Long-time behavior of chaotic quantum systems: complexity growth and entropy fluctuations

Nick Hunter-Jones

Stanford University

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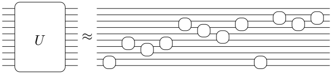
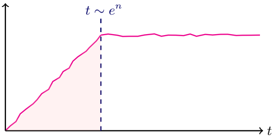
Quantum Information Entropy in Physics
Workshop at the YITP, Kyoto

Based on:
[Brandão, Chemissany, NHJ, Kueng, Preskill], PRX Quantum, 1912.04297
wip with Oszmaniec, Horodecki, 2203.verysoon
[Cotler, NHJ, Ranard], PRA, 2010.11922

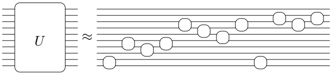
Based on:

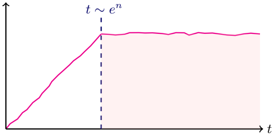
1) with F. Brandão, W. Chemissany, R. Kueng, J. Preskill
"Models of quantum complexity growth," arXiv:1912.04297



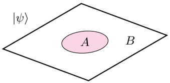


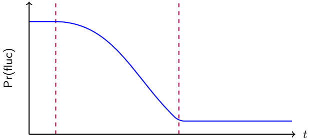
2) with M. Oszmaniec, M. Horodecki, "Saturation and recurrence of quantum complexity for random quantum circuits," arXiv:2203.hopefullyverysoon





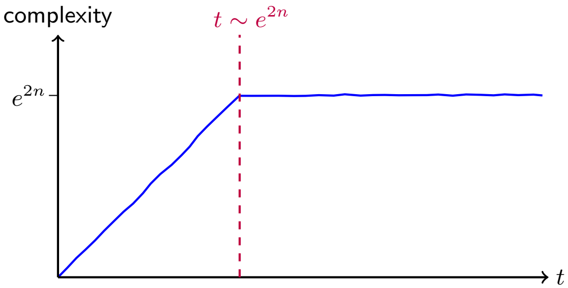
3) with J. Cotler, D. Ranard, "Fluctuations of subsystem entropies at late times," arXiv:2010.11922





Two long-time properties of many-body systems

1) Quantum complexity growth



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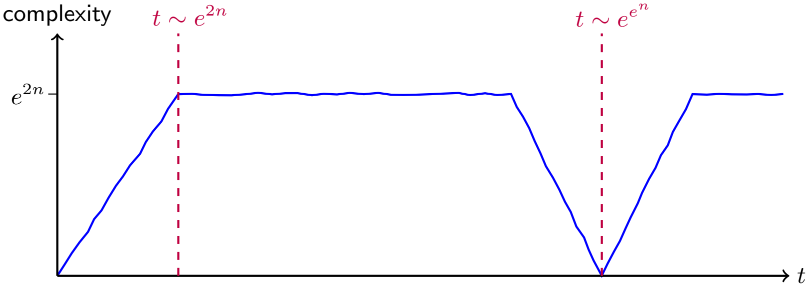
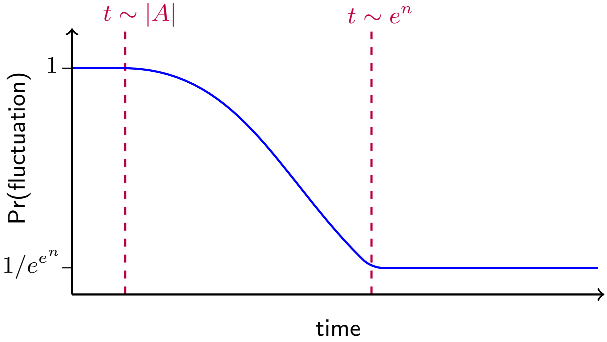


Two long-time properties of many-body systems

1) Quantum complexity growth

2) Subsystem entropy fluctuations



Overview

- Define quantum complexity
- Complexity by design
- Complexity of local random quantum circuits
- Complexity saturation and recurrence for RQCs
- Subsystem entropy fluctuations

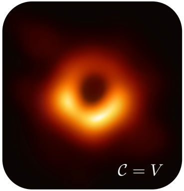
Quantum complexity growth

Quantum complexity

Quantum complexity is an important and well-established notion in QI

Recent interest in quantum many-body physics:

- distinguish topological phases of matter at zero temperature [Chen, Gu, Wen]
- describe regions behind black hole horizons in AdS/CFT [Susskind], [Stanford, Susskind]



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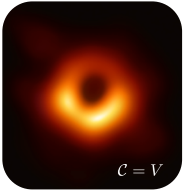
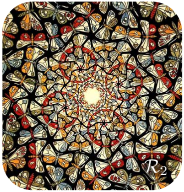


More generally, complexity growth is one universal aspect of real-time dynamics in strongly-interacting many-body systems

$\rightarrow$ relation to thermalization, quantum chaos, ...

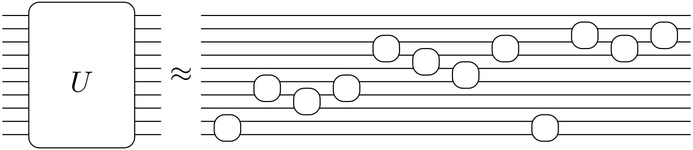


Complexity

some intuition

Complexity is a somewhat intuitive notion

The traditional definition involves building a circuit with gates drawn from a universal gate set, which implements the state or unitary to within some tolerance δ



We are interested in the minimal size of a circuit that achieves this

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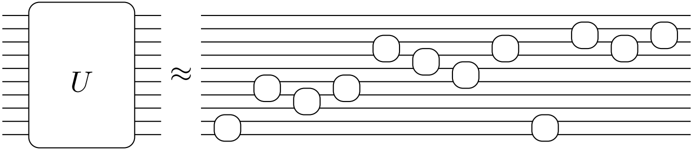


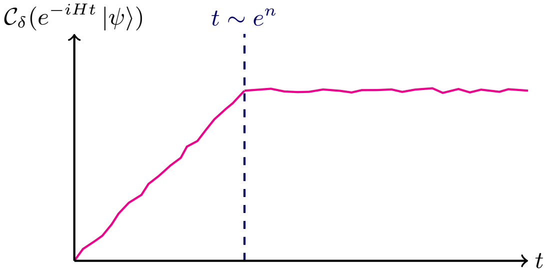
We are interested in the minimal size of a circuit that achieves this

Consider systems of n qudits (with local dim q), such that $d = q^n$

Complexity

some expectations

It is believed(/expected/conjectured) that the complexity of a simple initial state, grows (possibly linearly) under the time-evolution by a chaotic Hamiltonian



saturating after an exponential time

Complexity

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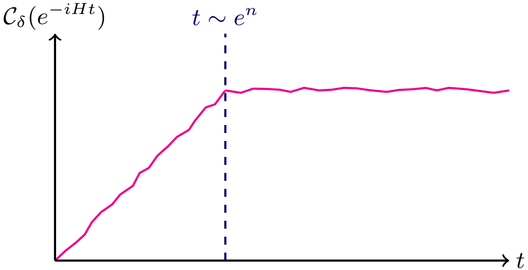
It is believed(/expected/conjectured) that the complexity e^{-iHt} grows (possibly linearly) for a chaotic Hamiltonian H



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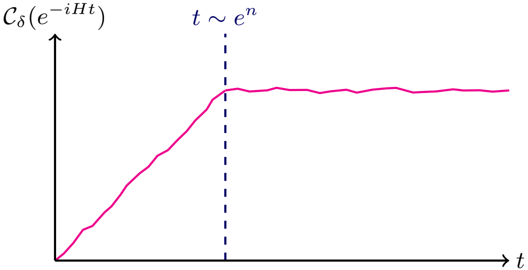
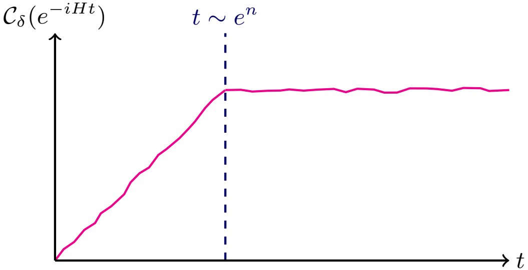


saturating after an exponential time

computing the quantum complexity analytically is very hard (especially for a fixed H)

Complexity

some expectations



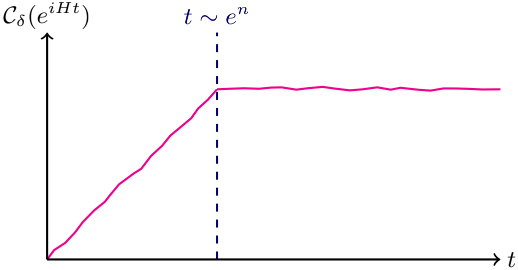
Why?

polynomial/linear growth: early time collisions should be rare; upper bounds on growth from Hamiltonian simulation algorithms

saturation: counting δ-balls in $U(d)$, doubly exp $\left(\sim (1/\delta)^{2^{2n}}\right)$ 'distinct' unitaries, and thus can reach any unitary with a depth $t \sim e^{2n}$ circuit

Complexity

some expectations



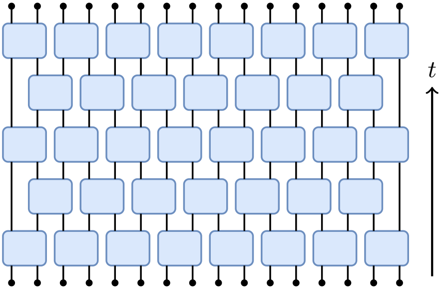
To make progress:

$\rightarrow$ use complexity theoretic assumptions to make statements about the complexity of a particular Hamiltonian evolution at exponentially long times [Aarsonson], [Susskind], [Bohdanowicz, Brandão]

$\rightarrow$ focus on ensembles of time-evolutions (RQCs)

Our goal

Consider random quantum circuits, on n qudits of local dimension q, evolving with staggered layers of 2-site unitaries, each drawn randomly from a gate set G

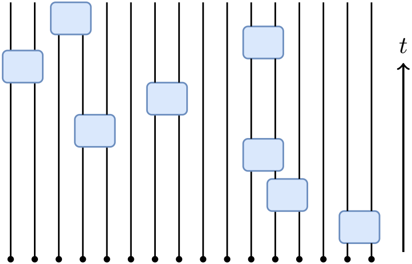


where evolution to time t is given by $U_t = U^{(t)} \dots U^{(1)}$

and try to prove the growth of complexity in this model

Our goal

Consider random quantum circuits, on n qudits of local dimension q, evolving with random nearest-neighbor 2-site unitaries, each drawn randomly from a gate set G



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Complexity growth in RQCs

Specifically, it has been conjectured that

Conjecture [Brown, Susskind], [Susskind]

Most local random quantum circuits of depth t have a complexity that scales *linearly* in t for an exponentially long time.

This sounds reasonable, but is hard to prove: one needs to show that collisions between circuits of subexponential size are rare.

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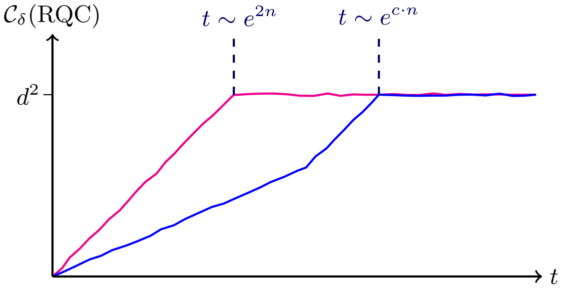
Another approach: focus on the *exact* circuit complexity
[Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern]

Complexity growth in RQCs

(some results)

We expect that complexity grows linearly in time, saturating after an exponential time

What we prove for RQCs on n qubits

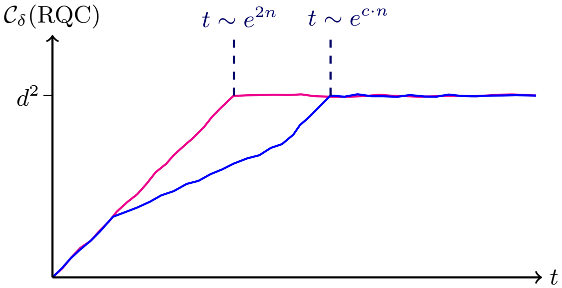


Complexity growth in RQCs

(some results)

We expect that complexity grows linearly in time, saturating after an exponential time

What we prove for RQCs on n qudits (large q)

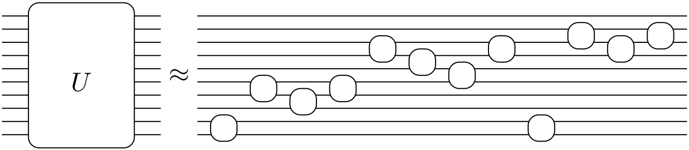


Unitary complexity

Consider a system of n qudits with local dimension q, where $d = q^n$.

Complexity of a unitary: the minimal size of a circuit, built from elementary 2-local gates, that approximates the unitary U

We assume the circuits are built from 2-local gates chosen from a universal gate set G. Let G_r denote the set of all circuits of size r



where $\square \in G$

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Complexity of a unitary

We say that a unitary $U \in U(d)$ has δ-complexity $\mathcal{C}_\delta(U) = r$ if and only if

$$r = \min\left\{r' : \exists V \in G_{r'} \text{ s.t. } \|U - V\| \leq \delta\right\}$$

(where the distance used is $\|\mathcal{U} - \mathcal{V}\|_\diamond$ and $\mathcal{U} = U(\rho)U^\dagger$)

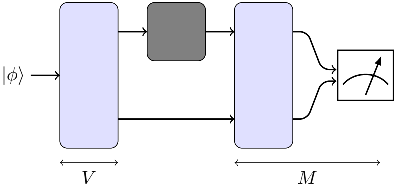
Complexity from measurements

We can consider an alternative (stronger) definition of the complexity of a state or unitary, in terms of an optimal distinguishing measurement

Roughly, the strong complexity of U is the minimal circuit required to implement an ancilla-assisted measurement capable of distinguishing $\mathcal{U}$ from the completely depolarizing channel $\mathcal{D}$

Task is to distinguish the channels with restricted state preparation and measurements as

$$\begin{aligned} \text{maximize} \quad & \left|\mathrm{Tr}\Big(M\big((\mathcal{U}\otimes\mathcal{I})|\phi\rangle\langle\phi| - (\mathcal{D}\otimes\mathcal{I})|\phi\rangle\langle\phi|\big)\Big)\right| \\ \text{subject to} \quad & M \in M_{r'}, \ \ |\phi\rangle = V\,|0\rangle\,, \ V \in G_r \end{aligned}$$



Complexity by design

We are interested in the complexity of random quantum circuits

To make progress we can derive some general statements about the complexity of unitary k-designs

But first, we need to define the notion of a unitary design

Unitary k-designs

Haar: (unique L/R invariant) measure on the unitary group $U(d)$

k-fold channel: $\Phi_{\mathcal{E}}^{(k)}(\mathcal{O}) \equiv \sum_i p_i U_i^{\otimes k}(\mathcal{O}) U_i^{\dagger \otimes k}$

exact k-design: $\Phi_{\mathcal{E}}^{(k)}(\mathcal{O}) = \Phi_{\text{Haar}}^{(k)}(\mathcal{O})$

but for general k, few exact constructions are known

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Approximate k-design

For $\epsilon > 0$, an ensemble $\mathcal{E}$ is an ϵ-approximate k-design if the k-fold channel obeys

$$\left\| \Phi_{\mathcal{E}}^{(k)} - \Phi_{\mathrm{Haar}}^{(k)} \right\|_{\diamond} \leq \epsilon$$

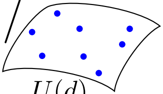
$\rightarrow$ designs are powerful

Unitary k-designs

If an ensemble of unitaries $\mathcal{E}$ forms an approximate k-design

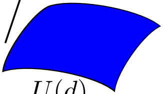
the average over $\mathcal{E}$ is close to the average over the full unitary group up to the k-th moment

$$\left\langle \ldots \underbrace{U \ldots U^\dagger}_{k \text{ of these}} \ldots \right\rangle$$



$U(d)$

$$\approx \left\langle \ldots \underbrace{U \ldots U^\dagger}_{k \text{ of these}} \ldots \right\rangle$$

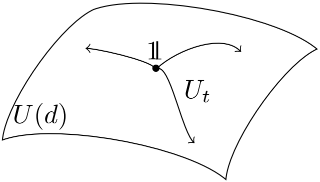


$U(d)$

Intuition for k-designs
(eschewing rigor)

How random is the time-evolution of a system compared to the full unitary group $U(d)$?

Consider an ensemble of time-evolutions at a fixed time t: $\mathcal{E}_t = \{U_t\}$
e.g. RQCs, Brownian circuits, or $\{e^{-iHt},\ H \in \mathcal{E}_H\}$ generated by disordered Hamiltonians



quantify randomness:
when does $\mathcal{E}_t$ form a k-design?
(approximating moments of $U(d)$)

Complexity by design

an exercise in enumeration

Consider an approximate unitary k-design $\mathcal{E}_k = \{p_i, U_i\}$

Can we say anything about the complexity of U_i's?

The structure of a design is sufficiently restrictive, can bound the complexity of design elements count the number of unitaries of a specific complexity

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Can prove the following two (informal) statements:

- with high prob, a unitary U drawn from an ϵ-approx k-design $\mathcal{E}$ has complexity $\mathcal{C}_\delta(U) \approx nk$
- an ϵ-approx k-design $\mathcal{E}$ contains an exp # ($\sim e^{nk}$) of *distinct* unitaries with this complexity

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an exercise in enumeration

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Complexity for unitary designs

With probability $\geq 1 - e^{-nk}$, a unitary $U \sim \mathcal{E}_k$ drawn from an ϵ-approximate k-design has

$$\mathcal{C}_\delta(U) \geq \frac{1}{\log n|G|}\big(nk \log q - \log(1+\epsilon) + k\log(1+\delta^2)\big)$$

RQCs and randomness

Consider local RQCs on n qudits, with gates drawn randomly from a universal gate set G

Now we need a powerful result from [Brandão, Harrow, Horodecki]

RQCs form approximate designs

For $k \leq \sqrt{d}$, the set of local random quantum circuits of depth t forms an ϵ-approximate unitary k-design if

$$t \geq ck^{11}(n + \log(1/\epsilon))$$

where c is a constant

i.e. RQCs of depth $t = O(nk^{11})$ form k-designs

Complexity by design

Now we can combine these two results to say something about the complexity of local random circuits

- with very high probability, a local RQC of depth t, has complexity $C_\delta \gtrsim n(t/n)^{1/11}$
- the set of depth t local RQCs contains an exp number of distinct unitaries with this complexity

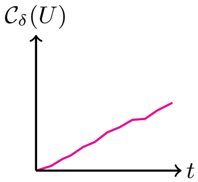
Complexity by design

Now we can combine these two results to say something about the complexity of local random circuits

- with very high probability, a local RQC of depth t, has complexity $C_\delta \gtrsim n(t/n)^{1/11}$
- the set of depth t local RQCs contains an exp number of distinct unitaries with this complexity

This establishes a polynomial relation between the growth of complexity and depth of the circuit up to exponential times $t \leq \sqrt{d} = q^{n/2}$

$\rightarrow$ but what we really want is linear growth

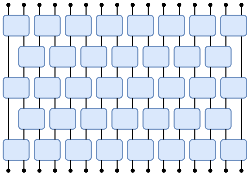


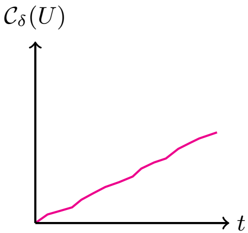
Complexity growth from design growth

What we have shown is that

$$\text{complexity} \sim k$$

Using [Brandão, Harrow, Horodecki], local RQCs form k-designs in $t = O(nk^{11})$ depth, gives complexity $\sim t^{1/11}$





Complexity growth from design growth

Using results about the design growth for various models we can prove statements about their complexity growth:

- Stochastic quantum Hamiltonian/Brownian quantum circuit
  [Onorati, Buerschaper, Kliesch, Brown, Werner, Eisert]

$$H(t) = \sum_{j<k} \sum_{\alpha,\beta} \mathcal{J}_{jk}^{\alpha\beta}(t)\, \sigma_j^\alpha \sigma_k^\beta$$

design depth $t = O(nk^{11})$ gives complexity growth $\sim t^{1/11}$

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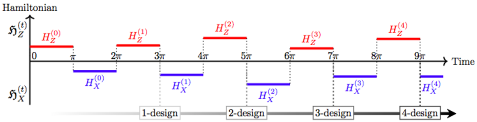
  design depth $t = O(nk^{11})$ gives complexity growth $\sim t^{1/11}$
- Nearly time-independent Hamiltonian dynamics
  [Nakata, Hirche, Koashi, Winter]



k-designs in $O(n^2k)$ steps (up to $k = \sqrt{n}$) gives short time complexity growth $\sim t$

RQCs and $t \sim k$

an appeal for linearity

To get a linear growth in complexity we need a linear growth in design

$$\text{complexity} \sim k \sim t$$

we had $t = O(nk^{11})$, but would need $t = O(nk)$

A lower bound on the k-design depth for these RQCs is $\Omega(nk)$

Can we prove that RQCs saturate this lower bound? (and are thus optimal implementations of k-designs)

Design growth in RQCs

Theorem (Design growth at large q) [NHJ]

RQCs on n qudits form ϵ-approximate k-designs when

$$t \geq 4nk + \log 1/\epsilon \quad \rightarrow \quad t = O(nk)$$

for some $q \geq q_0$, where q_0 depends on the size of the circuit

Theorem (Design growth for $q = \Omega(k^2)$) [Haferkamp, NHJ]

RQCs on n qudits with $q \geq 6k^2$ form ϵ-approximate k-designs when

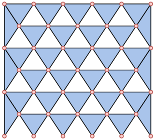
$$t \geq 18(2nk \log q + \log 1/\epsilon) \quad \rightarrow \quad t = O(nk \log k)$$

Designs from domain walls and gaps

Two approaches to computing the design depth for RQCs:

1) Partition function of a lattice model

$$Z = \sum_{\{\sigma\}}$$



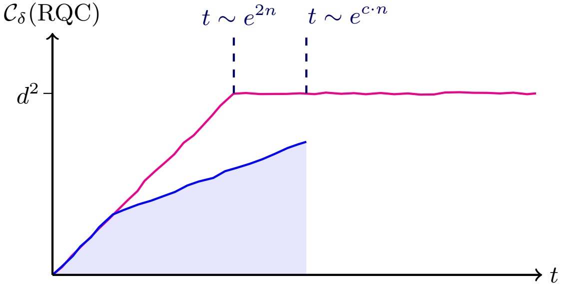
$$\leq \quad ?$$

2) Spectral gap of a local Hamiltonian

$$\Delta(H_{n,k}) \quad \geq \quad ?$$

Towards linear complexity growth

This makes some progress on the conjecture for random quantum circuits with large local dimension q



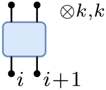
i.e. complexity is growing linearly in time t

Linear growth from small gaps

For RQCs, the spectral gap enters as [Brown, Viola], [Brandão, Horodecki]

$$\text{(distance to forming a design)} \leq d^{2k} \left(1 - \frac{\Delta(H_{n,k})}{n}\right)^{t}$$

where $H_{n,k}$ is a frustration-free Hamiltonian

$$H_{n,k} = \sum_{i=1}^{n} \Bigg(\mathbb{I} -$$


$$\Bigg)$$

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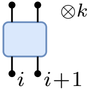
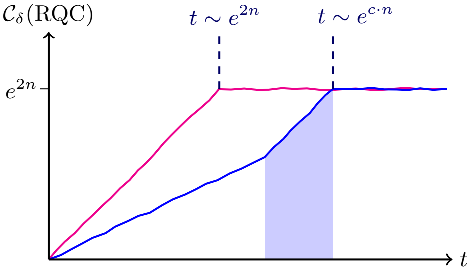
where $H_{n,k}$ is a frustration-free Hamiltonian

$$H_{n,k} = \sum_{i=1}^{n}\Bigg(\mathbb{I} -$$


$$\Bigg)^{\otimes k,k}$$

An exponentially-small, but k-ind, gap allows us to prove a linear complexity growth at late times

$(\Delta(H_{n,k}) \geq \Omega(e^{-c\cdot n}))$



Complexity saturation

How do we prove that complexity has saturated?

Haar random unitaries have maximal complexity, $C_\delta(U) \approx d^2$, but RQCs only approach Haar when $t \to \infty$

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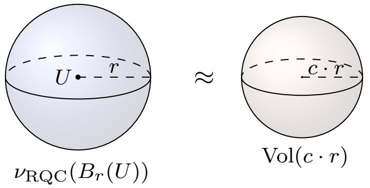
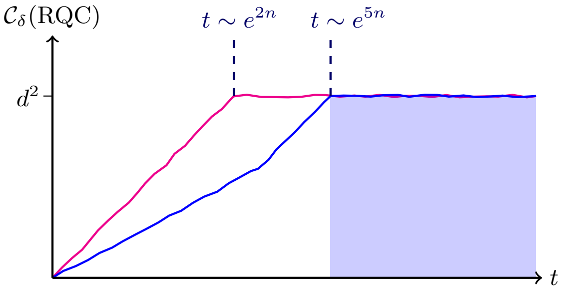
At exponential times ($t \sim e^{5n}$) RQCs equidistribute



(more formally, the measure assigned to balls by the ensemble of RQCs $\nu_{\mathrm{RQC}}(B_r(U)) \approx \mathrm{Vol}_{\mathrm{Haar}}(c \cdot r)$ for all $U \in U(d)$)

Complexity saturation

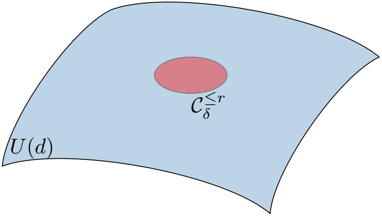
This allows us to show that

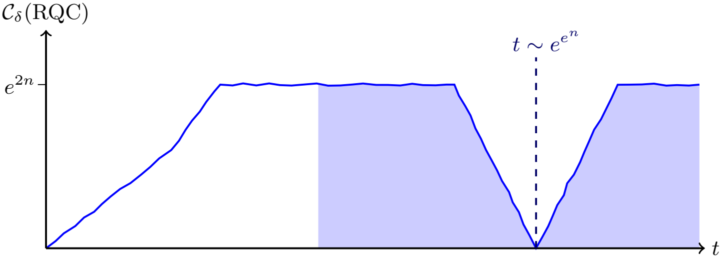


(can also prove that recurrences happen at doubly-exp times)

Explicit recurrence times

Once we achieve equidistribution, the probability of 'walking' to a particular unitary becomes $\approx$ that as prescribed by the Haar measure



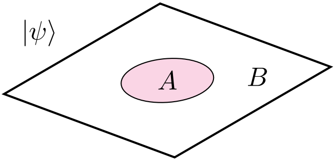


Subsystem entropy fluctuations

(a potential avatar of complexity)

Entropy fluctuations

Consider an n qubit system, initially in an unentangled state $|\psi\rangle$, which undergoes some evolution U_t (e.g. by e^{-iHt} for a chaotic H or RQC)



Consider the vN entropy ($S(\rho) = -\mathrm{tr}\,\rho \log \rho$) of a subsystem

$$\rho_A(t) = \mathrm{tr}_B\, U_t |\psi\rangle\langle\psi| U_t^\dagger$$

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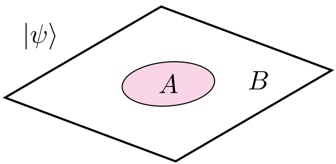

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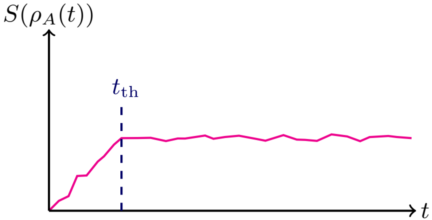

How often does the subsystem entropy fluctuate?

Entropy fluctuations

- How rare are entropy fluctuations after thermalization?
- How long must we wait (post-eq) to see an $O(1)$ fluctuation in the subsystem entropy $S(\rho_A(t))$?

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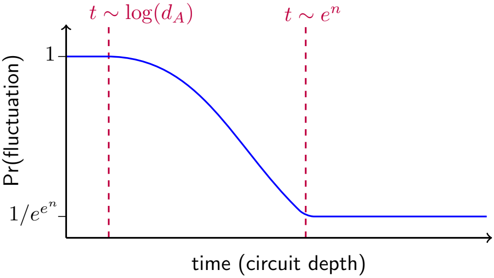
For RQCs, we prove that



Need to wait a doubly-exp long time to see a fluctuation

Entropy fluctuations

The (informal) theorem statements are

For 1D RQCs on n qubits of depth t, the entropy of the evolved state on the subsystem $\rho_A(t)$ obeys

$$\Pr\left(S(\rho_A(t)) \leq \log(d_A) - \delta\right) \lesssim \begin{cases} e^{-t} & t \leq e^n \\ e^{-e^n} & t > e^n \end{cases}$$

Let N_A^{ent} be the number of times t that a subsystem A satisfies $S(\rho_A(t)) \leq \log(d_A) - \delta$ for all times from $t = c_{\mathrm{th}} \log(d_A)$ up to $t = e^{c_{\mathrm{rec}} d}$, where $c_{\mathrm{th}} > 1$ and $c_{\mathrm{rec}} < 1$

For 1D RQCs on n qubits, and $n \geq \Omega(c_{\mathrm{th}} \log(d_A))$, the probability of an entropy fluctuation is bounded as

$$\Pr\left(N_A^{\mathrm{ent}} > 0\right) \lesssim \frac{1}{e^{\delta}} \frac{1}{d_A^{c_{\mathrm{th}}}}$$

(similar statements for the distance to the max mixed state)

Future science

- Can we prove anything about $\mathcal{C}_\delta(e^{-iHt})$ for a fixed Hamiltonian? or for an ensemble of Hamiltonians?
- Can we prove a linear design growth at small q (e.g. some constant local dimension) for an exponentially long times?
- Improved RQC gaps? would give closer to linear growth and earlier saturation time
- Connections between (the rarity of) subsystem entropy fluctuations and complexity growth in many-body systems?
- Study the pseudorandomness properties of other RQCs (e.g. charge conserving circuits [Khemani, Vishwanath, Huse], [Rakovszky, Pollmann, von Keyserlingk])
- Explore implications of strong definition of complexity (in terms of an optimal measurement) in holography and for many-body physics?

Thanks!