QICF22 — 18 October 2022

Detecting $\underbrace{\text{quantumness}}$ and entanglement with precessions

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Detecting quantumness in uniform precessions

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Detecting quantumness and $\underbrace{\text{entanglement}}_{\text{\{Upcoming extension\}}}$ with precessions

Pooja Jayachandran, …

Cite as: arXiv:????.????? [quant-ph]

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NUS National University of Singapore

What is quantum about a quantum harmonic oscillator?

Discrete Energy Levels

$$E_{n+1} - E_n = \hbar\omega$$

Zero-point Motion

$$E_0 = \frac{1}{2}\hbar\omega$$

Heisenberg Uncertainty

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Classical systems can exhibit the same behaviour

Bell Inequalities

With multipartite systems

Leggett–Garg, Noncontextual Inequalities

With sequential/compatible measurements

Negative "Probabilities"

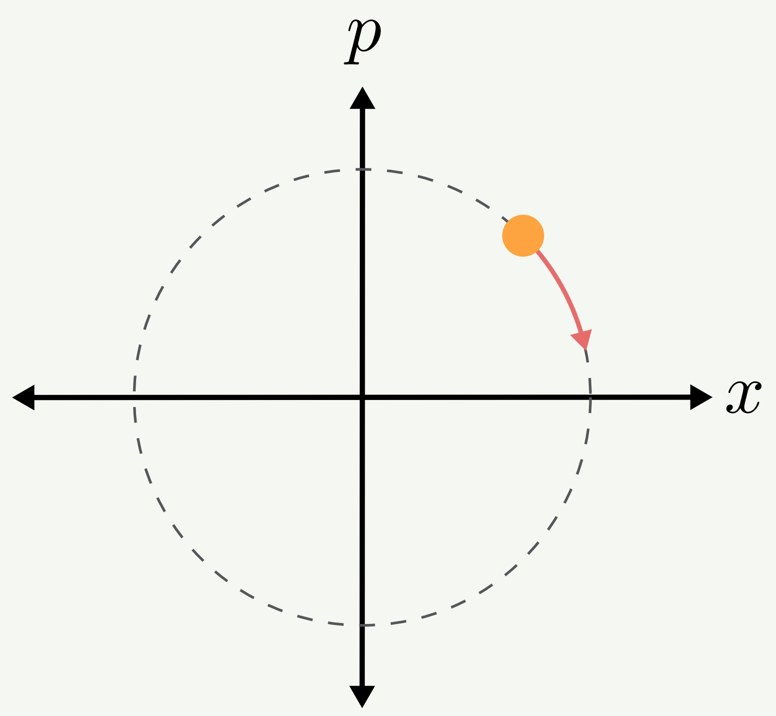
From state tomography

System must be quantum to exhibit these behaviours

What is **not** quantum about a quantum harmonic oscillator?

Classical Time Evolution

$$x(t) = x(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t)$$



Quantum Time Evolution

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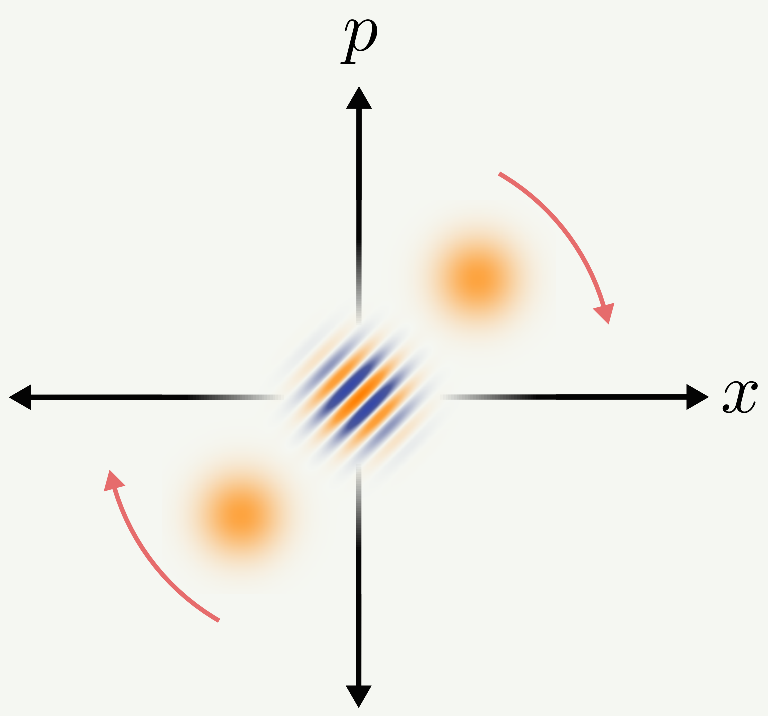
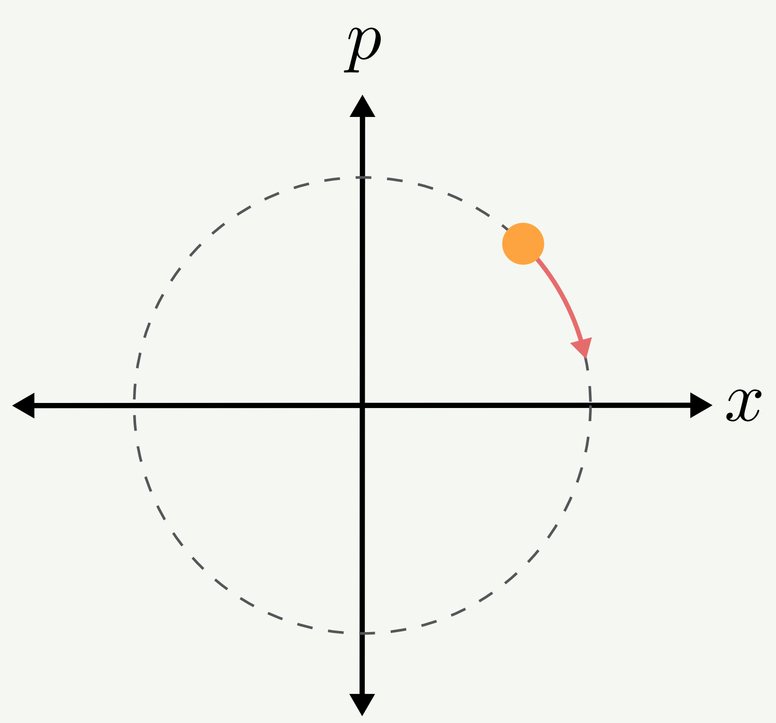


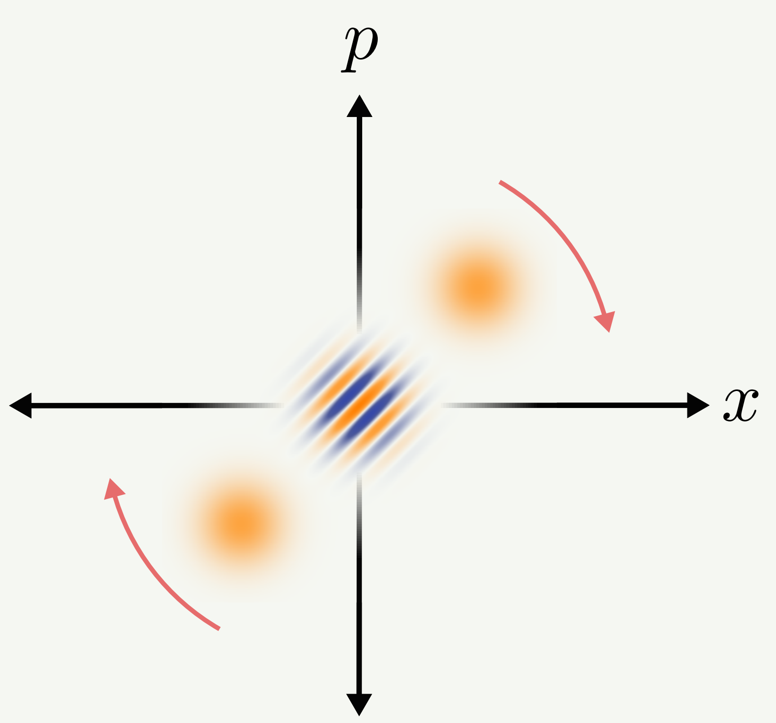
Time evolution of the quantum harmonic oscillator is as classical as it gets!

What is not quantum about a quantum harmonic oscillator?

Classical Time Evolution



Quantum Time Evolution



From Tsirelson's dusty arXivs, we ask the question:

How often is the coordinate of a harmonic oscillator positive?

Boris Tsirelson

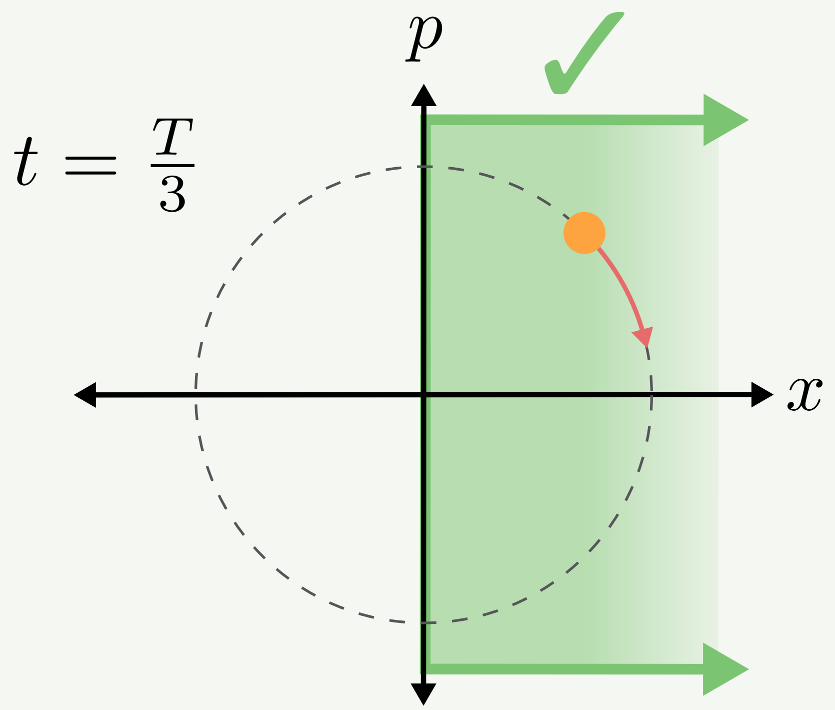
Cite as: arXiv:quant-ph/0611147

“How often is the coordinate of a harmonic oscillator positive?”

Assumptions: We have one (1) harmonic oscillator with period T

For each round,

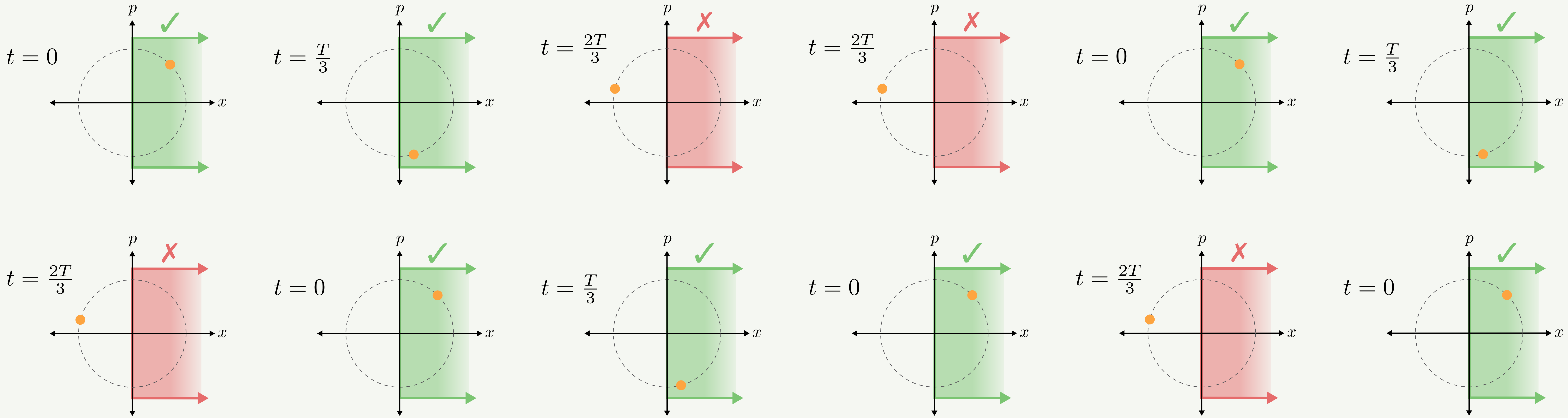
1. Prepare the system in some state
2. Wait for a duration $t \in \{0, \frac{T}{3}, \frac{2T}{3}\}$ (chosen randomly)
3. Measure the position of the system: Is $x(t) > 0$?



After many rounds: $P_3 =$ “*how likely is* $x > 0$?”

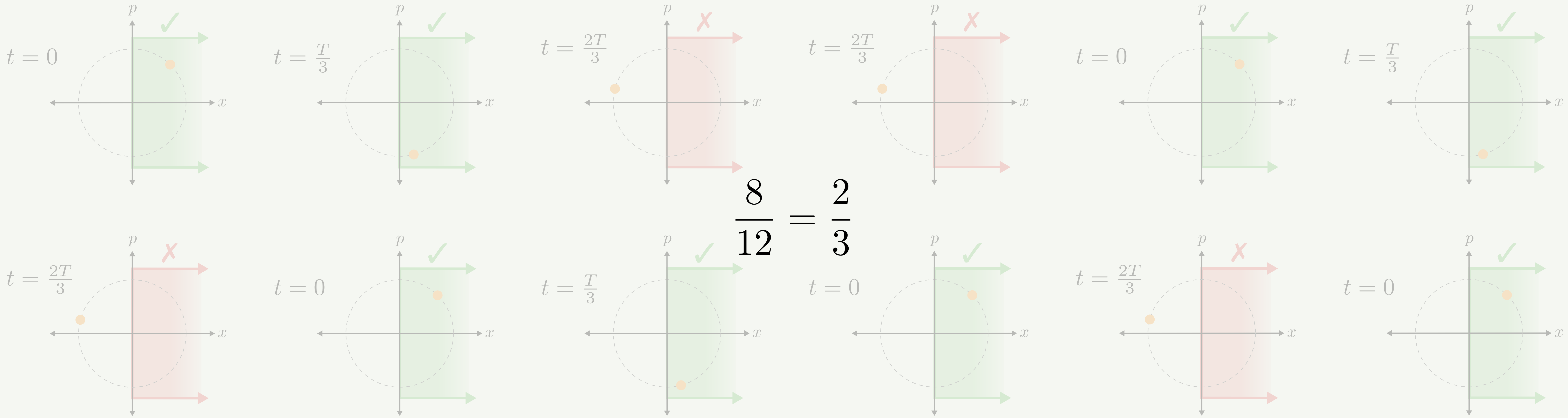
Comment: Each round has system reset, or performed on ensemble of identical oscillators — no sequential measurements!

“How often is the coordinate of a harmonic oscillator positive?”



After many rounds: how likely is $x > 0$?

“How often is the coordinate of a harmonic oscillator positive?”



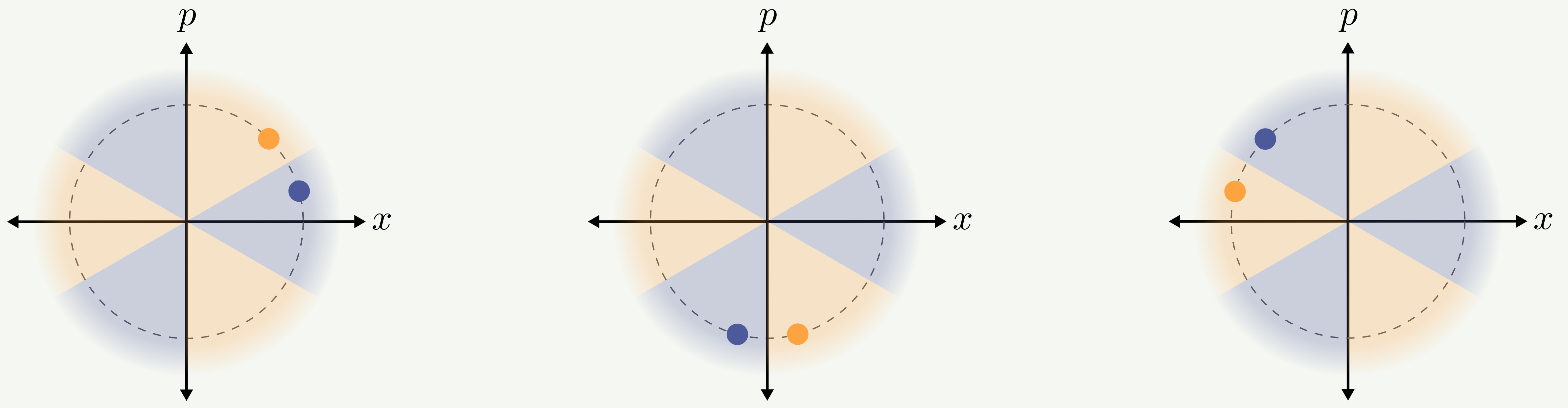
$$\frac{8}{12} = \frac{2}{3}$$

After many rounds: how likely is $x > 0$?

$$P_3 = \tfrac{1}{3}\,\mathrm{pos}(x(0)) + \tfrac{1}{3}\,\mathrm{pos}\big(x(\tfrac{T}{3})\big) + \tfrac{1}{3}\,\mathrm{pos}\big(x(\tfrac{2T}{3})\big)$$

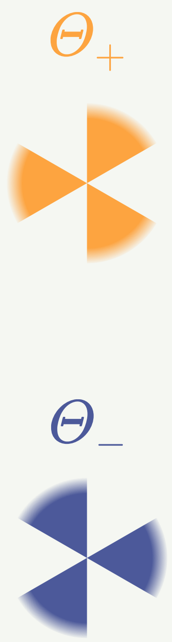
Classical bound of P_3

$t = 0$ $\quad$ $t = \frac{T}{3}$ $\quad$ $t = \frac{2T}{3}$



$P_3 = \frac{2}{3}$

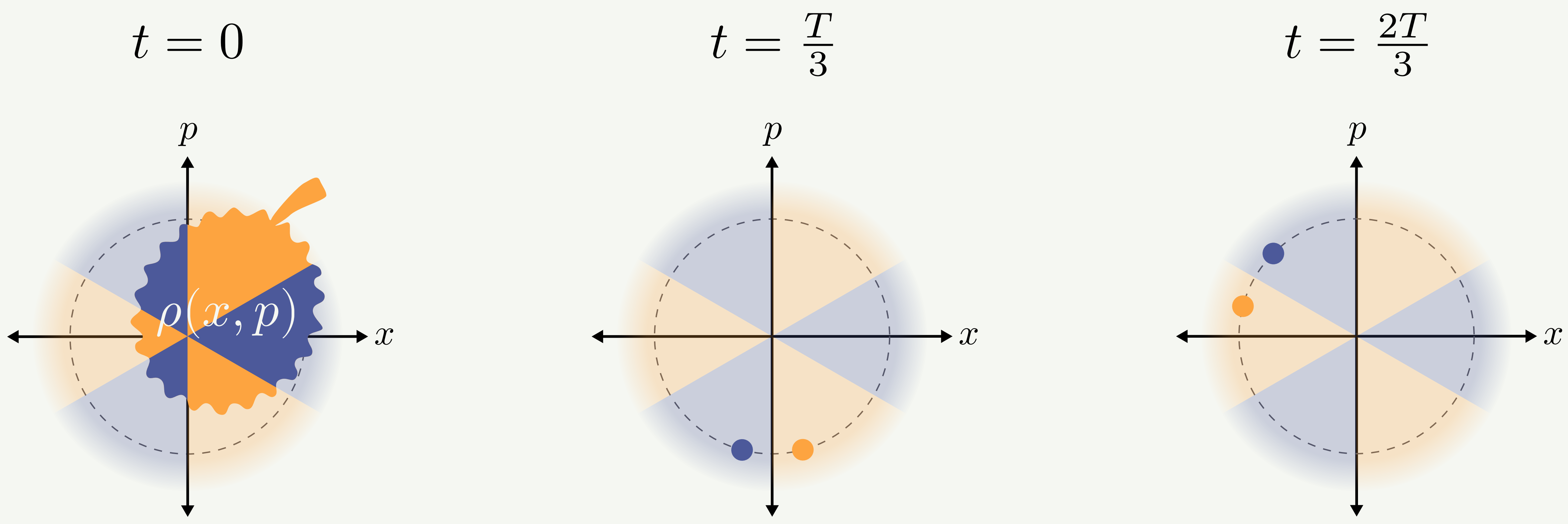
$P_3 = \frac{1}{3}$

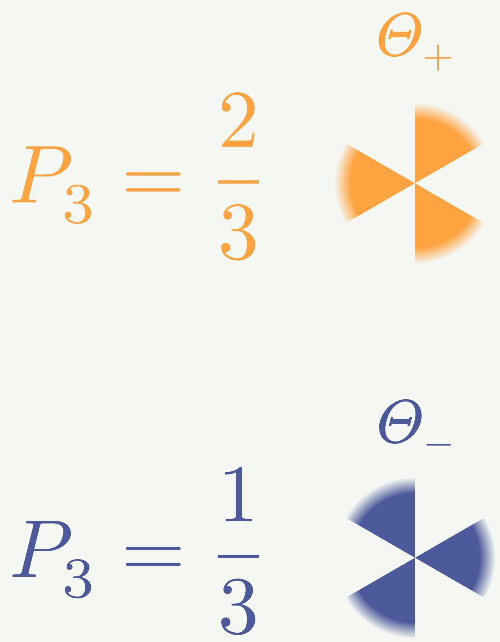


$$P_3 = \frac{1}{3}\,\mathrm{pos}(x(0)) + \frac{1}{3}\,\mathrm{pos}\big(x(\tfrac{T}{3})\big) + \frac{1}{3}\,\mathrm{pos}\big(x(\tfrac{2T}{3})\big)$$

For any classical state $\rho(x,p)$, $\quad \frac{1}{3} \leq P_3 \leq \frac{2}{3} \equiv \mathbf{P}_3^c$

Classical bound of P_3





For any classical state $\rho(x,p)$, define $\mathcal{I}_\pm \equiv \int_{\theta\in\Theta_\pm} \mathrm{d}\theta \int r\,\mathrm{d}r\,\rho(x,p)$

$$\boxed{P_3^c = \frac{2}{3}\mathcal{I}_+ + \frac{1}{3}\mathcal{I}_-}$$

The quantum case

Wigner function $W(x,p)$ takes the place of the joint probability density

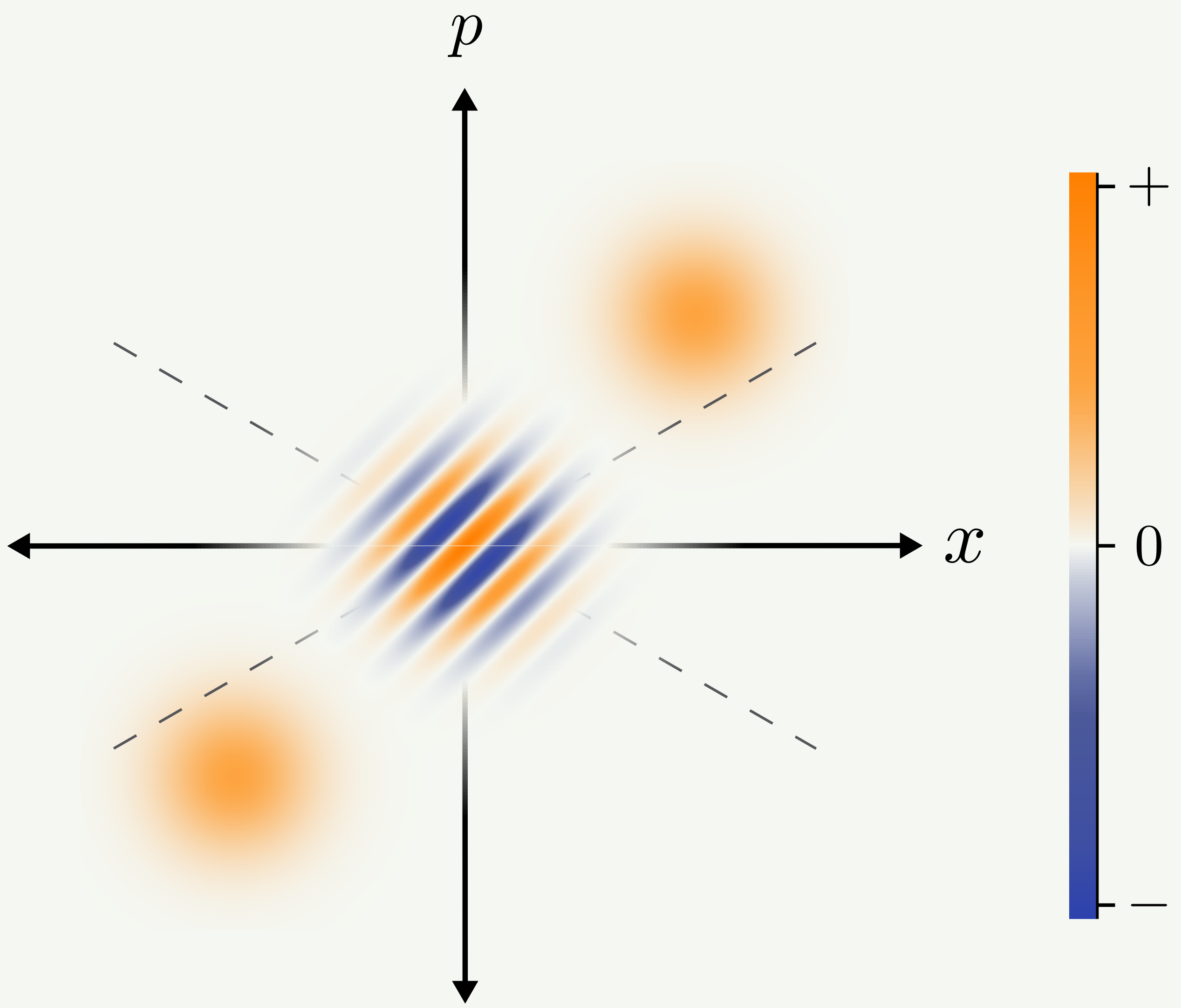
$$\Pr(x>0) = \int_0^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p\, W(x,p)$$

Time evolution is precession in phase space

Following the same arguments as before,

$$\mathcal{I}_\pm \equiv \int_{\theta\in\Theta_\pm} \mathrm{d}\theta \int r\,\mathrm{d}r\, W(x,p)$$

$$\boxed{P_3^\infty = \frac{2}{3}\mathcal{I}_+ + \frac{1}{3}\mathcal{I}_-}$$



Then, what's the difference between the two?

Violating the classical bound

Position and momentum are incompatible observables $[X, P] \neq 0$:
$W(x, p)$ is a *quasi*probability distribution

Negative values are allowed, if marginals are probabilities

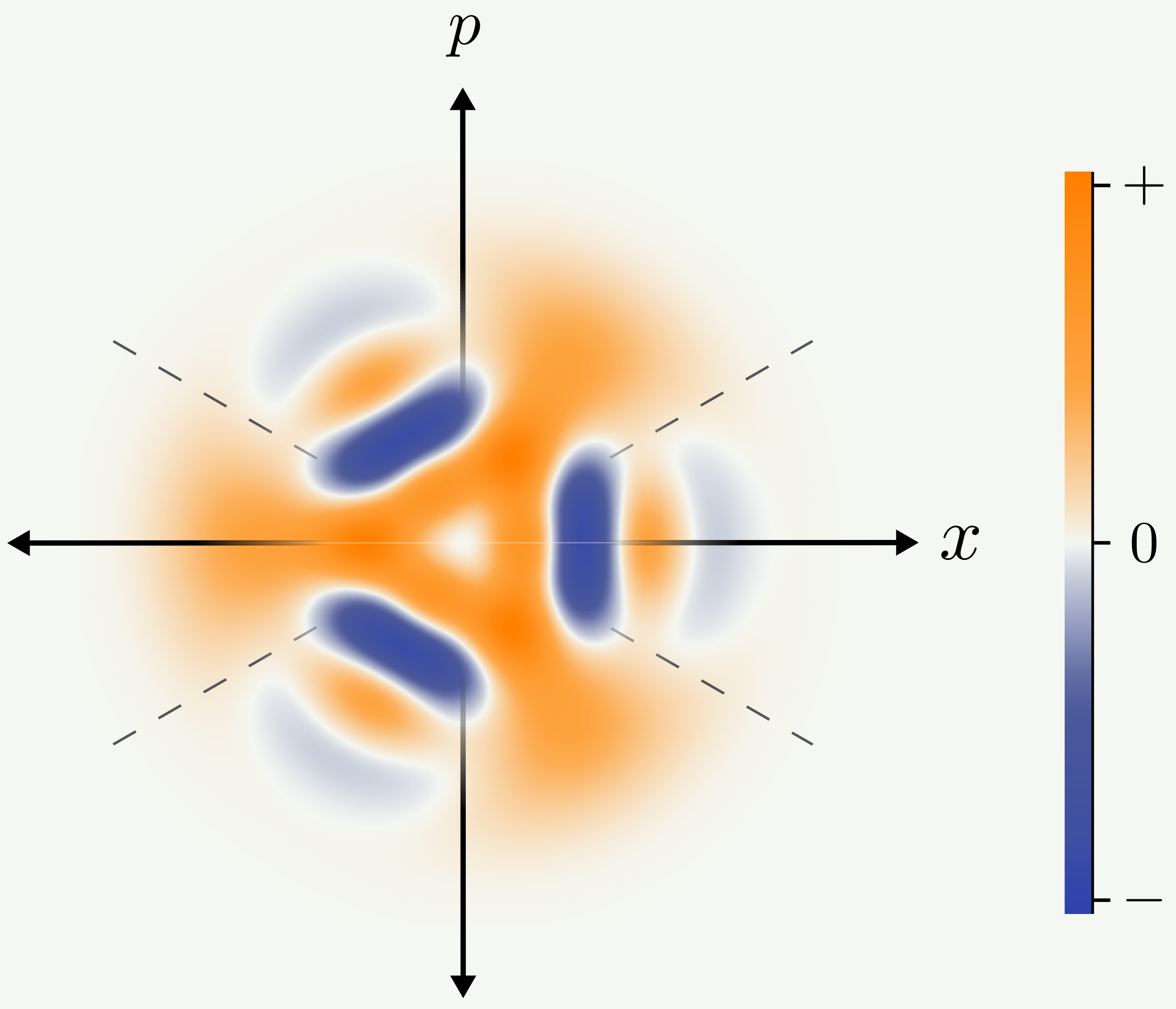
What if we concentrate the negativity into Θ_{-}?

$$\boxed{P_3^{\infty} = \frac{2}{3}\mathcal{I}_{+} + \frac{1}{3}\mathcal{I}_{-}}$$

$$\mathcal{I}_{-} < 0 \implies \mathcal{I}_{+} > 1$$

(from normalisation)

$$\implies P_3^{\infty} > \frac{2}{3} = \mathbf{P}_3^c$$



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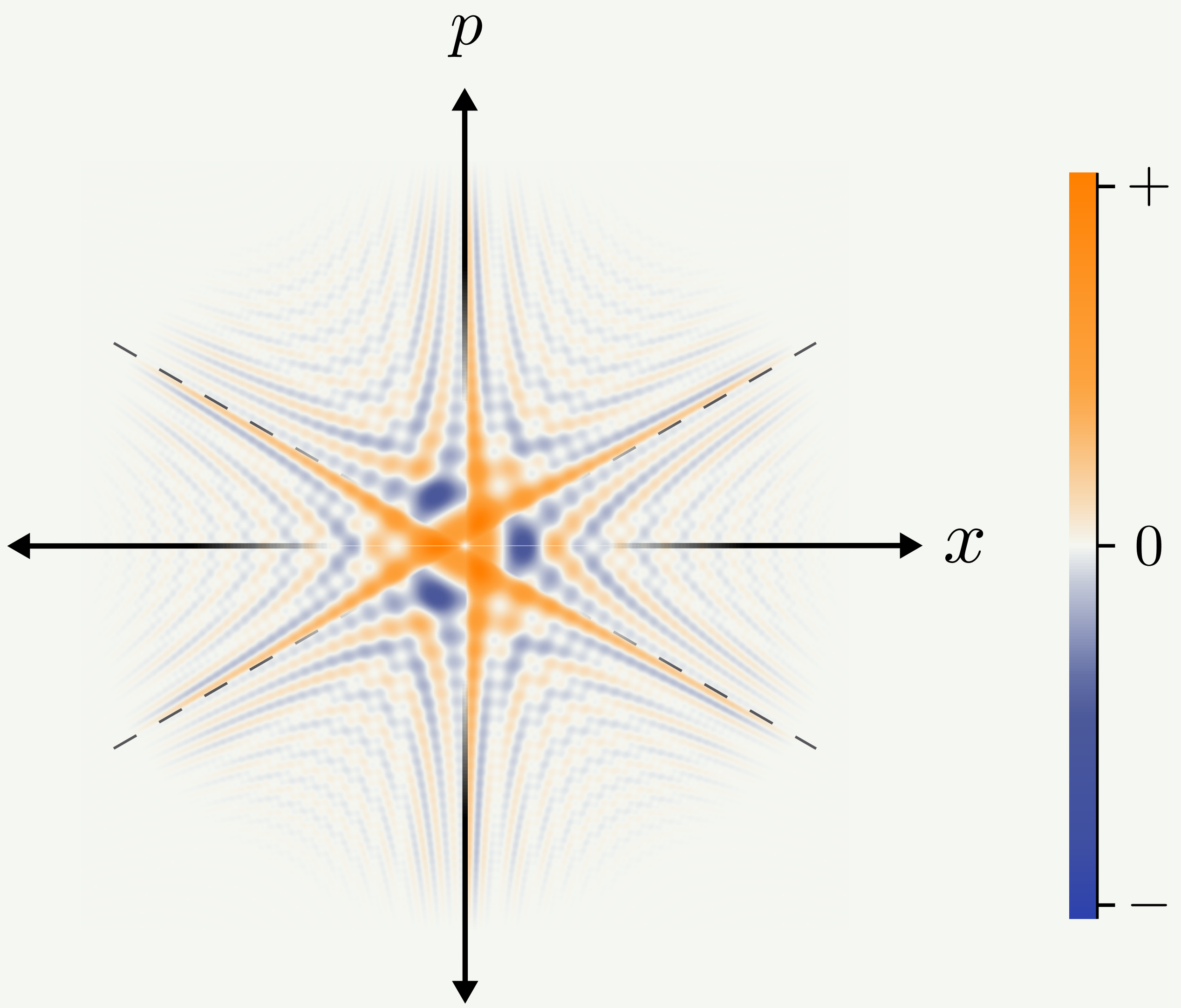


Largest quantum violation for $E < 2101\hbar\omega$

$$\mathbf{P}_3^\infty \gtrsim 0.709 > \mathbf{P}_3^c$$

Quantum harmonic oscillators can beat the classical bound!

Certifying quantumness with precessions in harmonic systems

Assumptions: 1. Dynamics of the system is a uniform precession
2. Period of the system is known

Protocol: For each round, 1. Prepare the system
2. Randomly wait a duration $t \in \{0, \frac{T}{3}, \frac{2T}{3}\}$
3. Measure the position $x(t)$

After many rounds: $P_3 =$ "*how likely is* $x > 0$*?*"

If $P_3 > \mathbf{P}_3^c = \frac{2}{3}$, the harmonic oscillator is quantum

Quantum–classical gap allows us to certify nonclassicality

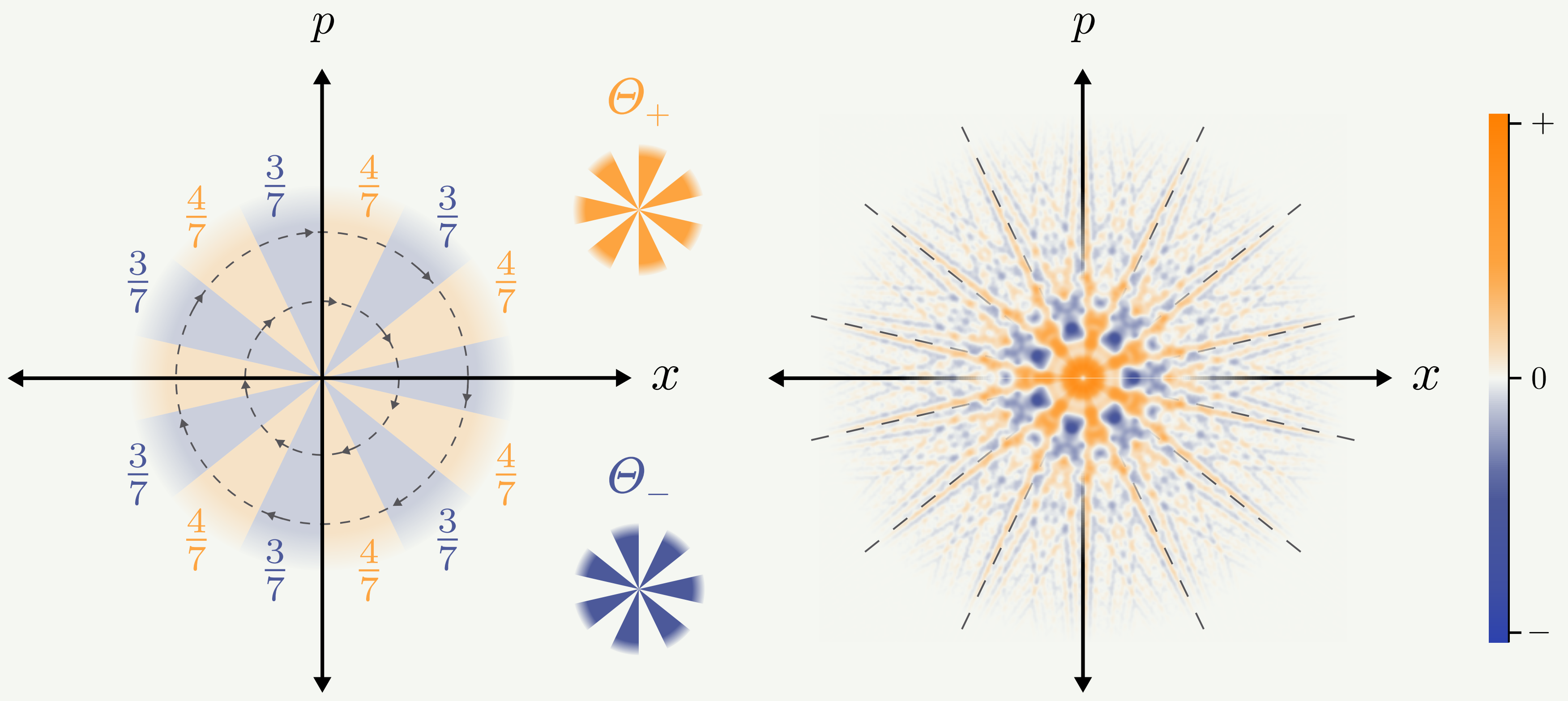
No need for sequential or simultaneous measurements!

Generalisation of protocol

Protocol with K times

$$P_K = \frac{1}{K}\sum_{k=0}^{K-1} \Pr\left(x\left(\tfrac{kT}{K}\right) > 0\right)$$

- Quantum–classical gap exists for all K odd
- Non-trivial (but loose) upper bound for $\mathbf{P}_K^\infty$

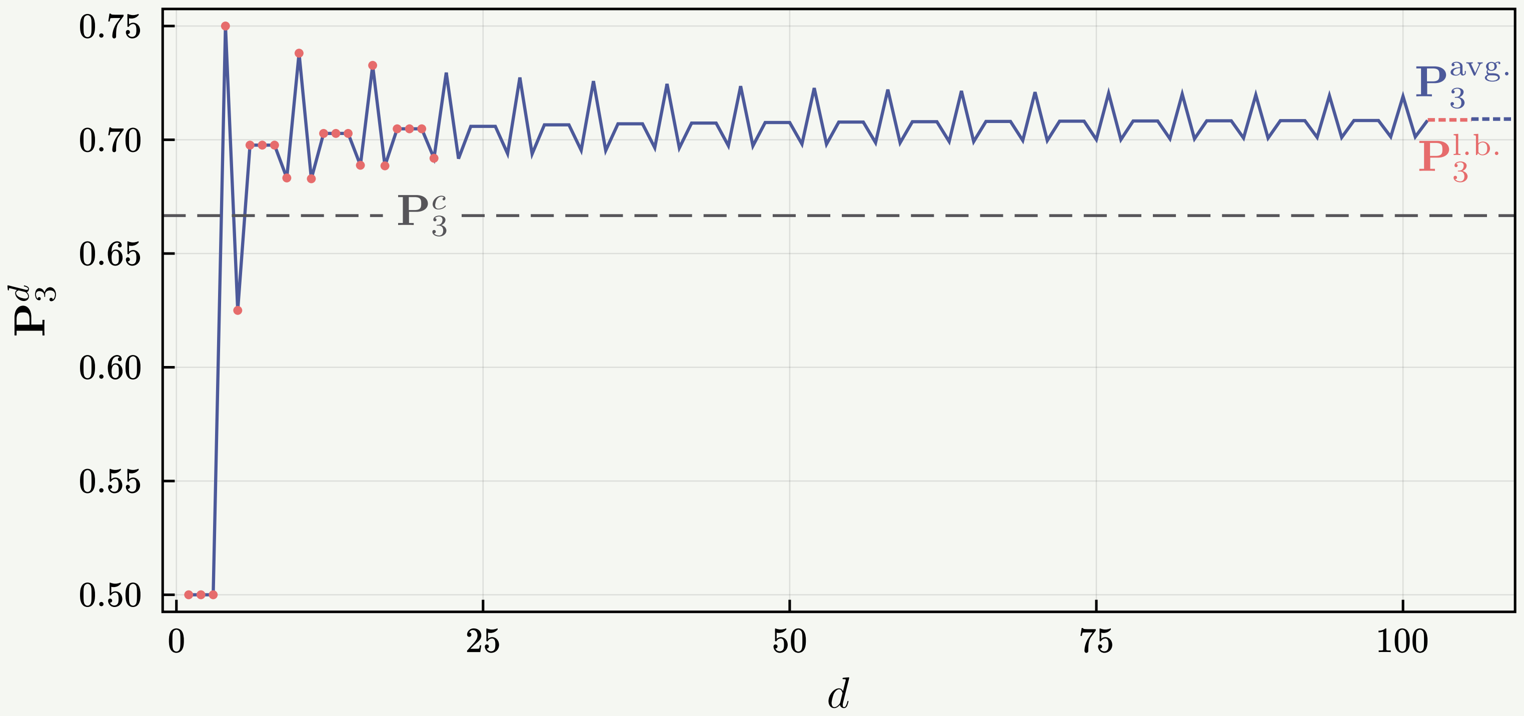


Precession in Real Space (Spins)

$$H = -\omega J_z$$

$$J_x(t) = \cos(\omega t)J_x(0) + \sin(\omega t)J_y(0)$$

- Gap exists for all j, excluding $j = 0, 1/2, 1, 2$
- $\mathbf{P}_3^4 = 3/4$: conjectured to be the largest



Generalisation of protocol

Uniform Precessions of Effective Oscillators

- Sum of variables precessing with the same frequency are uniformly precessing

$$H = -\omega\left(J_z^{(1)} + J_z^{(2)}\right) \qquad\qquad P_K = \frac{1}{K}\sum_{k=0}^{K-1}\Pr\left(J_x^{(1)}(t_k) + J_x^{(2)}(t_k) > 0\right)$$

- *Protocol with total angular momentum*: maximally-violating state is always entangled

$$H = \sum_{j=1}^{2}\left(\frac{p_j^2}{2m} + \frac{1}{2}m_j\omega_j^2 x_j^2\right) - g x_1 x_2 = \sum_{\sigma\in\{+,-\}}\frac{p_\sigma^2}{2\mu} + \frac{1}{2}\mu\omega_\sigma^2 x_\sigma^2 \qquad P_K = \frac{1}{K}\sum_{k=0}^{K-1}\Pr(x_\sigma(t_k) > 0)$$

- *Protocol with two harmonic oscillators*: does a violation in the normal mode tell us anything about the entanglement?

Witnessing entanglement with uniform precessions

Assumptions: 1. System consists of two harmonic oscillators

$$H = \sum_{j=1}^{2} \left(\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 x_j^2 \right) - g x_1 x_2 = \sum_{\sigma \in \{+,-\}} \frac{p_\sigma^2}{2\mu} + \frac{1}{2} \mu \omega_\sigma^2 x_\sigma^2$$

2. Period of the normal modes are T_σ

Protocol: For each round, 1. Prepare the system

2. Randomly wait a duration $t \in \left\{0, \frac{T_\sigma}{3}, \frac{2T_\sigma}{3}\right\}$

3. Measure the positions $x_1(t), x_2(t)$

After many rounds: $P_3 =$ "*how likely is* $x_\sigma > 0$?"

We already know that $P_3^c \leq \mathbf{P}_3^c = \frac{2}{3}$ for classical states

What about the values of P_3 for *separable* states?

Quantum violation of separable states

Consider a separable state and its corresponding Wigner function

$$\rho = \sum_k p_k \; \rho_1^{(k)} \otimes \rho_2^{(k)}$$

$$W_\rho(x_1, p_1; x_2, p_2) = \sum_k p_k W_{\rho_1^{(k)}}(x_1, p_1) W_{\rho_2^{(k)}}(x_2, p_2)$$

Consider the special case $x_+ \propto \left(\frac{m_1}{m_2}\right)^{\frac{1}{4}} x_1 + \left(\frac{m_2}{m_1}\right)^{\frac{1}{4}} x_2$

Then, the above state has the reduced Wigner function

$$W_{\mathrm{tr}_-(\rho)}(x_+, p_+) = \frac{1}{\pi\hbar} \sum_k p_k \, \mathrm{tr}\Big(\rho_1^{(k)} U(x_+, p_+) \rho_2^{(k)} U^\dagger(x_+, p_+)\Big)$$

for some unitary $U(x_+, p_+)$.

Quantum violation of separable states

$$\rho = \sum_k p_k \; \rho_1^{(k)} \otimes \rho_2^{(k)}$$

$$W_{\mathrm{tr}_-(\rho)}(x_+, p_+) = \frac{1}{\pi\hbar} \sum_k p_k \,\mathrm{tr}\Big(\rho_1^{(k)} U(x_+, p_+) \rho_2^{(k)} U^\dagger(x_+, p_+)\Big)$$

Since $W_\rho(x_+, p_+) \geq 0$ permits a classical description, it cannot violate the classical bound

$$\rho \text{ is separable } \implies P_3 \leq \mathbf{P}_3^c = \frac{2}{3}$$

or conversely

$$P_3 > \mathbf{P}_3^c = \frac{2}{3} \implies \rho \text{ is } \textit{not} \text{ separable}$$

If $P_3 > \mathbf{P}_3^c = \frac{2}{3}$, the harmonic oscillators are entangled

Other aspects of our entanglement witness

The criterion works for any K odd; entangled if $P_K > \mathbf{P}_K^c$

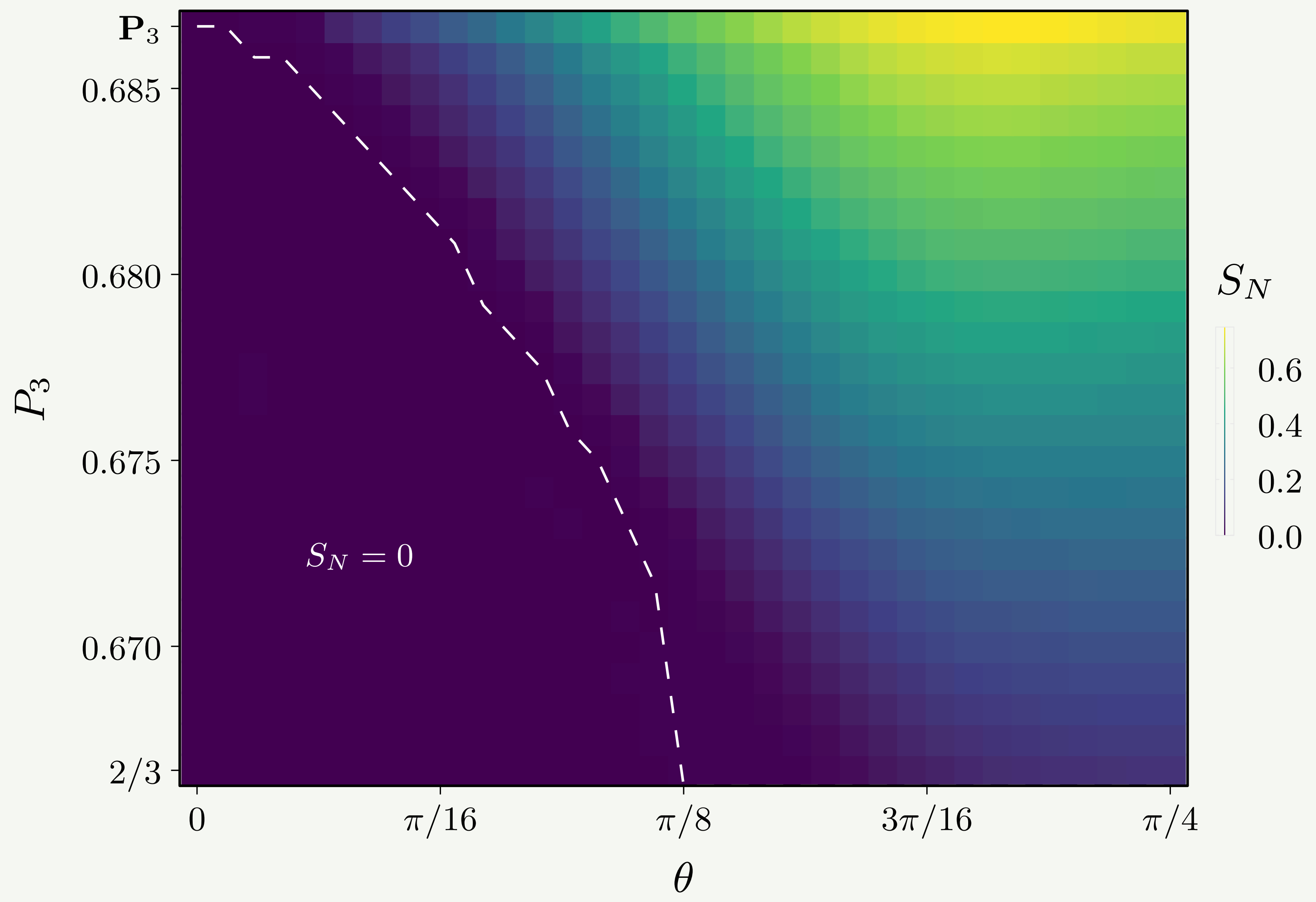
More generally, normal modes of the form

$$x_+ = \left(\frac{m_1}{m_2}\right)^{\frac{1}{4}} \cos\theta \; x_1 + \left(\frac{m_2}{m_1}\right)^{\frac{1}{4}} \sin\theta \; x_2$$

In other cases, larger violation required

θ fixed by system parameters

If uncoupled, θ can take on any value



Motivation for an alternative entanglement witness

Commonly used criteria for entanglement based on quantum uncertainty relations

For example, Duan et al. [arXiv:quant-ph/9908056]: for some real number c, define

$$u \equiv |c|\tilde{x}_1 + \frac{1}{c}\tilde{x}_2 \qquad v \equiv |c|\tilde{p}_1 - \frac{1}{c}\tilde{p}_2 \qquad \tilde{x}_j \equiv x_j\sqrt{\frac{m_j\omega_j}{\hbar}} \qquad \tilde{p}_j \equiv \frac{p_j}{\sqrt{m_j\hbar\omega_j}}$$

The state is entangled if $\left\langle(\Delta u)^2\right\rangle + \left\langle(\Delta v)^2\right\rangle < c^2 + \frac{1}{c^2}$

For any sufficiently pure classical states, $\left\langle(\Delta u)^2\right\rangle + \left\langle(\Delta v)^2\right\rangle \to 0$

Requires measurement precision set by $\hbar$, open to false positives by classical states

These issues become important as entanglement of mesoscopic/macroscopic objects become possible

Comparison with other entanglement witnesses

Generally, our criterion is useful in some cases, while other criteria might be useful in others

	Duan et. al. Phys. Rev. Lett. **84**, 2722	**Hillery & Zubiary** Phys. Rev. Lett. **96**, 050503	**Zhang et. al.** Phys. Rev. A **82**, 032323	**Ours**
False positives by classical states	✗	✗	✗	✓
Detects Gaussian entangled states	✓	✓	✓	✗
Detects some bound entangled states	✗	✗	✓	✓*
Detects family of states $\|\Psi_{\mathbf{n}}\rangle$	✗	✗	✗	✓

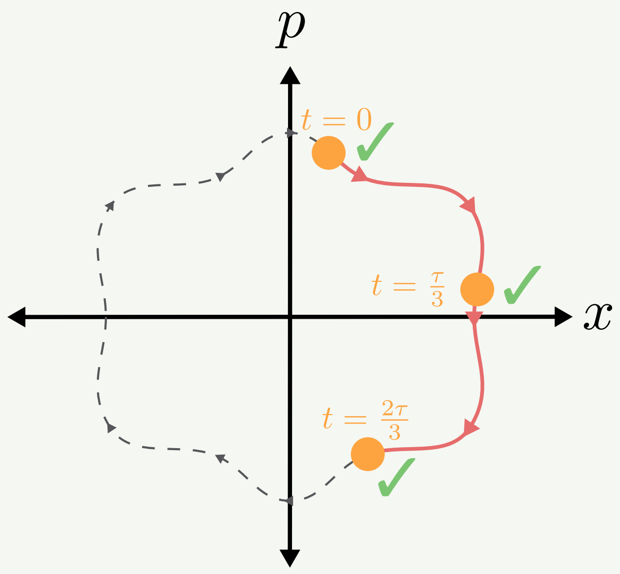
$$|\Psi_{\mathbf{n}}\rangle = \sum_{j=0}^{\mathbf{n}K} \sum_{n=\lfloor \frac{j}{K} \rfloor}^{\mathbf{n}} \psi_n \sqrt{\binom{nK}{j}} (\cos\theta)^{nK-j} (\sin\theta)^j \, |nK-j\rangle \otimes |j\rangle \,, \quad |\psi_0| < 1$$

Certifying quantumness with precessions in... anharmonic systems?

No fixed period T**:** Replace T with some choice of $\mathcal{T}$; measure at times $t \in \{0, \frac{\mathcal{T}}{3}, \frac{2\mathcal{T}}{3}\}$

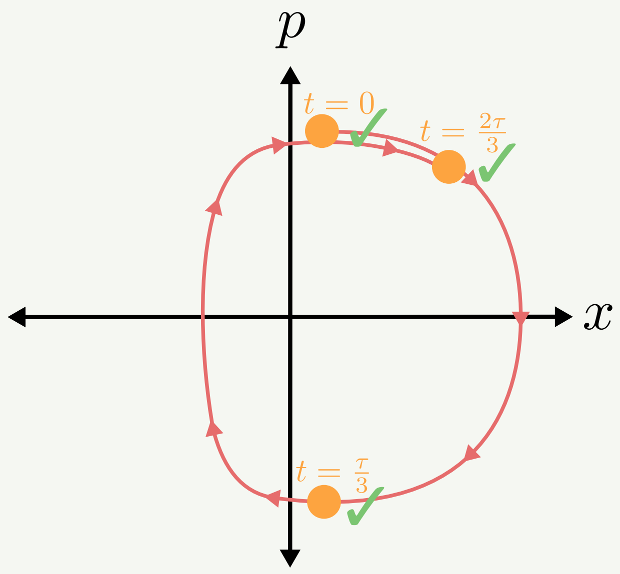
Classical bound can be 1: When period of oscillation is too large or too small

$2\mathcal{T}/3 \leq \Delta t_{0\to 0}(E)$



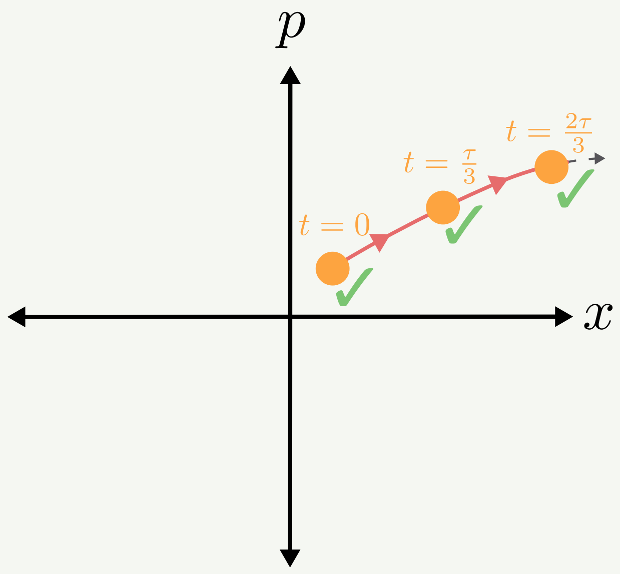
All measurements occur before particle leave positive plane

$2\mathcal{T}/3 \geq T(E)$



Particle completes its cycle between each measurement

Non-oscillating



Particle does not oscillate, same as $\Delta t_{0\to 0}(E) = \infty$ *or* $\mathcal{T}(E) = \infty$

Limit energy range $E_{\min} < E < E_{\max}$ where these "bad cases" do not happen

Certifying quantumness with precessions in anharmonic systems

Assumptions: 1. Dynamics of the system is given by a Hamiltonian $H(x,p)$

2. Parameters $(\omega_0, \alpha, \dots)$ of the system are known

3. Energy of the system is bounded $E_{\min} < E < E_{\max}$

Protocol: For each round, 1. Prepare the system

2. Randomly wait a duration $t \in \{0, \frac{\mathcal{T}}{3}, \frac{2\mathcal{T}}{3}\}$

3. Measure the position $x(t)$

After many rounds: $P_3 =$ "*how likely is* $x > 0$*?*"

$\boxed{\textbf{If } P_3 > \mathbf{P}_3^c = \frac{2}{3}\textbf{, the anharmonic system is quantum}}$

If a "*good*" choice of $\mathcal{T}$, $E_{\min}$, $E_{\max}$ can be made!

Anharmonic example: Kerr-like Hamiltonian

$$H(x,p) = H_0(x,p) + \frac{\alpha}{2\hbar\omega^2} H_0^2(x,p) \qquad \text{where} \qquad H_0(x,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2$$

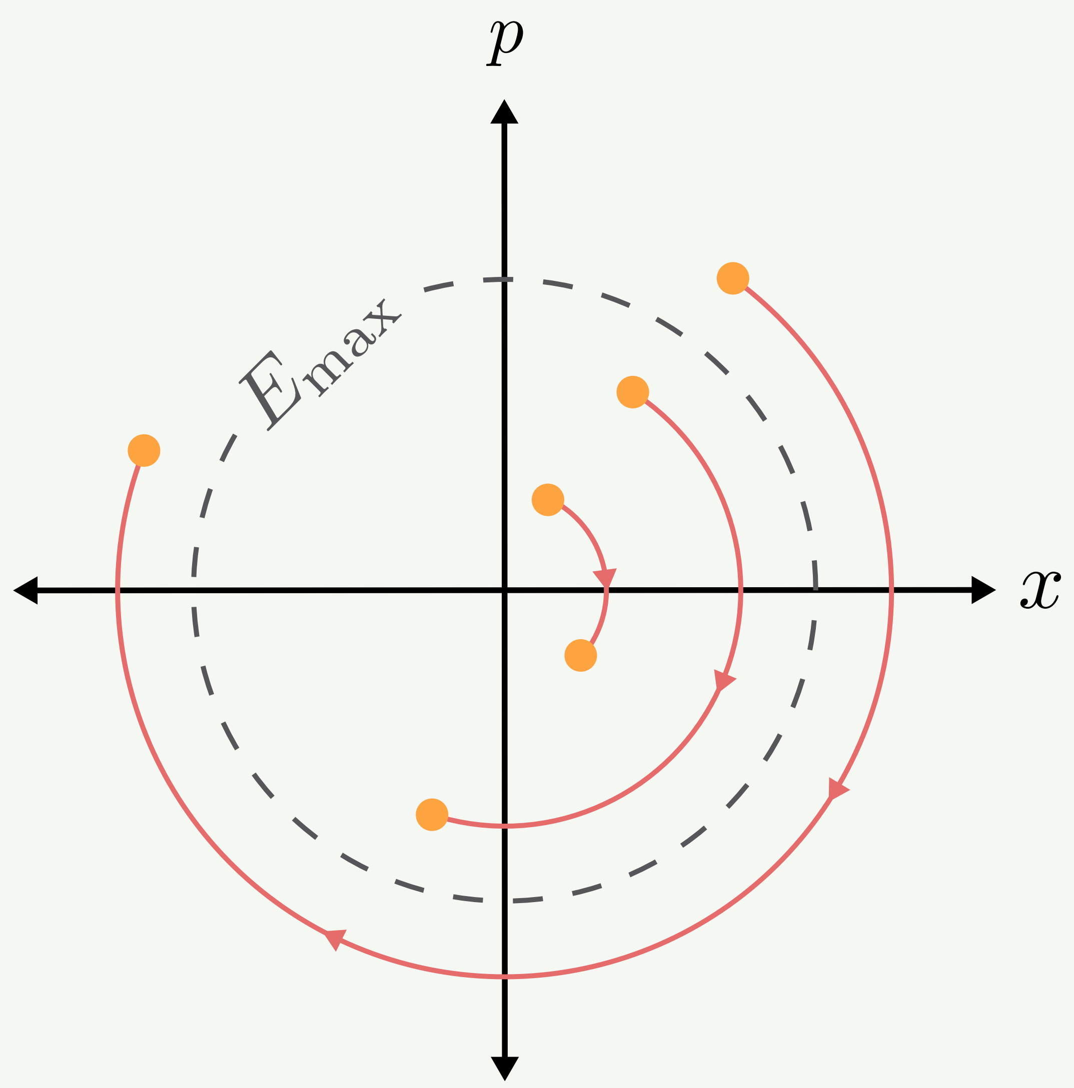
Describes systems with Kerr nonlinearities, transmon systems in the dispersive regime

Classical solution given by

$$x(t) = x(0)\cos(\omega(E)t) + \frac{p(0)}{m\omega}\sin(\omega(E)t)$$

With energy-dependent frequency

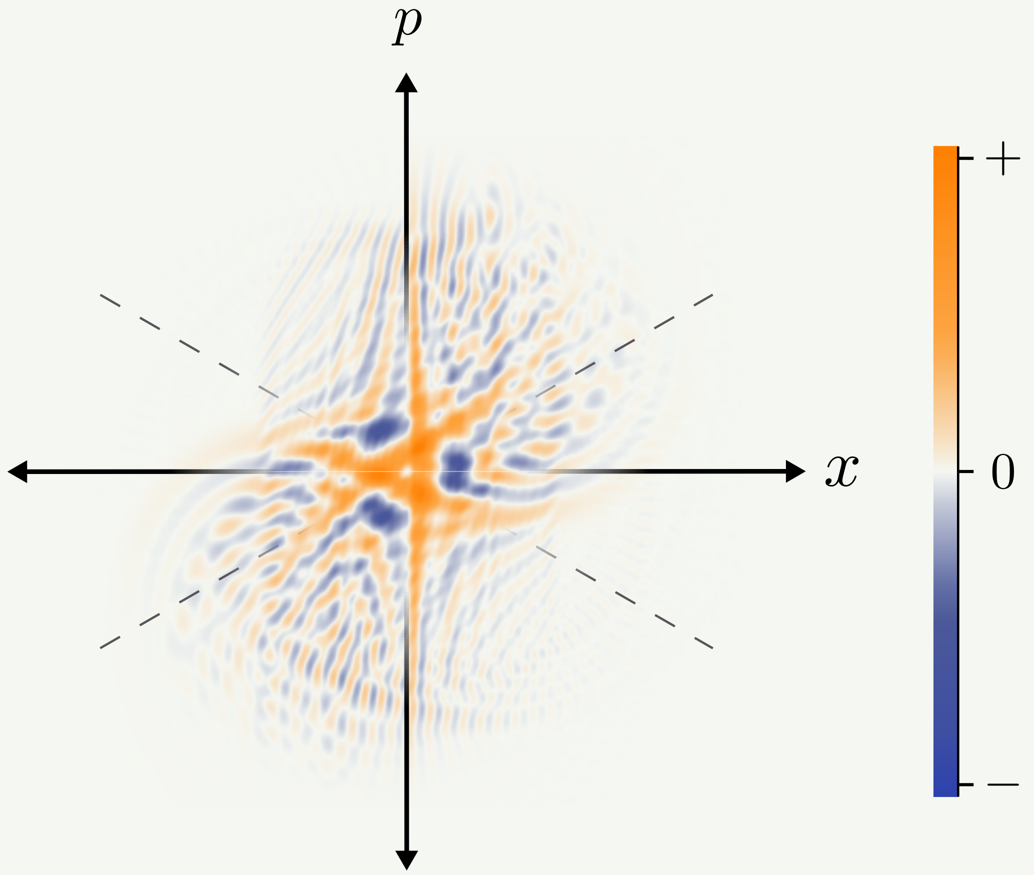
$$\omega(E) = \sqrt{1 + \frac{2\alpha E}{\hbar\omega_0^2}}\,\omega_0$$



Anharmonic example: Kerr-like Hamiltonian

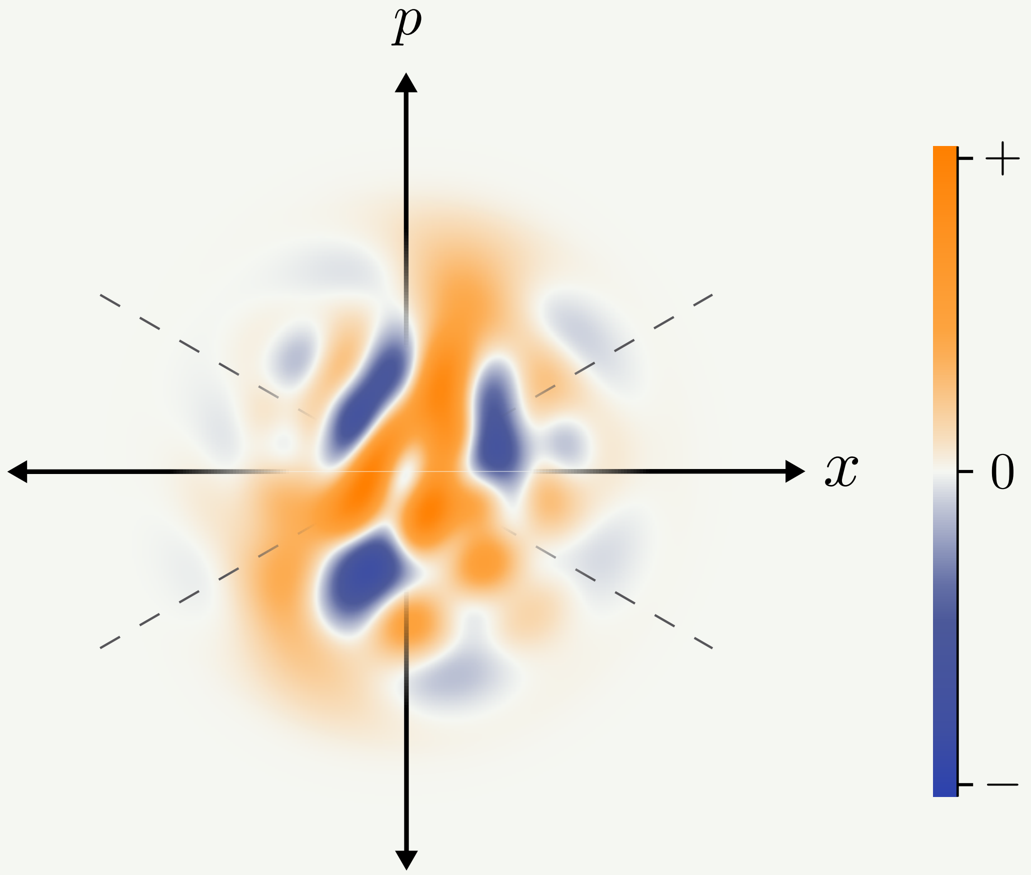
Performing the protocol with $\mathcal{T} = \frac{2\pi}{\omega_0}$ and $E_{\max} = \frac{5\hbar\omega_0^2}{8\alpha}$, classical bound unchanged

$\alpha/\omega_0 = 0.005$



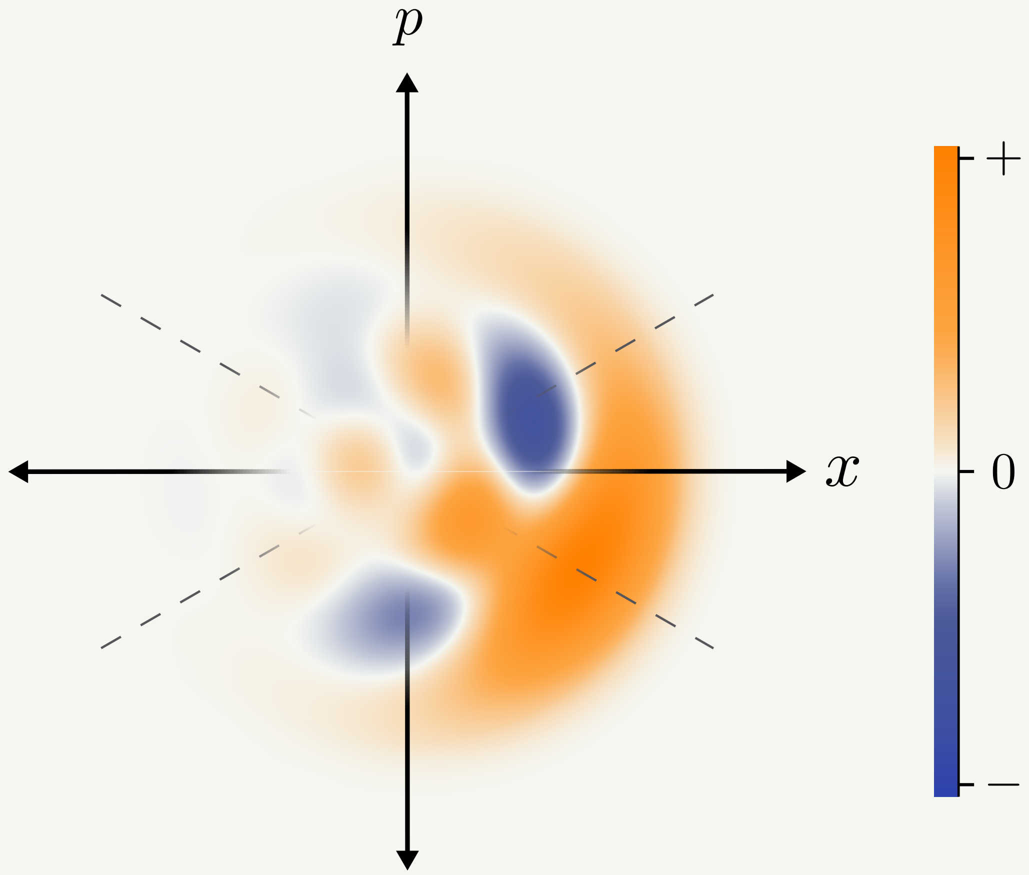
$\mathbf{P}_3^\infty = 0.703$

$\alpha/\omega_0 = 0.05$



$\mathbf{P}_3^\infty = 0.685$

$\alpha/\omega_0 = 0.1$

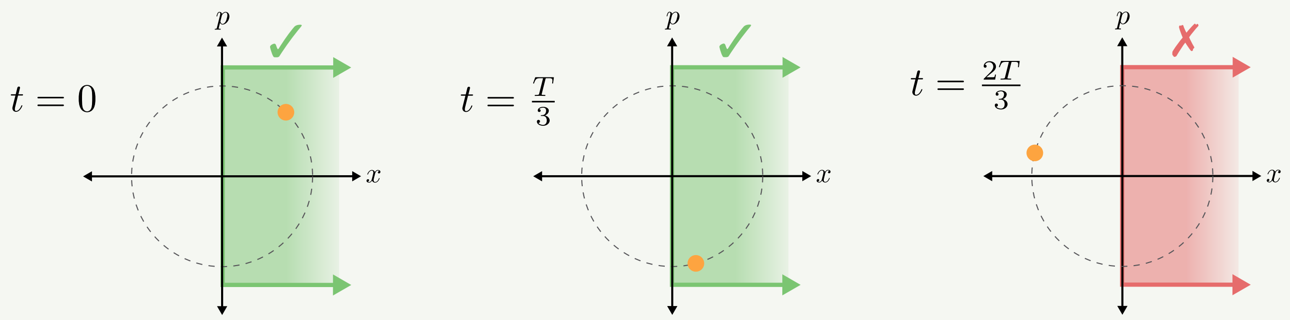


$\mathbf{P}_3^\infty = 0.674$

No quantum gap when $\alpha/\omega_0 \geq 1/9$

Conclusion

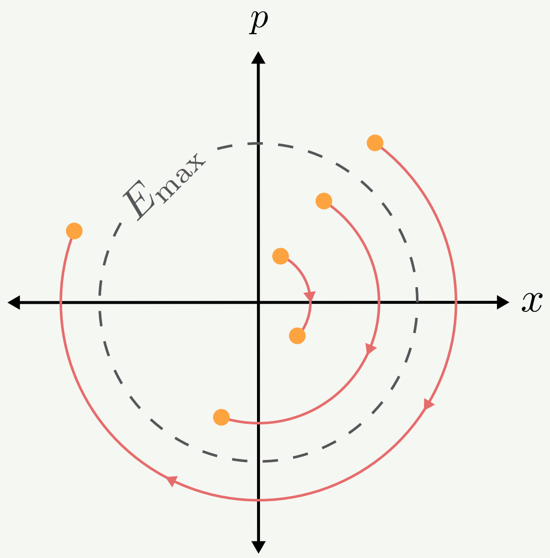
- **To detect quantumness**: Ask "*How often is the coordinate of a uniformly-precessing variable positive?*"



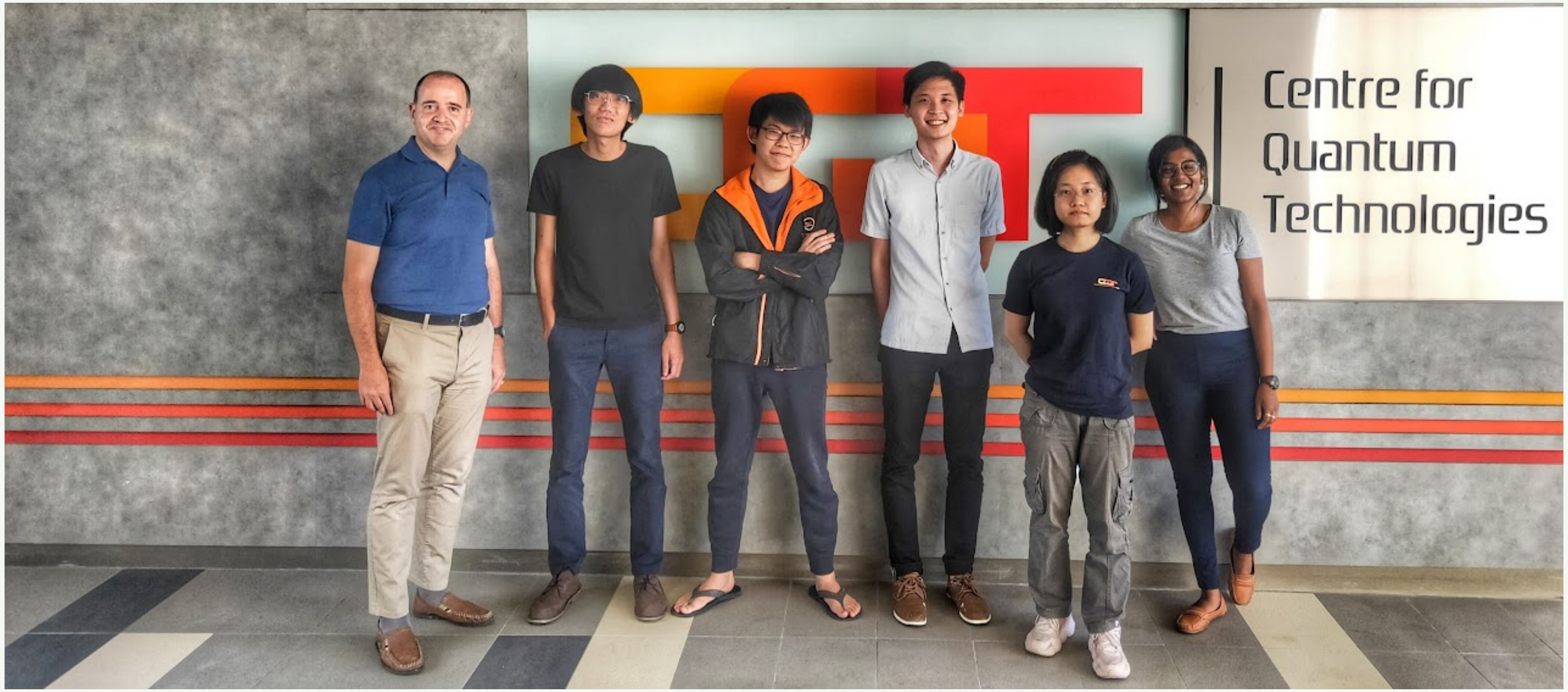
- **To witness entanglement**: Weighted sum of coordinates of two harmonic oscillators

$$P_K = \frac{1}{K} \sum_{k=0}^{K-1} \Pr(\sqrt{w_1} x_1(t_k) + \sqrt{w_2} x_2(t_k) > 0)$$

- **For anharmonic systems**: Include assumption of energy bounds



Thank you for your attention!



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