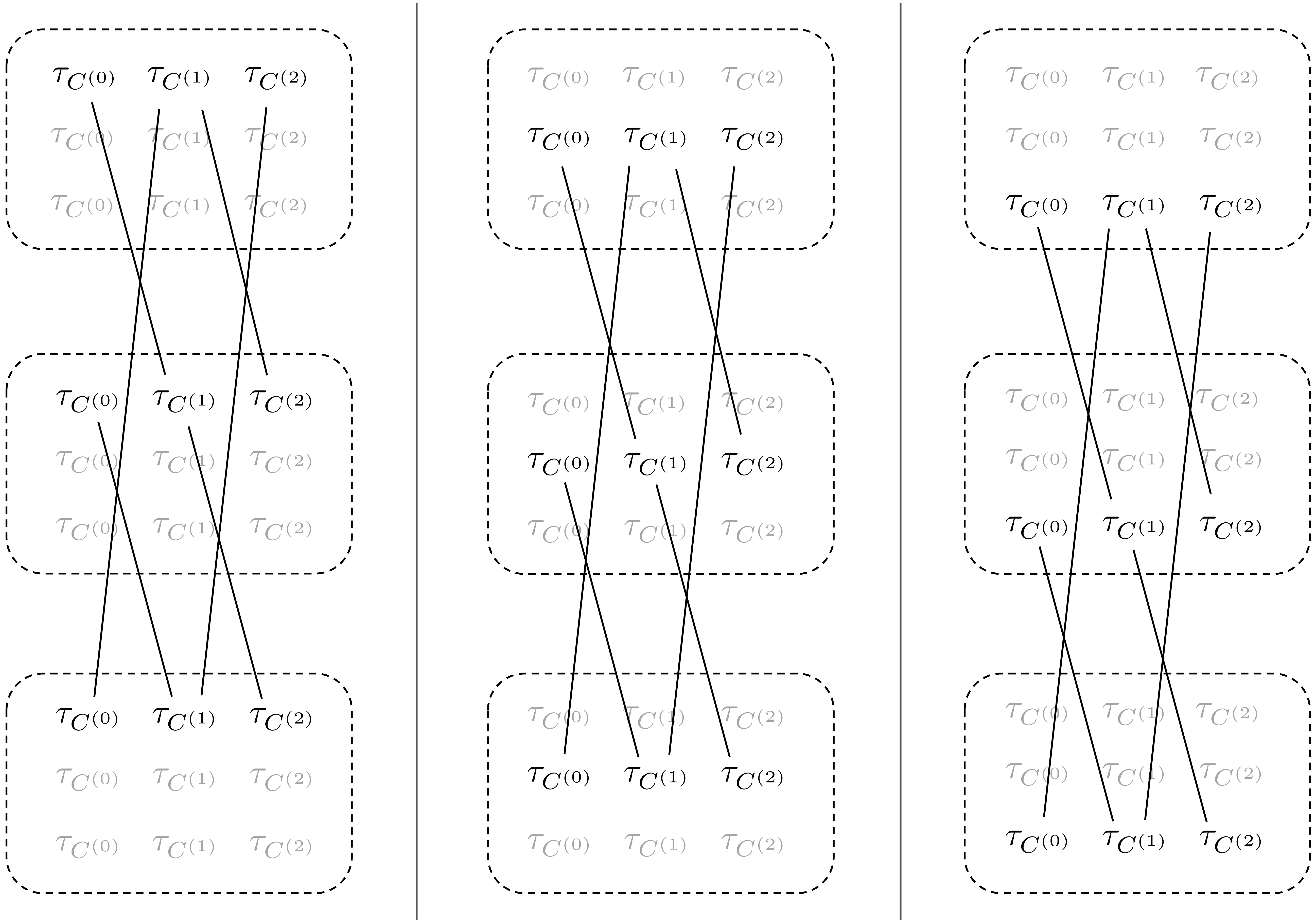
Partial Reusability of Marginal Catalysts

Given multiple copies of marginal catalysts, we can partially reuse them.



e.g., given three copies, we can run the same transformation six times.

Partial Reusability of Marginal Catalysts



K^n copies of marginal catalysts $\tau_{C^{(0)}} \otimes \ldots \otimes \tau_{C^{(K-1)}}$ allow $(n+1)K^n$ transformations.

Proof Sketch: Two-Level Amplification Protocol

Amplify coherence in two-level system with marginal catalysts. [Ding, Fu, Fan, PRA, '21]

Coherent states $\Sigma(\eta) := \dfrac{\mathbb{I} + \eta X}{2}$

Catalysts $\Gamma(\eta) := \dfrac{1}{2}\left(\mathbb{I} + \dfrac{\sqrt{3}\eta}{2}X + \dfrac{4-\eta^2}{6}Z\right)$

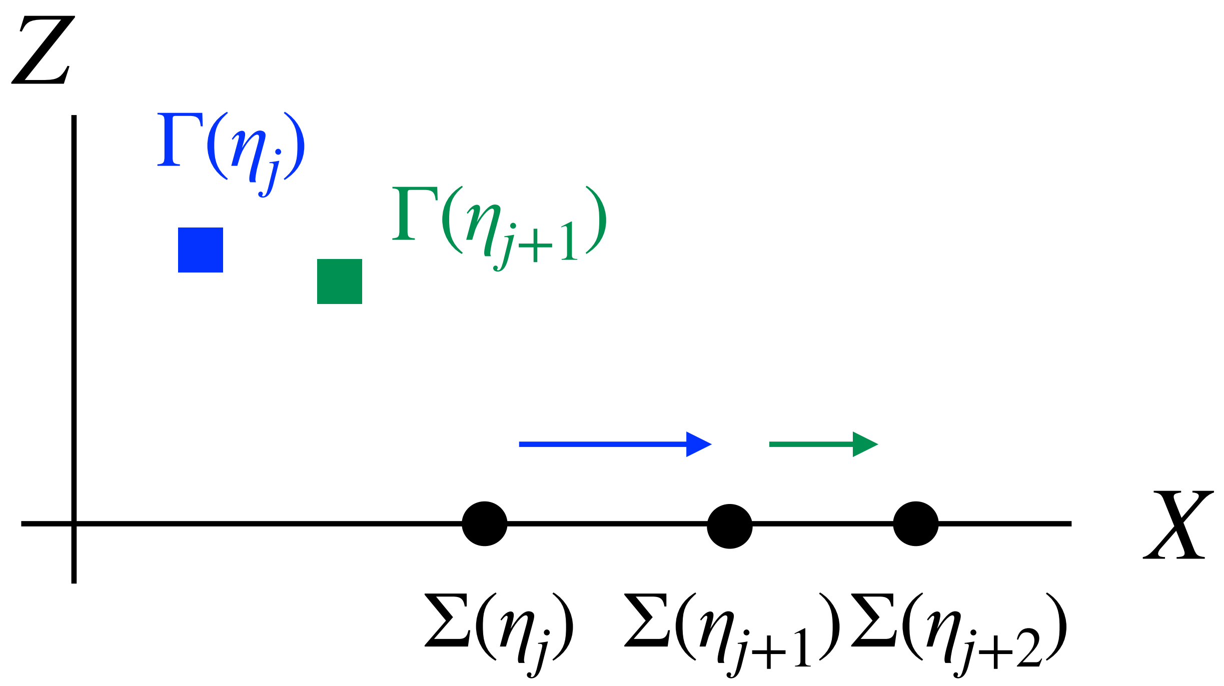
Covariant operation $\mathscr{E}$ with Kraus operators

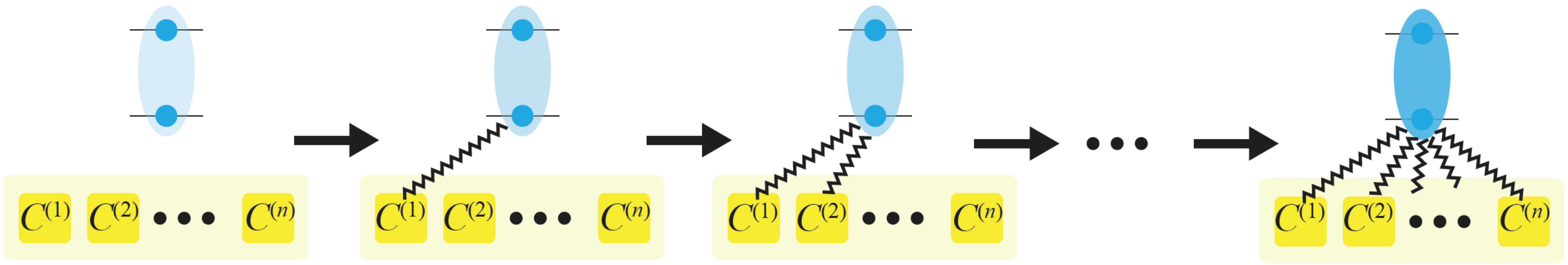
$$K_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/4 & \sqrt{3}/4 & 0 \\ 0 & \sqrt{3}/4 & 3/4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad K_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Tr}_C\, \mathscr{E}(\Sigma(\eta_j) \otimes \Gamma(\eta_j)) = \Sigma(\eta_{j+1})$$

$$\mathrm{Tr}_S\, \mathscr{E}(\Sigma(\eta_j) \otimes \Gamma(\eta_j)) = \Gamma(\eta_j)$$

$$\eta_{j+1} := \frac{\eta_j(25 - \eta_j^2)}{24}$$

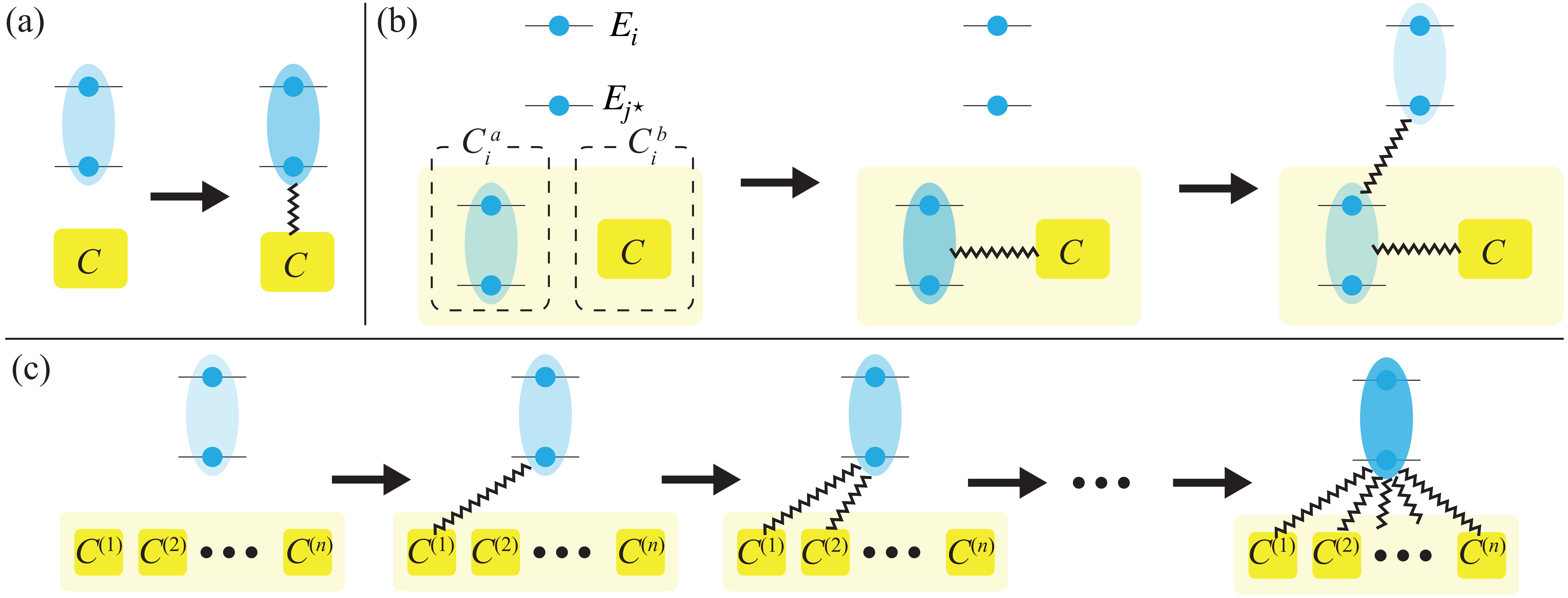




Proof Sketch : Preparing Coherent Resource

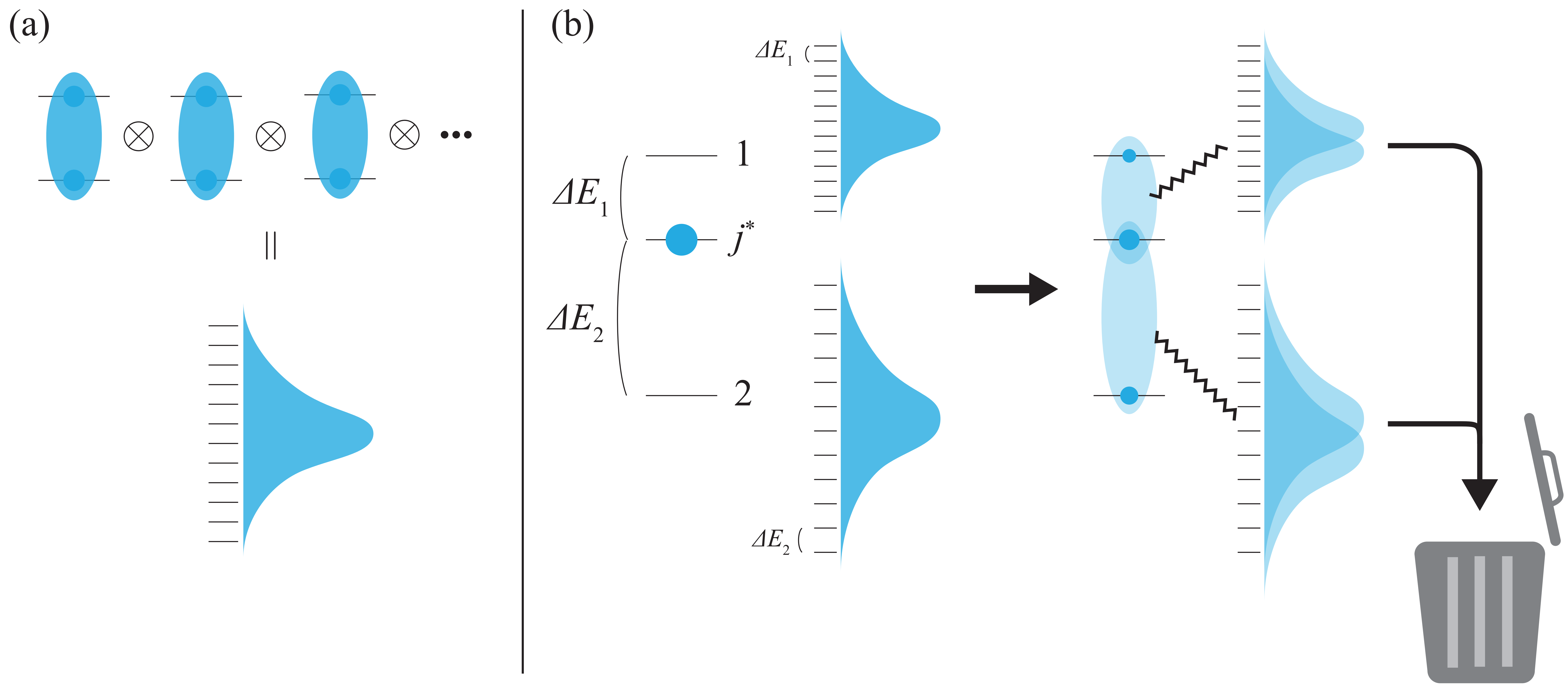
For the system Hamiltonian $H = \sum_i E_i |i\rangle\langle i|$, fix an arbitrary reference energy level $E_{j^\star}$.

For another energy level E_i ,



Do this many times for every i

Proof Sketch : Creating Desired State



We can approximate an action of an arbitrary unitary V on $|j^\star\rangle$ by $\mathrm{Tr}_R\left[U\left(|j^\star\rangle\langle j^\star| \otimes \eta_R\right)U^\dagger\right]$ with a covariant unitary U and a resource state η_R.

How Could This Be Possible?

Is coherence generated for free? ⟶ "Negative" coherence is hidden in correlation!

Resource measure $\mathfrak{R}$ is *superadditive* if $\mathfrak{R}(\rho_{12}) \geq \mathfrak{R}(\rho_1) + \mathfrak{R}(\rho_2), \forall \rho_{12}$

Examples:

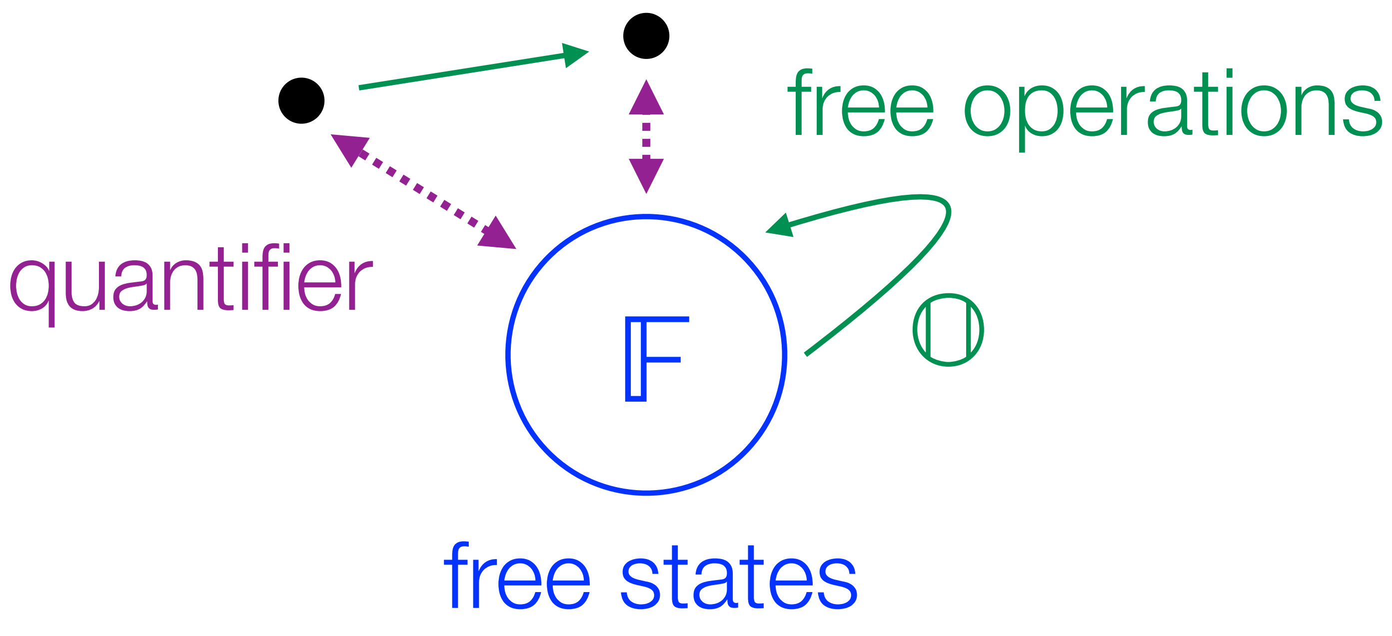
- Quantum thermodynamics: Free energy
- Speakable coherence: Relative entropy of (speakable) coherence
- Entanglement: Entanglement entropy

Every coherence measure violates superadditivity. [Marvian, Spekkens, PRL '19]

Even if $\mathfrak{R}(\rho) \leq \mathfrak{R}(\rho')$, it can be that $\mathfrak{R}\left(\rho \otimes \tau_{C^{(0)}} \cdots \otimes \tau_{C^{(K-1)}}\right) \geq \mathfrak{R}\left(\rho' \otimes \underline{\tau_{C^{(0)} \ldots C^{(K-1)}}}\right)$

Correlation reduces the total coherence

Restriction of Marginal Catalysts



We can put a universal restriction on the catalytic resource transformation at the level of *general resource theories.*

- $\mathfrak{R}(\rho_{12}) \geq \mathfrak{R}(\rho_1) + \mathfrak{R}(\rho_2),\ \forall \rho_{12}$ superadditivity
- $\mathfrak{R}(\rho_1 \otimes \rho_2) = \mathfrak{R}(\rho_1) + \mathfrak{R}(\rho_2),\ \forall \rho_1, \rho_2$ product additivity

If ρ can be transformed to ρ' by a marginal-catalytic or correlated-catalytic free transformation, then every super additive, product-additive resource measure satisfies $\mathfrak{R}(\rho) \geq \mathfrak{R}(\rho')$

If there exists even a single superadditive, product-additive, and faithful resource measure, arbitrary resource transformation is impossible.

e.g., quantum thermo, speakable coherence, entanglement

Achievable Catalytic Transformation

We discussed necessary conditions for catalytic transformation. **What about sufficiency?**

Asymptotic transformation from ρ to ρ' with $\mathbb{O}$

$$\exists \Lambda^{(n)} \in \mathbb{F} \text{ s.t. } \lim_{n\to\infty} \| \Lambda^{(n)}(\rho^{\otimes n}) - \rho'^{\otimes n} \|_1 = 0$$

Asymptotic transformation can be converted to single-shot catalytic transformation.

c.f. [Shiraishi, Sagawa, PRL, '21]

If ρ can be transformed to ρ' by an asymptotic free transformation, then ρ can be transformed to ρ' by a free transformation with correlated and marginal catalysts.

Completely characterize the catalytic transformation of

- Quantum thermodynamics with Gibbs-preserving operations [Shiraishi, Sagawa, PRL '21]
- LOCC pure state transformation [Kondra et al., PRL '21] [Kipka-Bartosik, Skrzypczyk, PRL '21]
- Speakable coherence

Operational meaning of relative entropy measures in terms of single-shot transformation.

Coherence Manipulation with Correlated Catalysts

$\rho \otimes \tau_C \to \rho'_{SC}$ such that $\mathrm{Tr}_C\, \rho'_{SC} = \rho'$ and $\mathrm{Tr}_S\, \rho'_{SC} = \tau_C$

Can we also realize an arbitrary coherence transformation with correlated catalysts?

We at least need some initial coherence due to the coherence no-broadcasting theorem.

Conjecture

If all energy differences for nonzero off-diagonals of ρ' can be expressed as a linear combination of those of ρ, then ρ can be transformed to ρ' by a correlated-catalytic covariant operation with an arbitrarily small error.

Could be a key toward the complete characterization of the capability of **thermal operations with correlated catalysts.**

We can show a weaker version of this statement with a broader class of operations.

Summary

- Marginal-catalytic covariant operation can realize arbitrary state transformation.
- This is possible by a peculiar property of coherence measure. We derived a general restriction on such an anomalous transformation valid for general resource theories.
- Presented achievable condition in relation to asymptotic transformation and precise characterization for some theories in combination with the above restriction.
- Put forward a conjecture that covariant operations with a correlated catalyst can realize a state transformation as long as the sufficient initial coherence is present.

Outlook

- Prove or disprove the conjecture on correlated-catalytic covariant transformation.
- Characterization of correlated-catalytic and marginal-catalytic thermal transformations.
  - Can we show that if there is nonzero coherence in the initial state, then transformation rule is governed by the free energy?
  - Can we extend the result for semiclassical transformation with marginal catalysts to a quantum setting?
- Extension to general groups.

Thank you!