General Quantum Resource Theories

Maximal Resources, Catalytic Replication, and Asymptotically Consistent Measures

Hayata Yamasaki [1, 2], **Kohdai Kuroiwa** [3]
1: IQOQI Vienna, Austrian Academy of Sciences, 2: Atominstitute, TU Wien, 3: University of Waterloo

hayata.yamasaki@gmail.com @hayatayamasaki

YITP international workshop, Quantum Information Entropy in Physics, 24th March, 2022

https://arxiv.org/abs/2002.02458 Quantum 4, 355 (2020)
https://arxiv.org/abs/2103.05665 Physical Review A Letter 104, L020401 (2021)

Quantum resource theories (QRTs)

A unified framework for analyzing inherent quantum properties

Quantum properties = Resources to overcome restriction on quantum operations

Example: entanglement, coherence, magic states, non-Gaussianity, athermality…

Central topics:

- **Manipulation**: How can we transform resources by free operations?
- **Quantification**: How can we quantify resources?

E. Chitambar, G. Gour, Rev. Mod. Phys. 91, 025001 (2019)

Challenge in Establishing General QRTs

What are universal properties of quantum resources shared among QRTs?

General QRTs: Investigating universal properties shared among many resources

Previous works on general QRTs with **limited applicability**

Mathematical assumptions have been considered to make the analysis tractable

e.g. uniqueness of a maximal resourceful state, convexity, finite-dimensionality

E.g., F.G.S.L. Brandão et al. (2015), Z.W. Liu et al. (2019), R. Takagi et al. (2019), B. Regula et al. (2020),…

Challenge: **Physically well-motivated resources** may not satisfy assumptions to make analysis tractable

- **Non-uniqueness of max resources**: magic on qutrits, coherence with physically incoherent operations
- **Non-convexity**: non-Gaussianity, quantum discord, quantum Markov chain
- **Infinite-dimensionality**: non-Gaussianity, entanglement in quantum field theory

This work: General QRTs with Minimal Axioms

Aim: To investigate manipulation/quantification of as general resources as possible

A QRT is specified by free operations

Axioms on free operations: (no convexity or finite-dimensionality imposed)

1. Free operations can be used at any time in any order. (Closed under **composition**)
2. Free operations can be used regardless of other systems. (Closed under **tensor product**)
3. Doing nothing is free. (The **identity map** is free)
4. Ignoring systems is free. (The **trace** is free)

\+ **Compact sets** of states and free operations

(The choice of free operations specifies free states) $\sigma : \text{free} \Leftrightarrow \forall \rho \xrightarrow{\text{free operation}} \sigma$

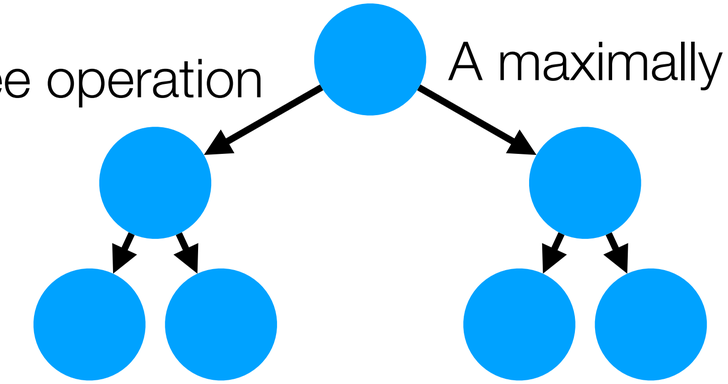
Main results: Universal structures of resources with full **generality** (next slides)

https://arxiv.org/abs/2002.02458

https://arxiv.org/abs/2103.05665

Result 1/3: Existence of Maximally Resourceful States

Crucial role as the unit in quantifying resources



An order relation

$\mathcal{E}(\rho) = \sigma$, $\mathcal{E}$: free$\Leftrightarrow \rho \succ \sigma$

Maximal = Most resourceful among comparable states

Problem

- **Non-unique** in general: magic on qutrits, coherence with physically incoherent operations
- **Existence is not obvious**: infinite-dimensional case

Theorem: Maximally resourceful states always exist in all QRTs in our framework.

Proof technique: a mathematical theorem from the theory of topology and ordered sets

+ the assumption of **compactness** of states and free operations

https://arxiv.org/abs/2002.02458

Result 2/3: Catalytic Replication of Resources

A novel form of catalytic transformation in general QRTs

Catalytic replication: asymptotic transformation of a state into many copies

$$\rho \xrightarrow{\text{asymptotic}} \rho^{\otimes r}$$

r : conversion rate

Proposition: In catalytic replication, $r(\rho \to \rho) = 1 \text{ or } \infty$

Our main contributions: Discovery of **catalytically replicable states**

- Naturally, free states are infinitely replicable
- **Indeed, we discover a case where a resource state is infinitely replicable**

  Example of QRTs with catalytically replicable states: $\{|0\rangle\langle 0|, \mathrm{CNOT}, \mathrm{id}, \mathrm{Tr}\}$, QRT of imaginality

https://arxiv.org/abs/2002.02458

Result 3/3: Consistent Measures of Resources

A good and broadly applicable way of quantifying quantum resources

Resource measure: a family of functions of states with **monotonicity** under free operations

Uniqueness inequality for entanglement [M. J. Donald, M. Horodecki, O. Rudolph (2001)]

Distillable entanglement:

Distillation into **ebits**

$$E_{\mathrm{D}}(\rho) \leqq E(\rho) \leqq E_{\mathrm{C}}(\rho)$$

Entanglement cost:

Formation from **ebits**

Any measure with **conventional properties**:

normalization, asymptotic continuity, additivity

Problem: but not simply applicable to QRTs with **non-unique maximally resourceful states**

Result: Uniqueness inequality for general resources

Distillable resource:

Distillation into the **hardest resource**

$$R_{\mathrm{D}}(\rho) \leqq R(\rho) \leqq R_{\mathrm{C}}(\rho)$$

Resource cost:

Formation from the **easiest resource**

(Applicable even to infinite-dimensional cases)

https://arxiv.org/abs/2002.02458

Asymptotically Consistent Resource Measures

No resource measure may satisfy these properties simultaneously

Problem (our result):

In QRT of magic on qutrits, no measure satisfies the conventional properties simultaneously

normalization, asymptotic continuity, additivity

→ These properties can be inconsistent in general

Solution: **Asymptotically consistent resource measures**

$R(\rho) \geqq R(\sigma) r(\rho \to \sigma)$: consistent with asymptotic conversion rate $r(\rho \to \sigma)$

Theorem: Any asymptotically consistent measure under an appropriate normalization condition satisfies the uniqueness inequality $R_{\mathrm{D}}(\rho) \leqq R(\rho) \leqq R_{\mathrm{C}}(\rho)$

https://arxiv.org/abs/2002.02458

https://arxiv.org/abs/2103.05665

Wide Applicability of Consistent Measures

Applicable even to QRTs without convexity and finite-dimensionality

Consistency with asymptotic conversion rate $R(\rho) \geqq R(\sigma) r(\rho \to \sigma)$

Example: **Regularized relative entropy of resources** $R_{\mathrm{R}}(\rho) = \lim_{n\to\infty} \frac{1}{n} \min_{\sigma:\ \mathrm{free}} \{D(\rho^{\otimes n} || \sigma)\}$

Wide applicability = A theoretical foundation for quantitative studies

- **All convex and finite-dimensional** QRTs
- **Non-convexity**: quantum discord, quantum Markov chain, (Counterexample in general)
- **Infinite-dimensionality**: non-Gaussianity (for a convex version)

Proof technique: subadditivity & asymptotic continuity of relative entropy of resources

https://arxiv.org/abs/2103.05665

Conclusion

Framework of QRTs in highly general form, covering non-convex & infinite-dimensional cases

Main results

1. Proof of **existence of maximally resourceful states** in general framework
2. Manipulation: **Catalytic replication** of resource states
3. Quantification: Investigation of **asymptotically consistent** resource measures

- **Broadly applicable to QRTs of physically well-motivated resources**

  including magic on qudits, non-Gaussianity, non-Markovianity, discord, entanglement with infinite-dim
- Foundation for quantitatively studying a much broader class of quantum properties

  through a unified approach of QRTs

https://arxiv.org/abs/2002.02458 Quantum 4, 355 (2020)
https://arxiv.org/abs/2103.05665 Physical Review A Letter 104, L020401 (2021)

This work was supported by CREST (Japan Science and Technology Agency) JPMJCR1671, Crossministerial Strategic Innovation Promotion Program (SIP) (Council for Science, Technology and Innovation (CSTI)), JSPS Overseas Research Fellowships, JSP PRESTO Grant Number JPMJPR201A, Mike and Ophelia Lazaridis, and research grants by NSERC.

hayata.yamasaki@gmail.com @hayatayamasaki

Thank you for your attention.