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# Entanglement transport and thermalization in an isolated many-body system

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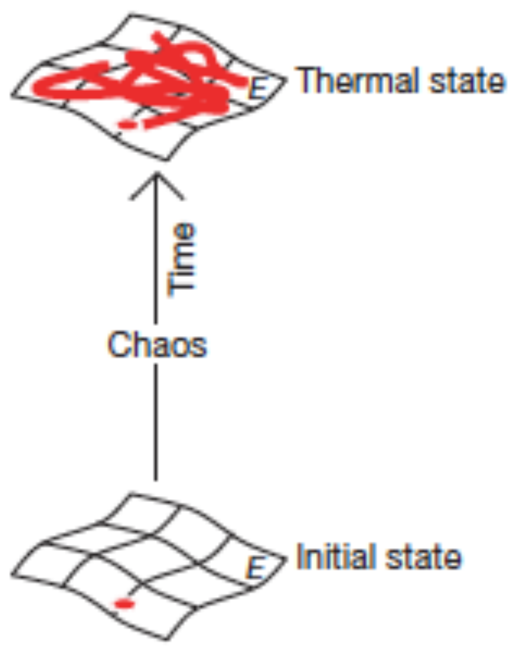
# Outline

- Introduction: Thermalization of isolated systems, transport of entanglement
- Model
- Time-evolution of entanglement entropy
- Discussion on transport of entanglement and thermalization
- Conjecture for the condition of thermalization
- Summary

# Introduction I: Thermalization of isolated systems

Classical system: ergodicity




chaotic nonlinear dynamics

Quantum system

?

unitary linear dynamics

What is the mechanism of thermalization in quantum systems?

# Thermalization in an isolated quantum system

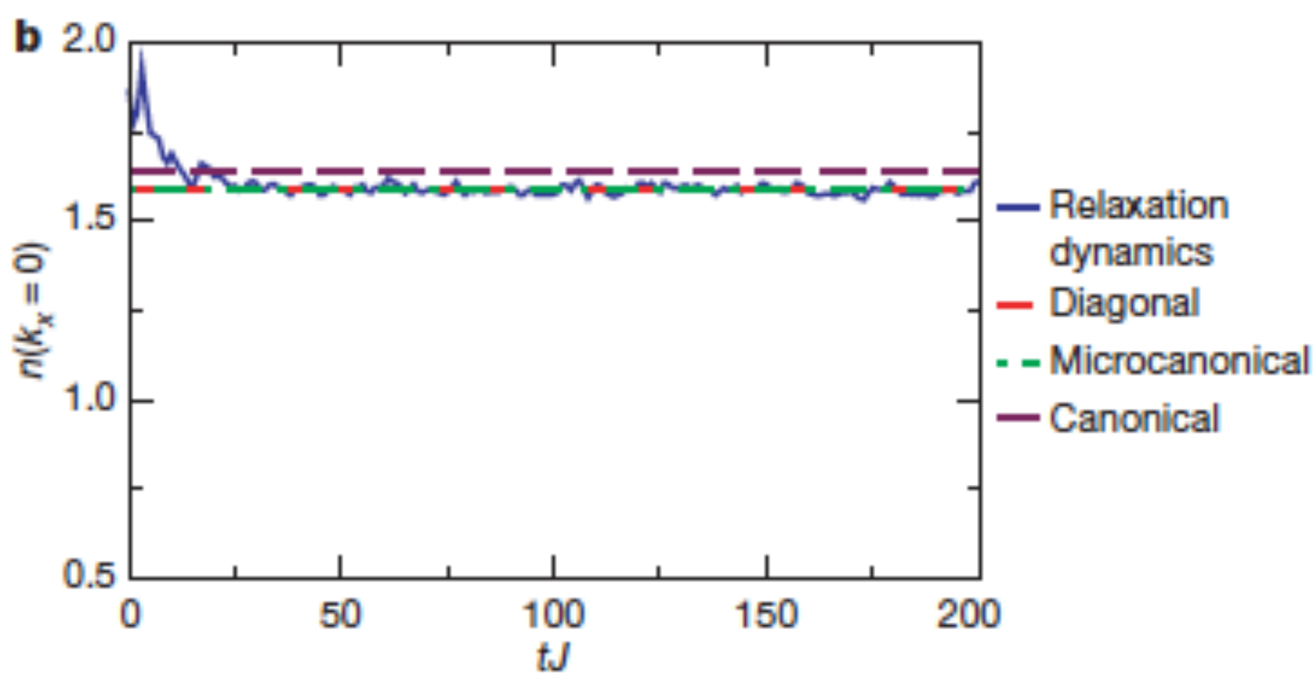
$$|\psi(0)\rangle = \sum_n a_n |n\rangle \quad \Longrightarrow \quad |\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \sum_n a_n e^{-iE_n t}|n\rangle$$

$$\langle A(t)\rangle = \langle\psi(t)|A|\psi(t)\rangle = \sum_{n,m} a_n^* a_m e^{i(E_n - E_m)t}\langle n|A|m\rangle$$

long-time average $\quad \overline{\langle A(t)\rangle} = \sum_n |a_n|^2 A_{nn} \equiv \langle A\rangle_{\mathrm{diagonal}}$

microcanonical average $\quad \langle A\rangle_{\mathrm{microcan}} \equiv \dfrac{1}{\mathcal{N}_{E_0,\Delta E}} \sum_{|E_n - E_0| < \Delta E} A_{nn}$

$E_0$ : average energy




M. Rigol et al., Nature 452, 854 (2008)

**Thermalization**

$$\langle A\rangle_{\mathrm{diagonal}} = \langle A\rangle_{\mathrm{microcan}}$$

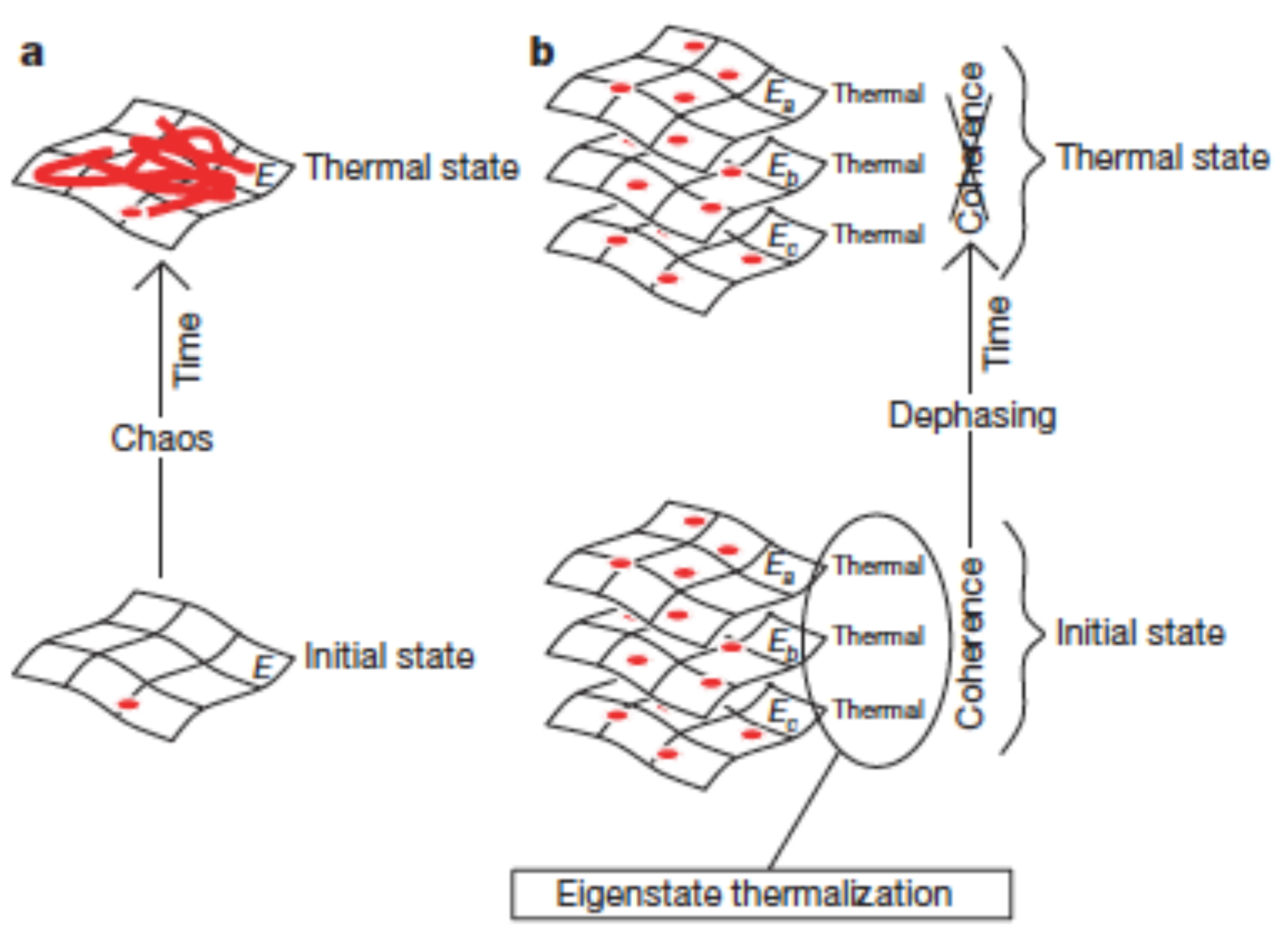
# Eigenstate thermalization hypothesis

$$\boxed{\langle n | A | n \rangle = \langle A \rangle_{\text{microcan}}}$$

J. M. Deutsch (1991); M. Srednicki (1994)

Eigenstate thermalization hypothesis (ETH):

Initial state implicitly contains a thermal state




ETH $\Rightarrow$ thermalization

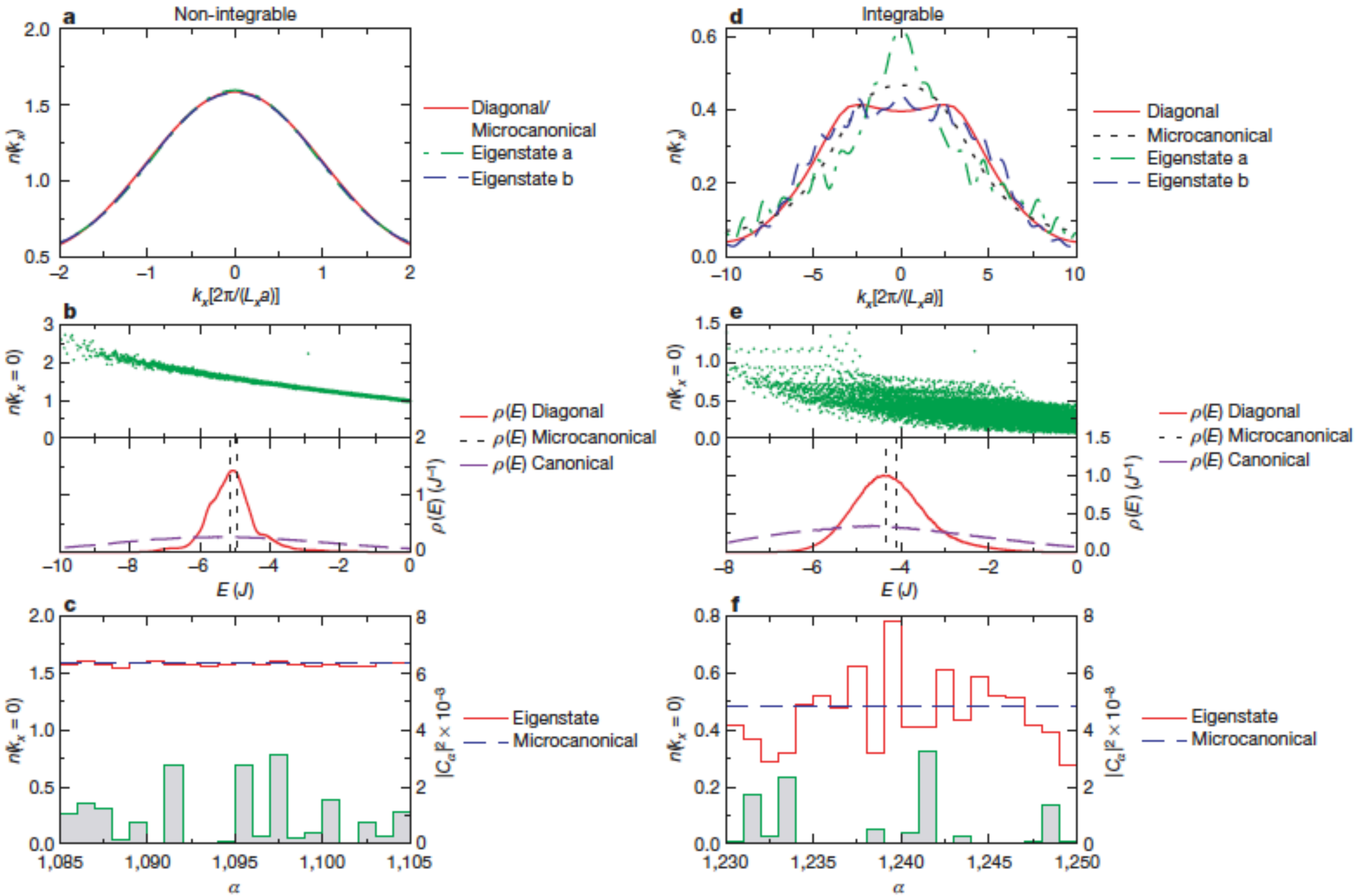
$$\langle A \rangle_{\text{diagonal}} = \sum_n |a_n|^2 A_{nn}$$

$$= \langle A \rangle_{\text{microcan}} \sum_n |a_n|^2$$

$$= \langle A \rangle_{\text{microcan}}$$

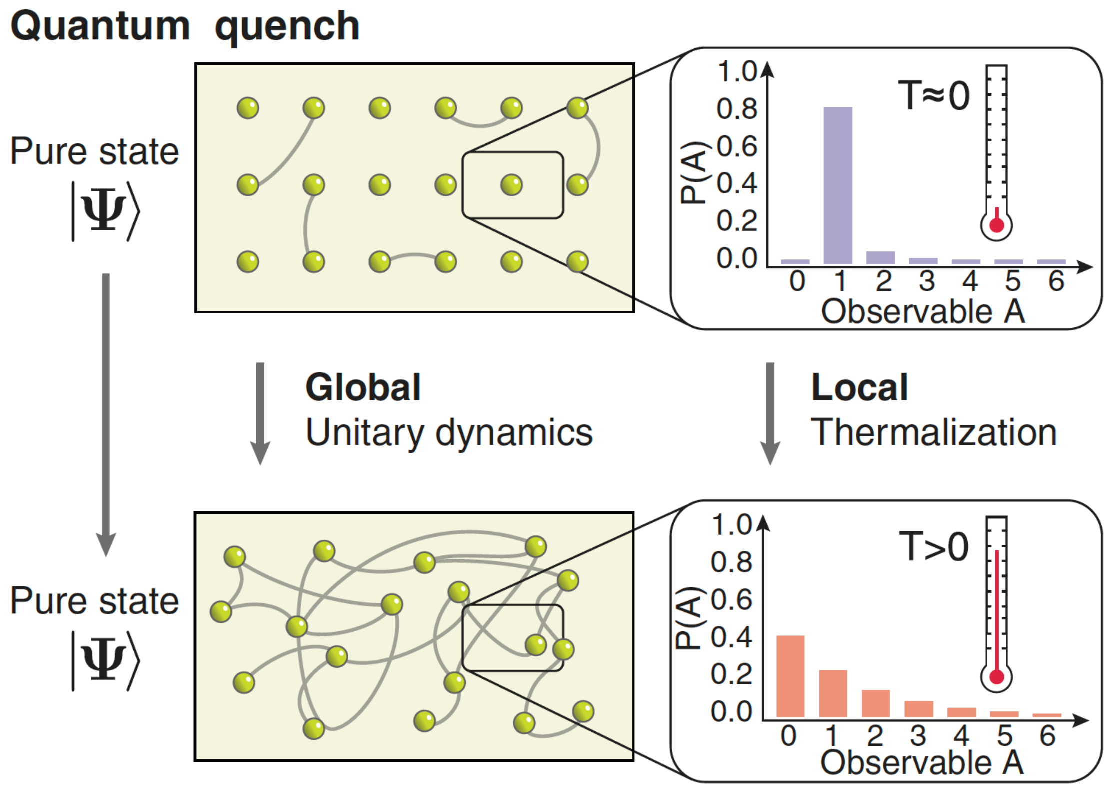
# Eigenstate thermalization hypothesis

M. Rigol et al., Nature 452, 854 (2008)




A non-integrable system obeys ETH and thermalizes

# Thermalization through entanglement




example: two qubits

$$|\psi\rangle_{\mathrm{AB}} = \frac{1}{\sqrt{2}}\left(|00\rangle_{\mathrm{AB}} + |11\rangle_{\mathrm{AB}}\right)$$

: entangled state

density matrix: $\rho_{\mathrm{AB}} = |\psi\rangle_{\mathrm{AB}}\langle\psi|$

trace out over B: $\rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\rho = \frac{1}{2}(|0\rangle_{\mathrm{A}}\langle 0| + |1\rangle_{\mathrm{A}}\langle 1|)$

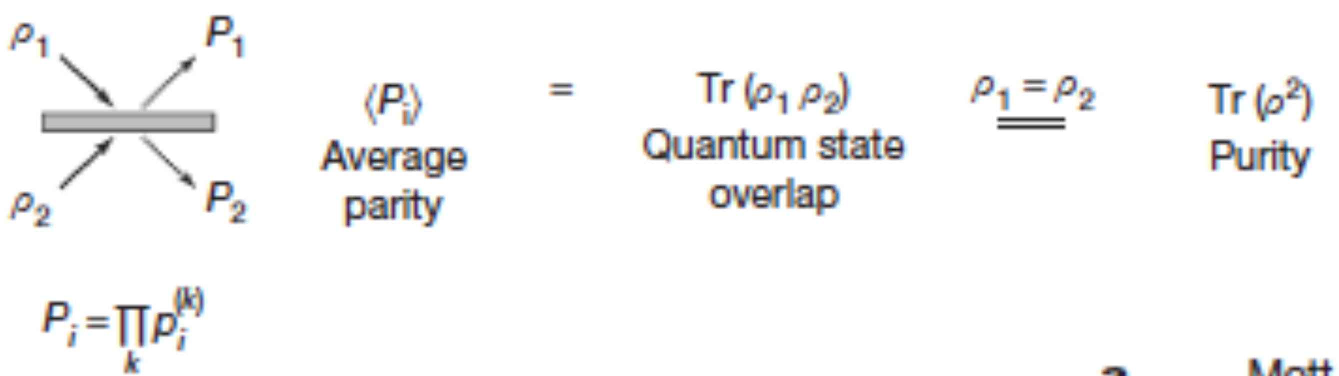
mixed state ➡ finite T

entanglement entropy: $S_{\mathrm{A}} = -\mathrm{Tr}(\rho_{\mathrm{A}} \log \rho_{\mathrm{A}}) = 1$

quantum purity: $\mathrm{Tr}(\rho_{\mathrm{A}}^2) = \frac{1}{2} < \mathrm{Tr}(\rho_{\mathrm{AB}}^2) = 1$

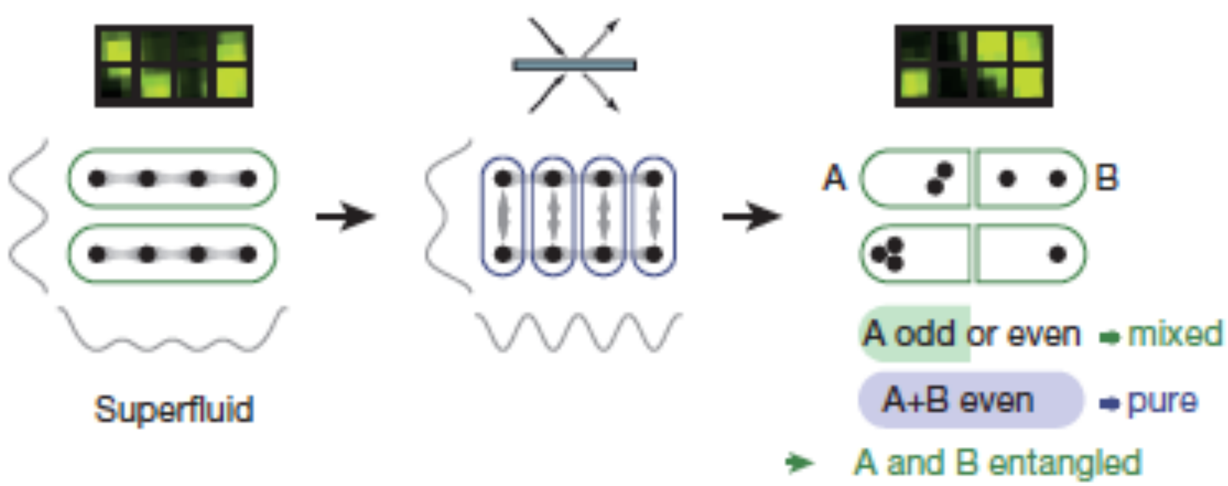
# Measuring entanglement entropy

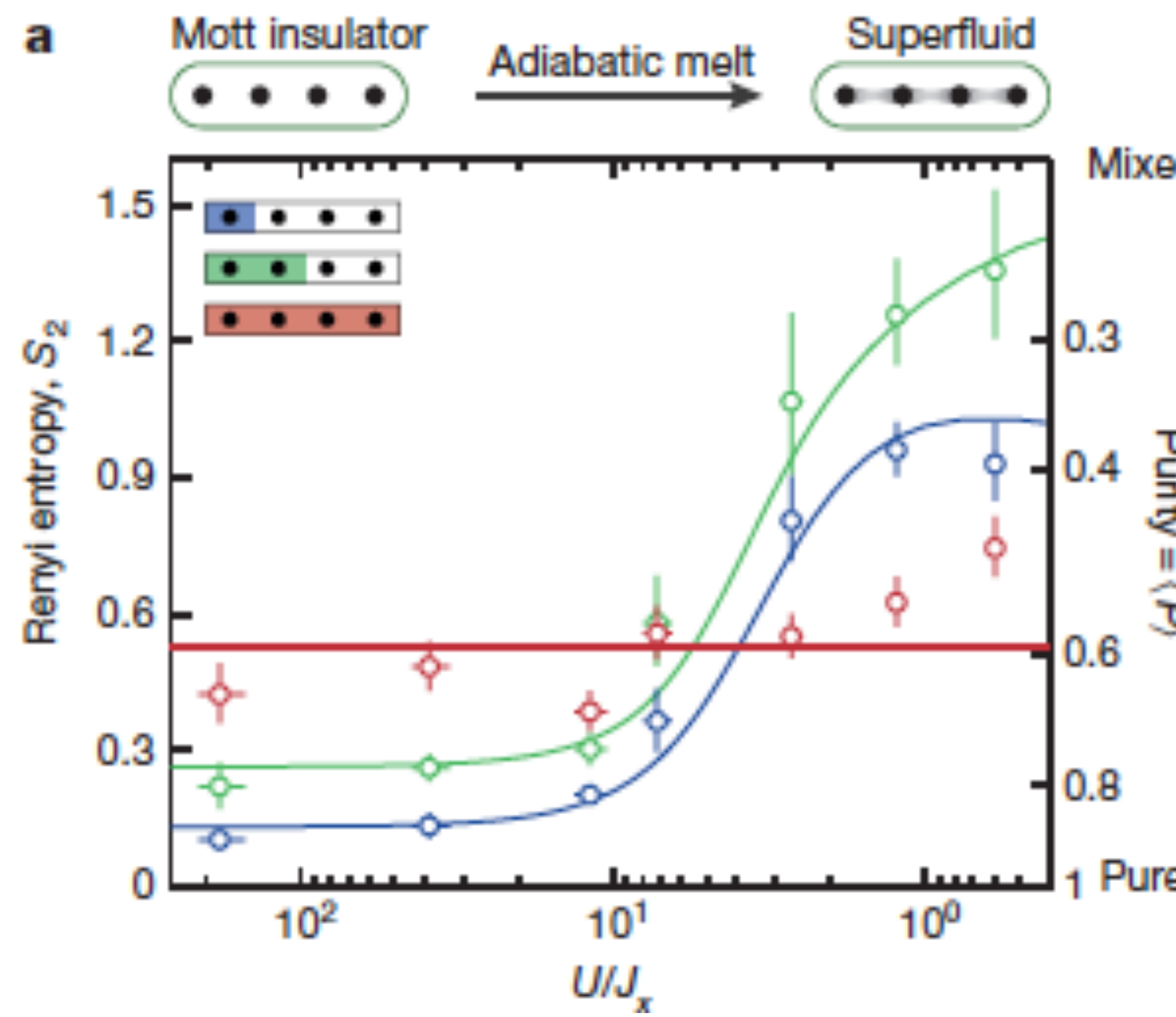
2nd order Renyi entropy: $S_2(A) = -\log \mathrm{Tr}(\rho_A^2)$




$$\boxed{\langle P_i \rangle = \mathrm{Tr}(\rho^2)}$$

quantum gas microscope is used to directly image the number parity of atoms on each lattice site

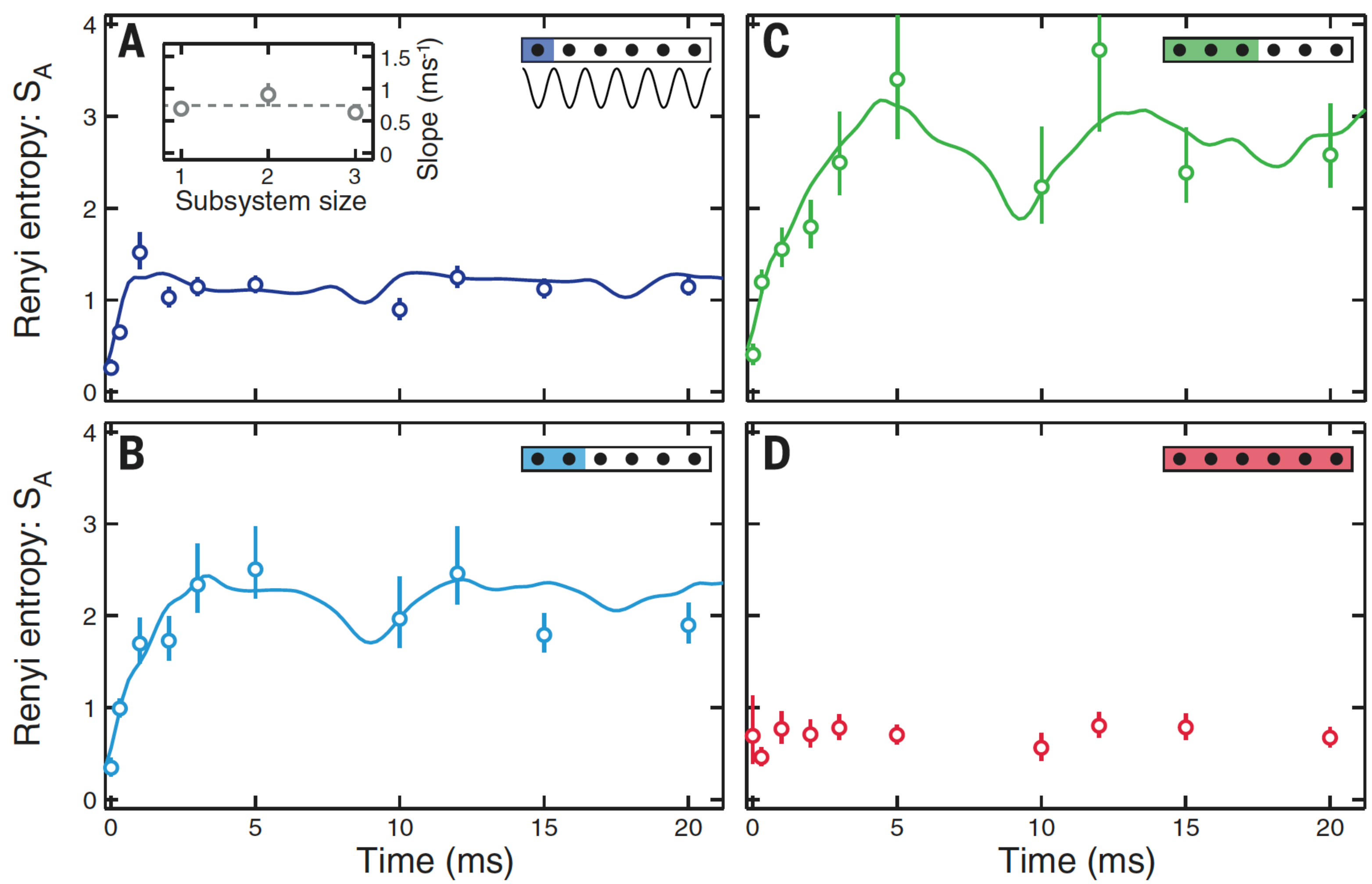



R. Islam et al., Nature **528**, 77 (2015)

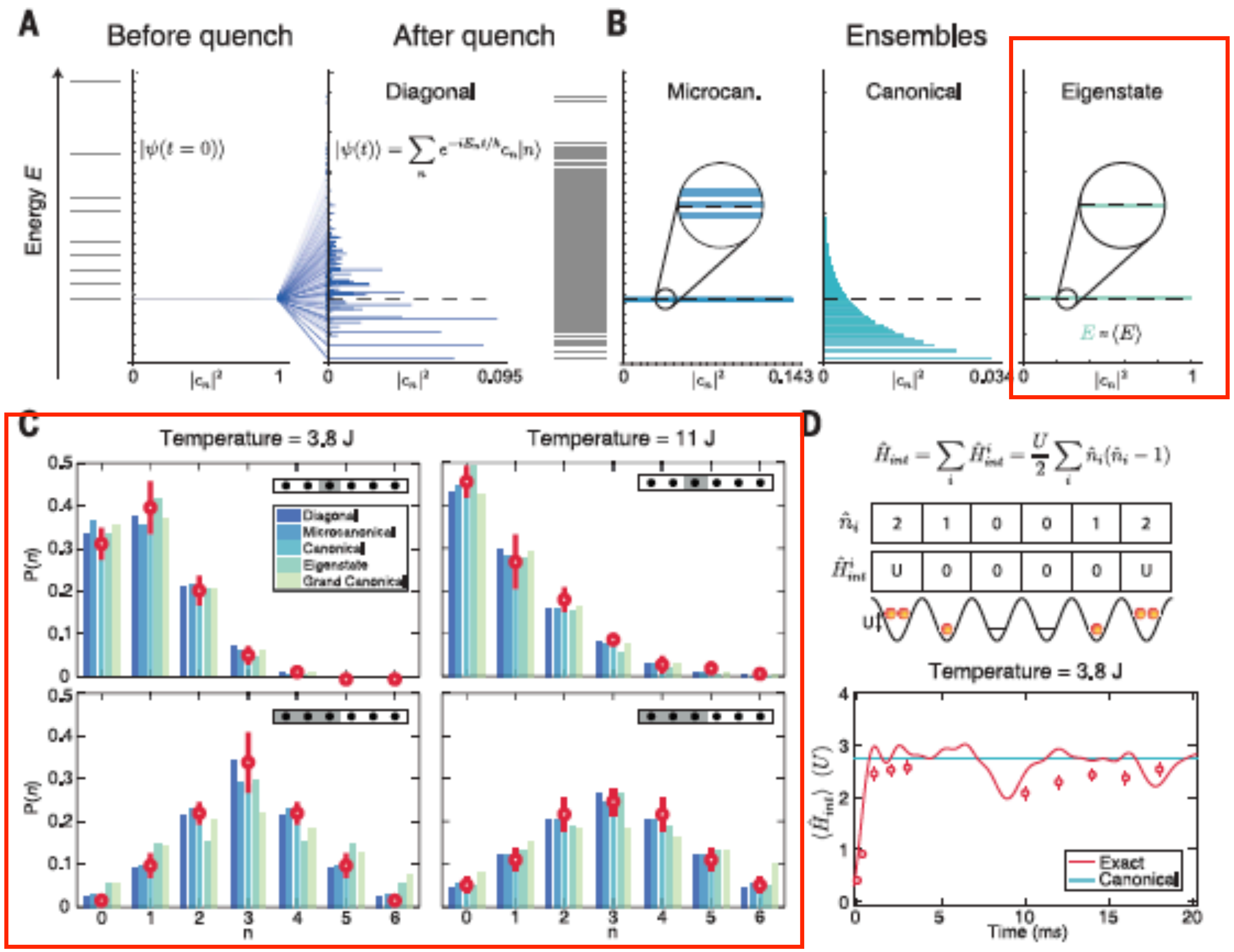
# Entanglement and quantum thermalization in atomic gases

Dynamics of bosonic atoms in an optical lattice after quench from Mott regime $J/U \ll 1$ to weakly interacting superfluid regime ($J/U \sim 1$)




A. M. Kaufman, *et al.*, *Science* **353**, 794 (2016).

# Entanglement and quantum thermalization in atomic gases




A. M. Kaufman, *et al.*, *Science* **353**, 794 (2016).

agreement with eigenstate ensemble indicates consistency with ETH

# Introduction 2: Transport of entanglement

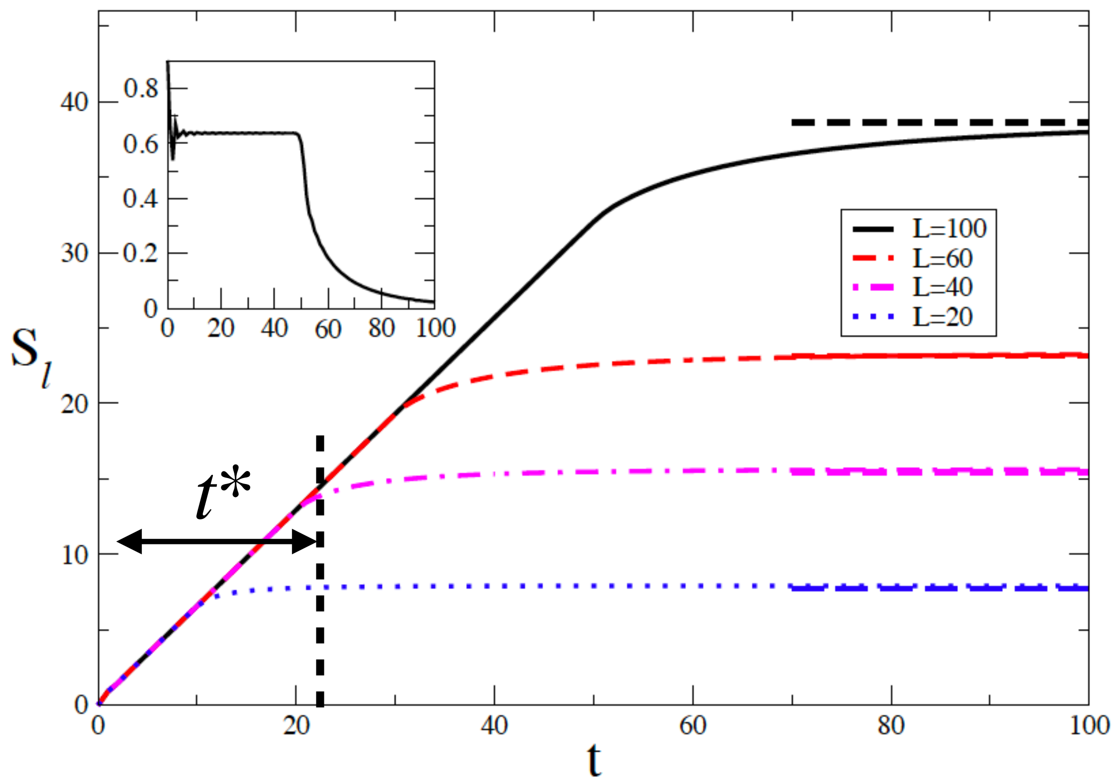
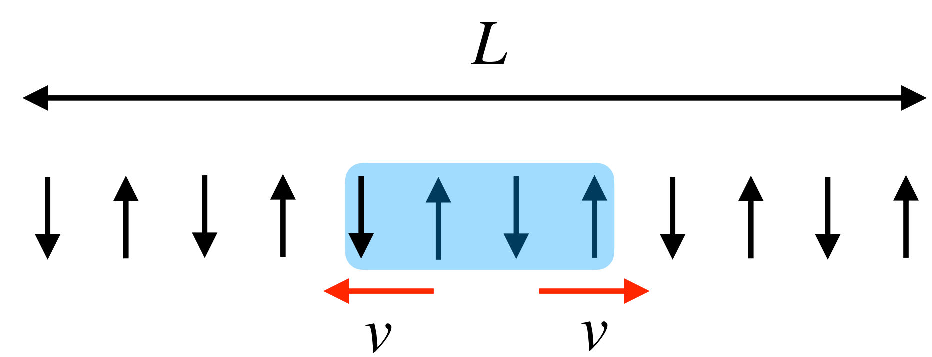



FIG. 3. Entanglement entropy for the quench from $h_0 = \infty$ to $h = 1$, for various $\ell$. The dashed lines are the leading asymptotic results for large $\ell$, cf. Eq. (3.19). The inset shows the derivative with respect to the time of $S_{100}(t)$.

$$H_I(h) = -\frac{1}{2}\sum_j [\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z]$$




$$t^* = \frac{L}{2v}$$

$v$: propagation speed of entanglement

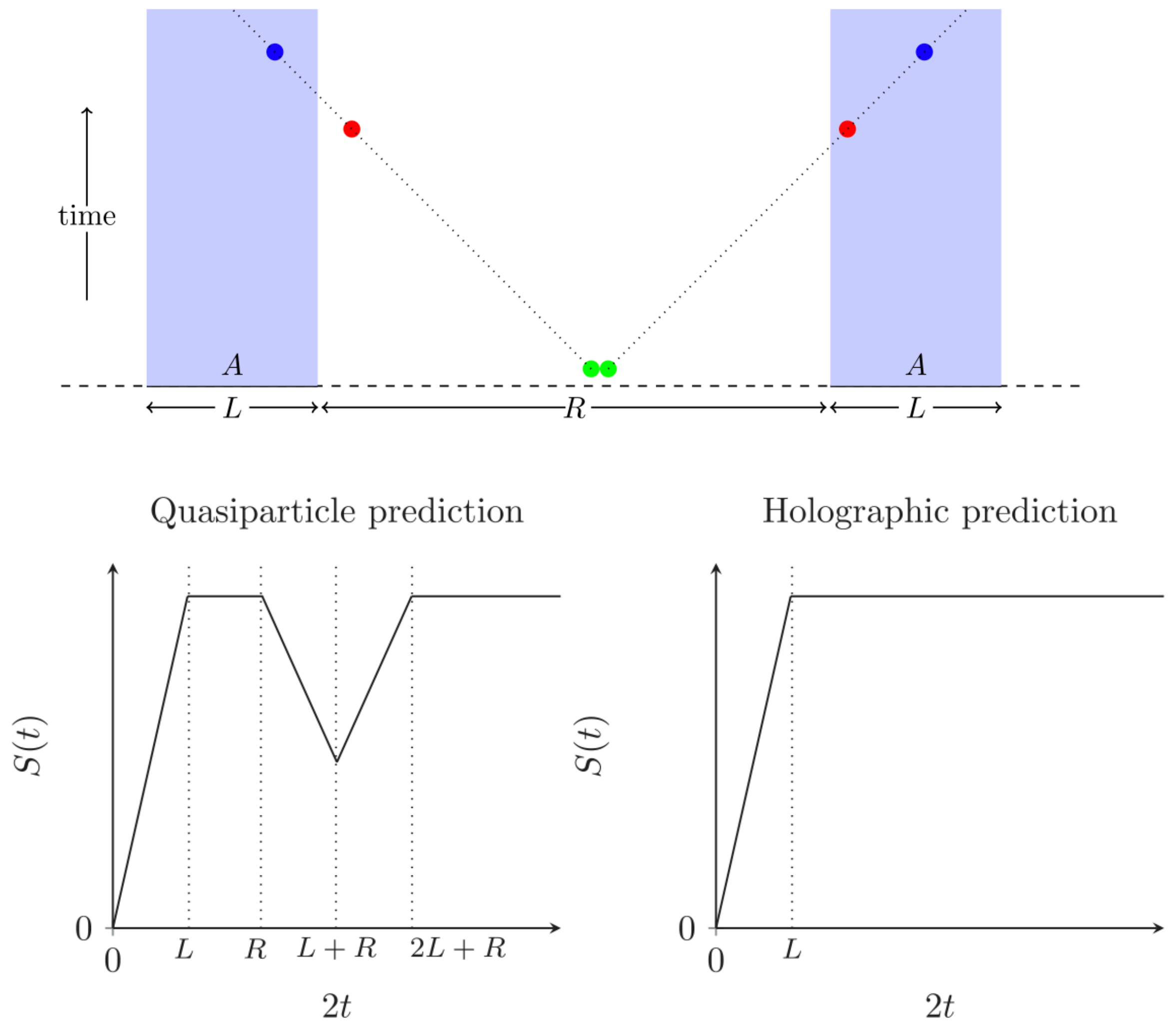
Calabrese and Cardy (2005), Kim and Huse (2013)

- Entanglement is not a conserved quantity:

  It can be locally created by the Ising-type interaction.

- Interpretation is based on the quasi-particle picture

# Introduction 2: Transport of entanglement




S. Leichenauer and M. Moosa PRD 2015.

$$S(t) = 2s_{\text{eq}} \times \begin{cases} 2t, & t \le \frac{L}{2}, \\ L, & \frac{L}{2} < t < \frac{R}{2}, \\ L - (t - \frac{R}{2}), & \frac{R}{2} < t < \frac{L+R}{2}, \\ L + (t - L - \frac{R}{2}), & \frac{L+R}{2} < t < \frac{2L+R}{2}, \\ L, & t > \frac{2L+R}{2}. \end{cases}$$

Result differs from the behavior expected by free quasi-particle.

Entangle can be generated even acausally separated systems

J. Koga, G. Kimura, and K. Maeda PRA 2018.

**Question**

How can we characterize propagation of entanglement?

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- Introduction: Thermalization of isolated systems, transport of entanglement
- $\boxed{\text{Model}}$
- Time-evolution of entanglement entropy
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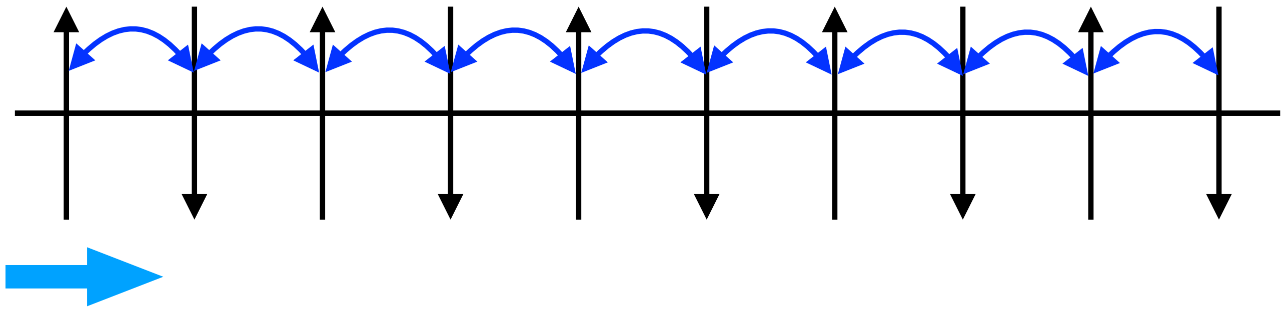
# Model

- Transverse Ising (TI) model: Ising int. + transverse filed



$$H_{\text{trans}} = -J\sum_i \sigma_i^z \sigma_{i+1}^z + h_x \sum_i \sigma_i^x \qquad J = 1, \text{ with various } h_x$$

Integrable system

Jordan-Wigner transformation

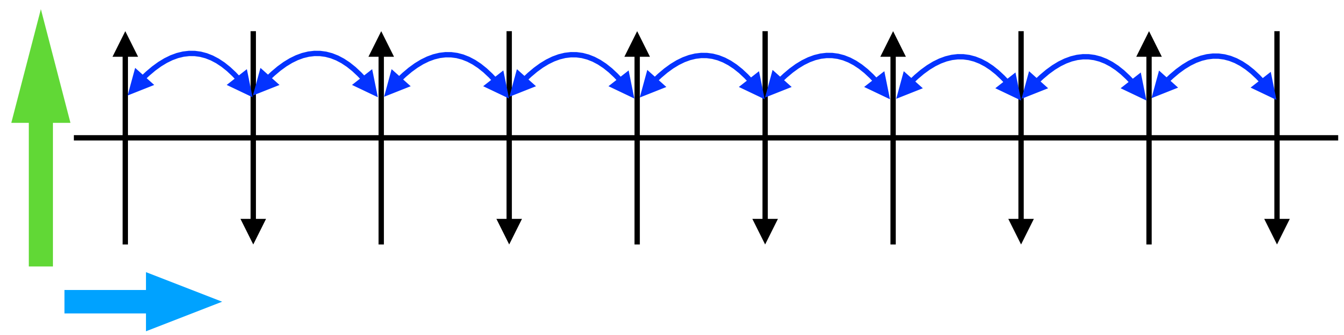
$$H = \sum_k \varepsilon_k \left( \gamma_k^\dagger \gamma_k - \frac{1}{2} \right)$$

"Free fermions"

$$\varepsilon_k = 2\left(J^2 + h_x^2 - 2h_x J \cos k\right)$$

# Chaostic extensions

- Chaotic Ising (CI) model: Transverse filed + longitudinal field

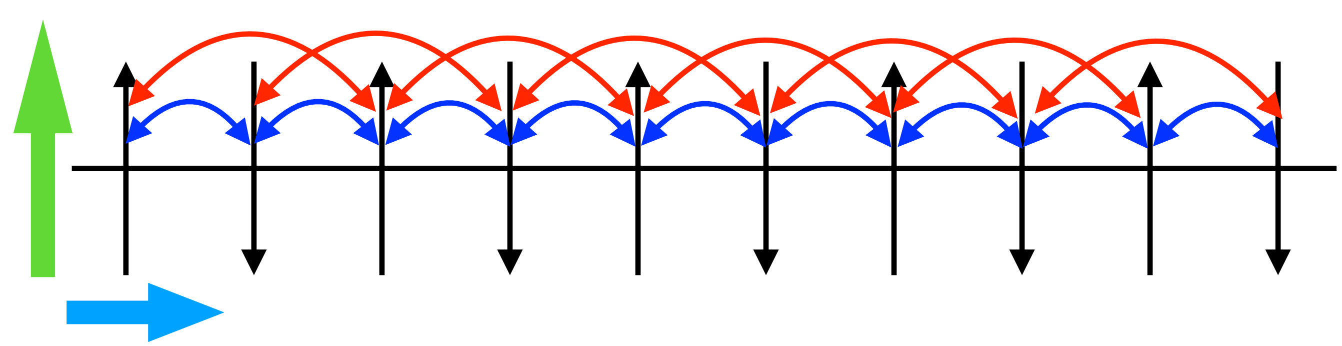


$$H_{\text{chaos}} = -J\sum_i \sigma_i^z \sigma_{i+1}^z + h_x \sum_i \sigma_i^x + h_z \sum_i \sigma_i^z$$

non-integrable

$$J = 1,\ h_x = 1.05,\ h_z = 0.5$$

- Extended chaotic Ising (ECI) model: chaotic Ising model + NNN interaction

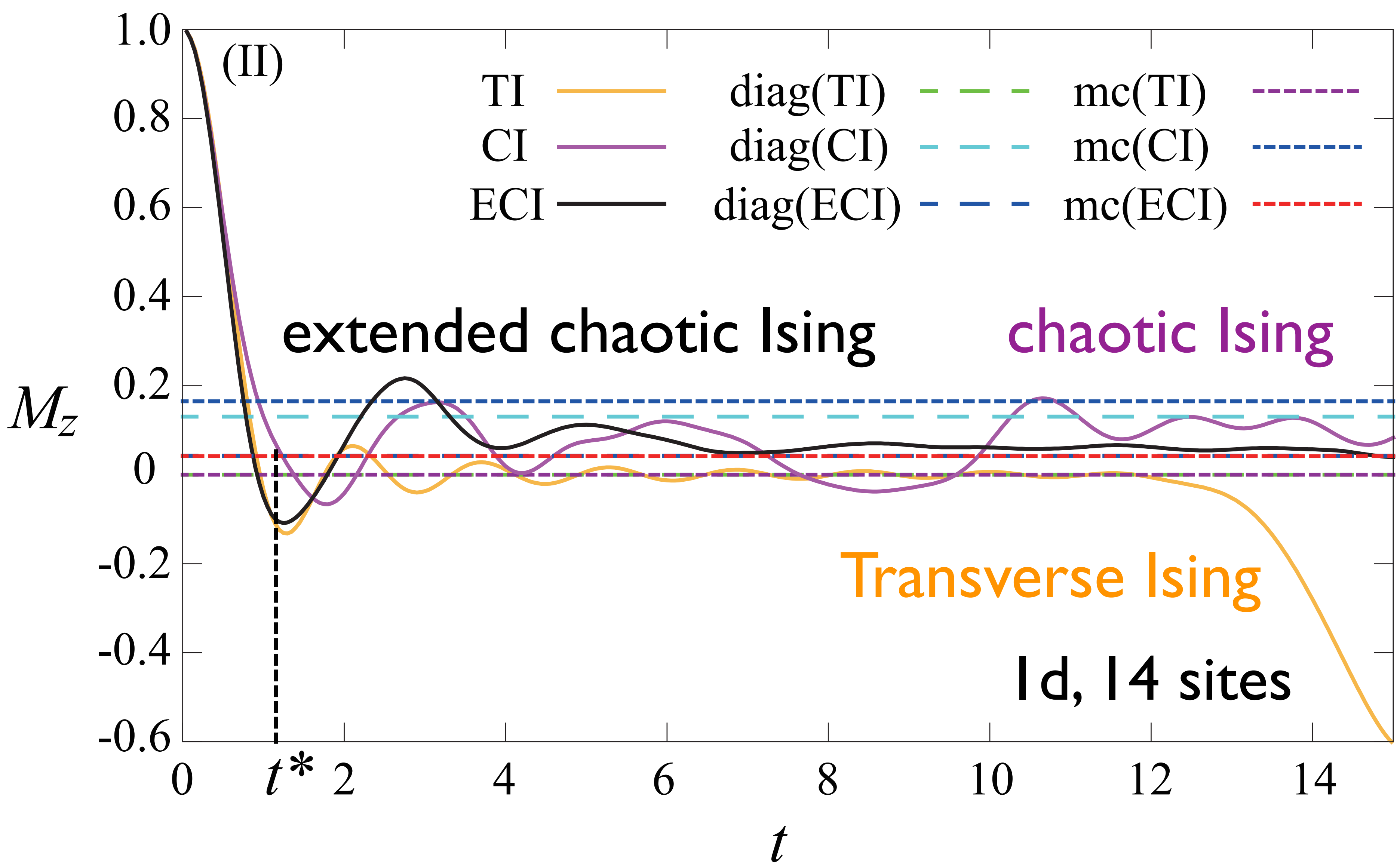


$$H_{\text{exchaos}} = -J\sum_i \sigma_i^z \sigma_{i+1}^z - J'\sum_i \sigma_i^z \sigma_{i+2}^z + h_x \sum_i \sigma_i^x + h_z \sum_i \sigma_i^z$$

$$J = 1,\ J' = 0.8,\ h_x = 1.05,\ h_z = 0.5$$

non-integrable

# One-site magnetization



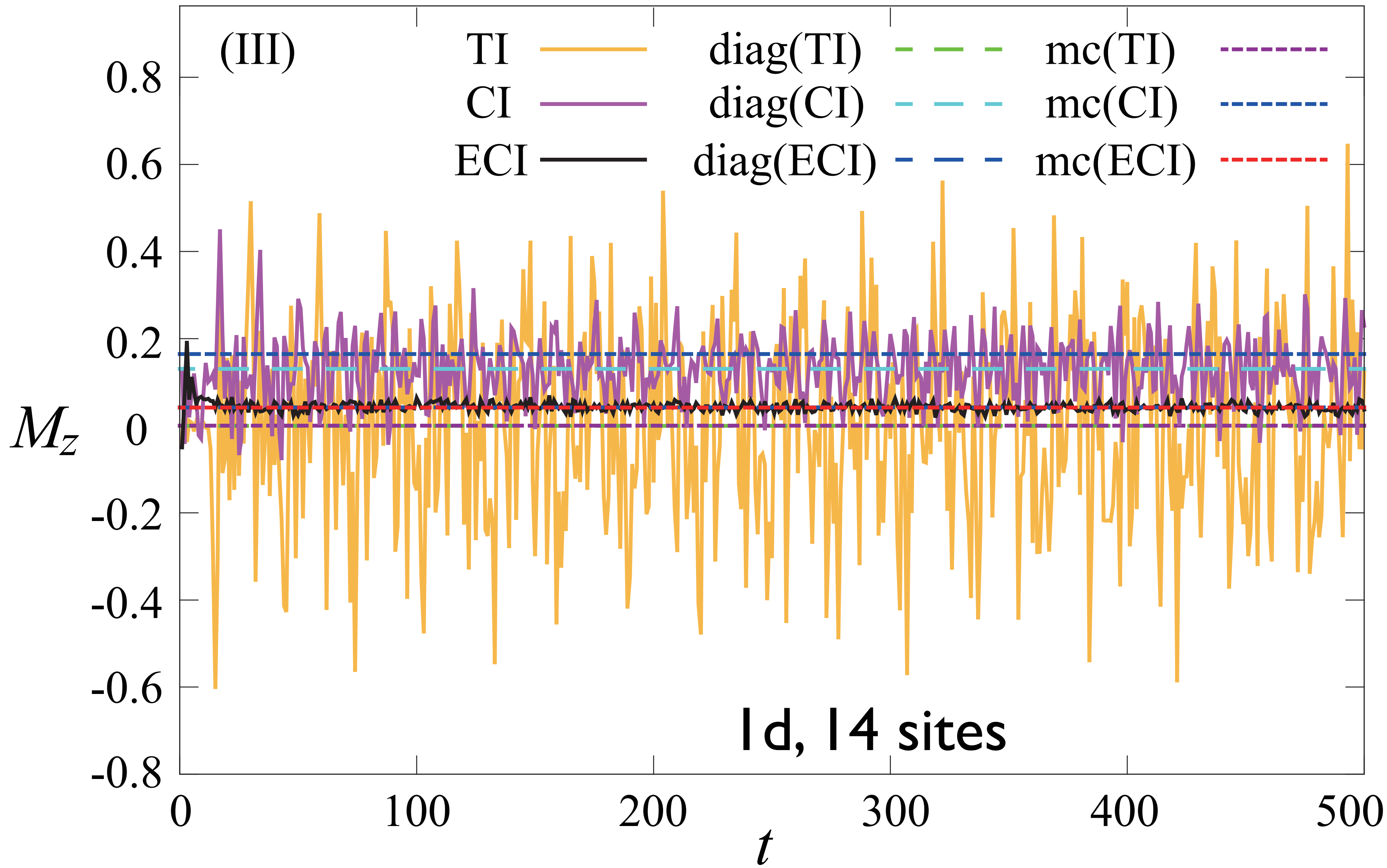

$$\langle A \rangle_{\text{microcan}}(E_0) = \frac{1}{\mathcal{N}_{E_0,\Delta E}} \sum_{|E_n - E_0| < \Delta E} \langle n | A | n \rangle$$

$$|\psi(t)\rangle = \sum_n a_n e^{-iE_n t} |n\rangle$$

$$\langle A \rangle_{\text{diagonal}} = \sum_n |a_n|^2 \langle n | A | n \rangle$$

# Thermalization

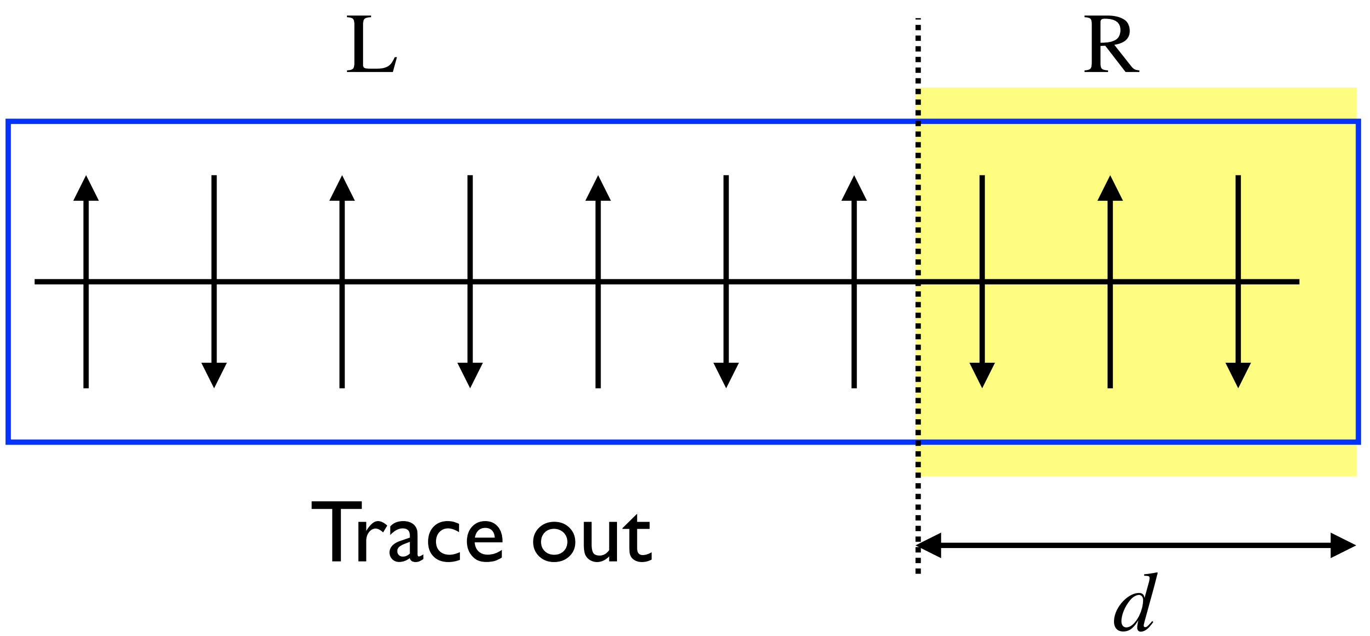
Long time behavior




- Extended chaotic Ising model thermalizes due to strong non-integrability, while transverse Ising, chaotic Ising model does not show thermalization.

# Entanglement entropy

Entanglement entropy for the subsystem R




Initial density matrix: $\rho_0 = |\psi_0\rangle\langle\psi_0|$

Density matrix at $t$: $\rho_{\mathrm{tot}}(t) = e^{-iHt}\rho_0 e^{iHt}$

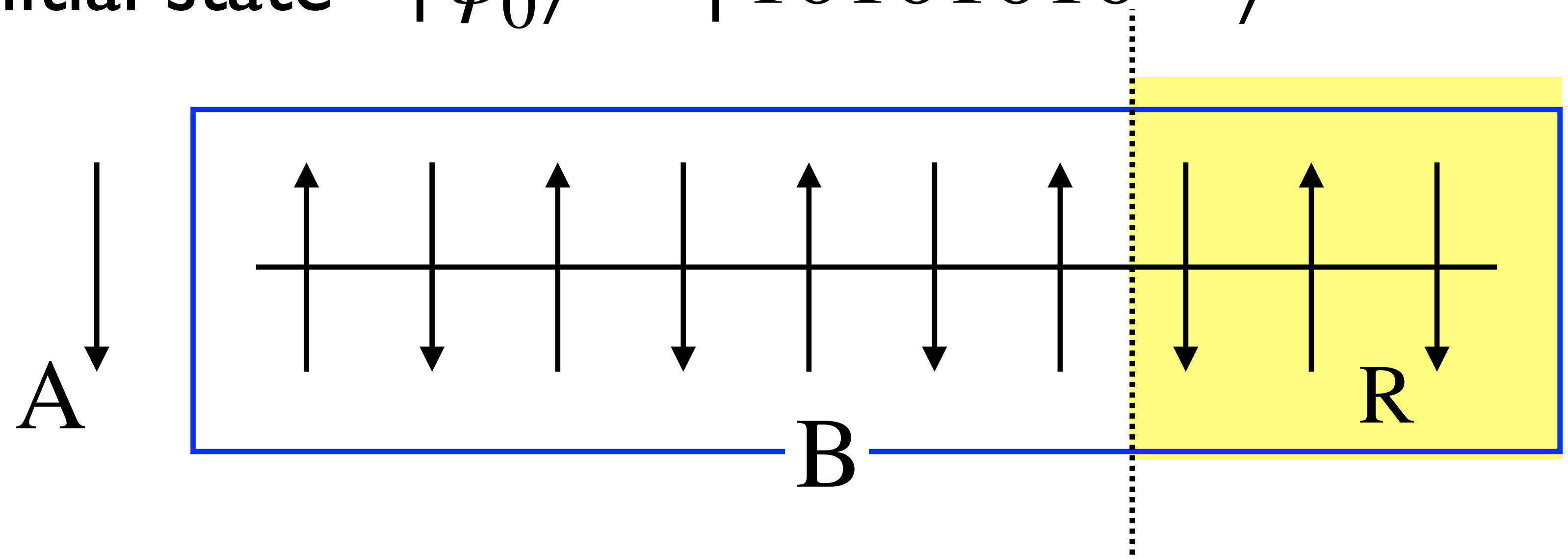
Reduced density matrix: $\rho_{\mathrm{R}} = \mathrm{Tr}_{\mathrm{L}}\rho_{\mathrm{tot}}$

Entanglement entropy: $S_{\mathrm{R}} = -\,\mathrm{Tr}(\rho_{\mathrm{R}} \log \rho_{\mathrm{R}})$

## Setup

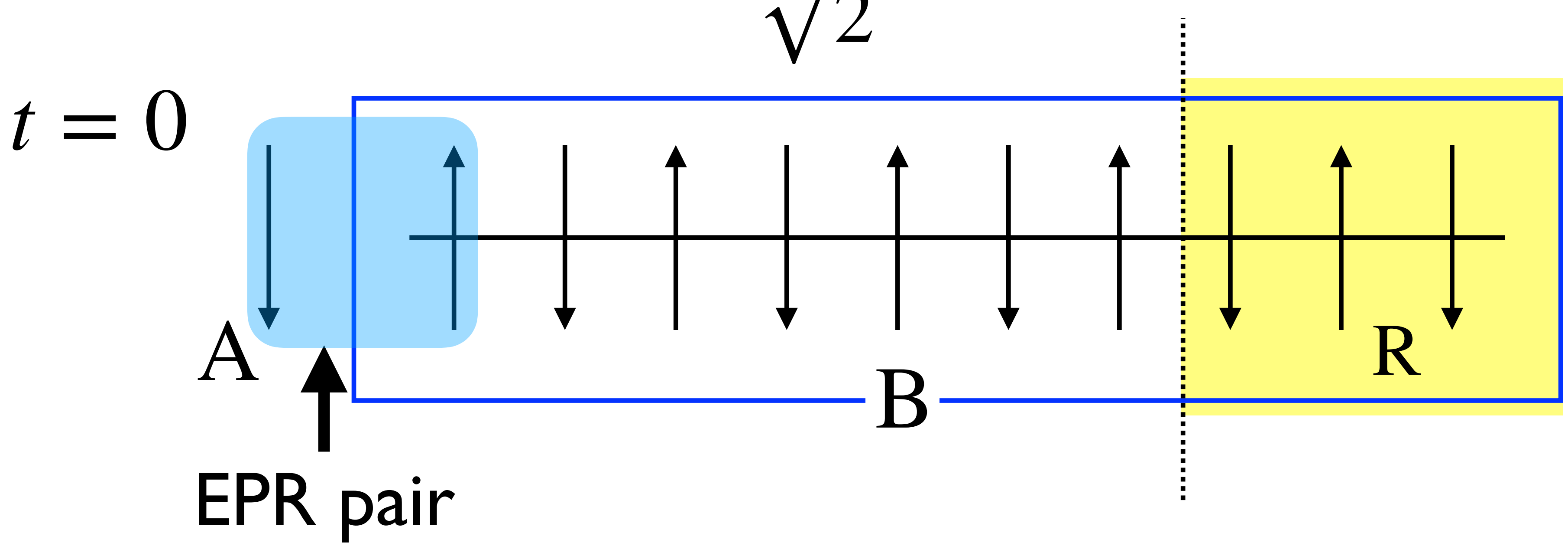
Iyoda and Sagawa, Phys. Rev. A **97**, 042330 (2018).

Case I: initial state $|\psi_0\rangle = |10101010\cdots\rangle$




Neel

Case II: initial state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \otimes |101010\cdots\rangle$



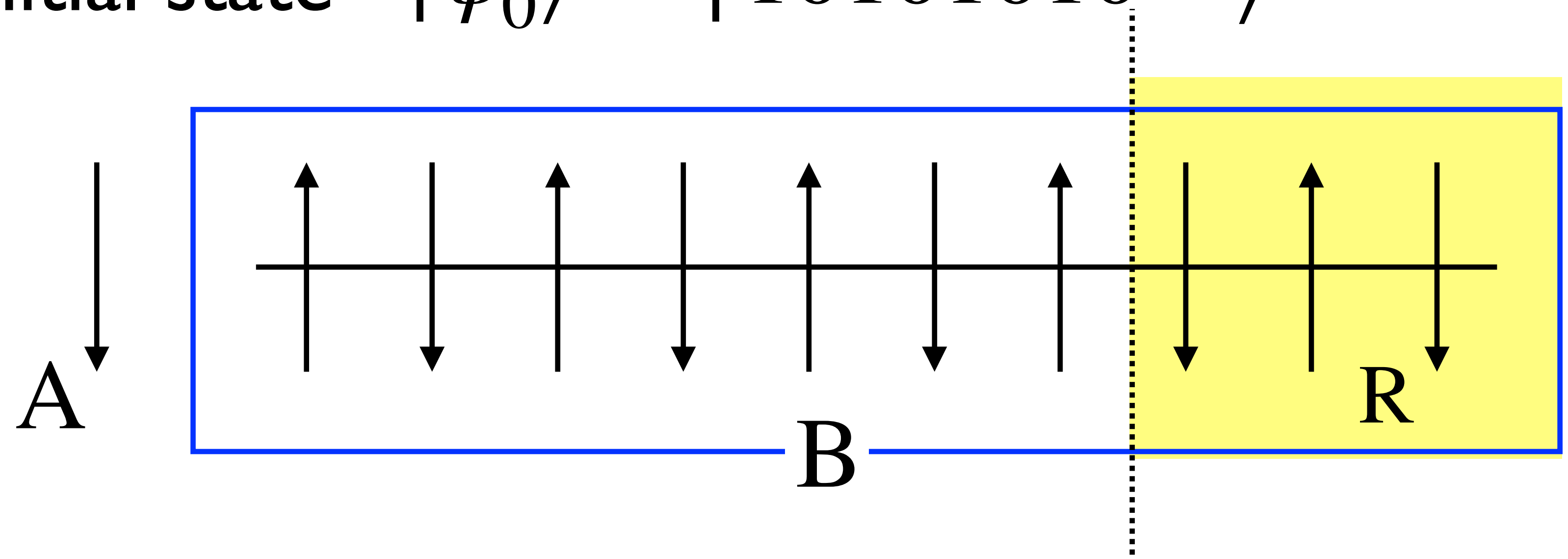

EPR

only subsystem B evolves in time

## Setup

Iyoda and Sagawa, Phys. Rev. A **97**, 042330 (2018).

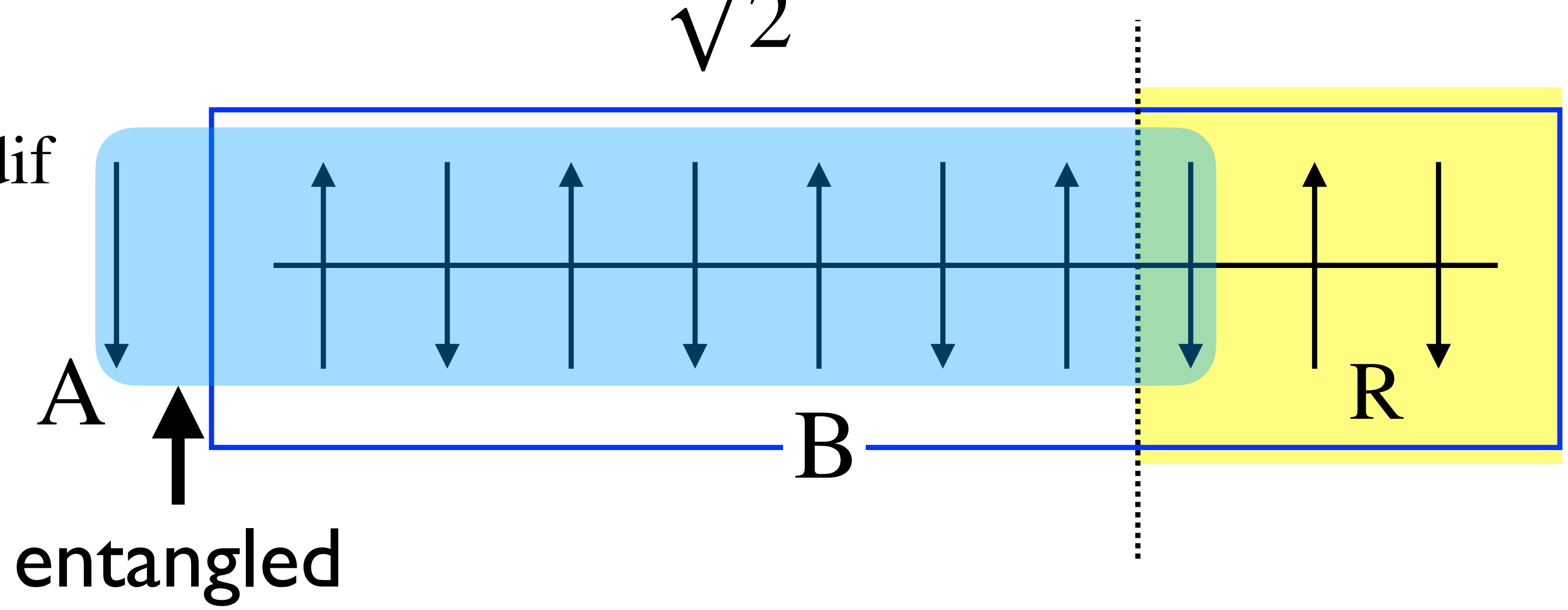
Case I: initial state $|\psi_0\rangle = |10101010\cdots\rangle$




Neel

Case II: initial state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \otimes |101010\cdots\rangle$

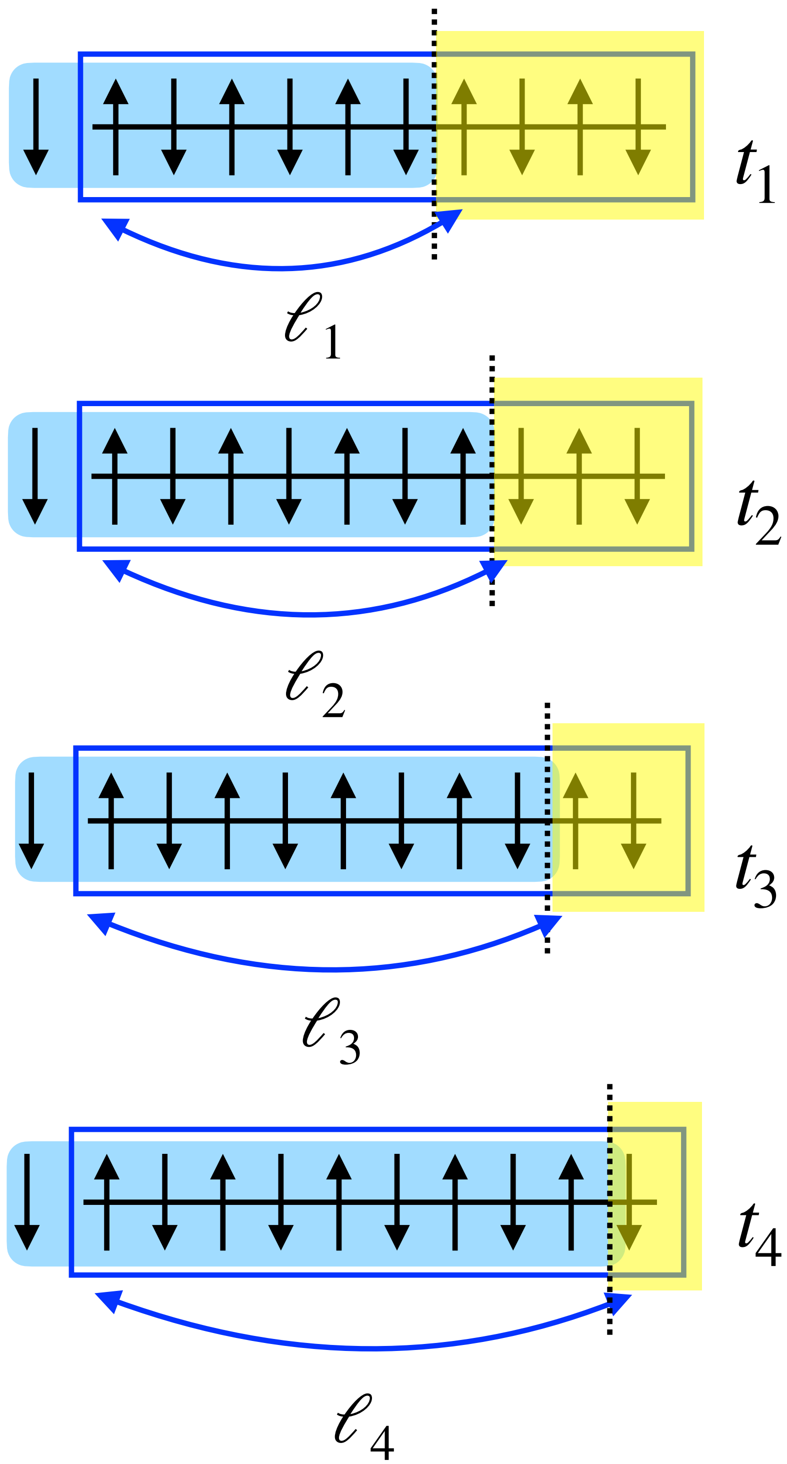
$t = t_{\mathrm{dif}}$

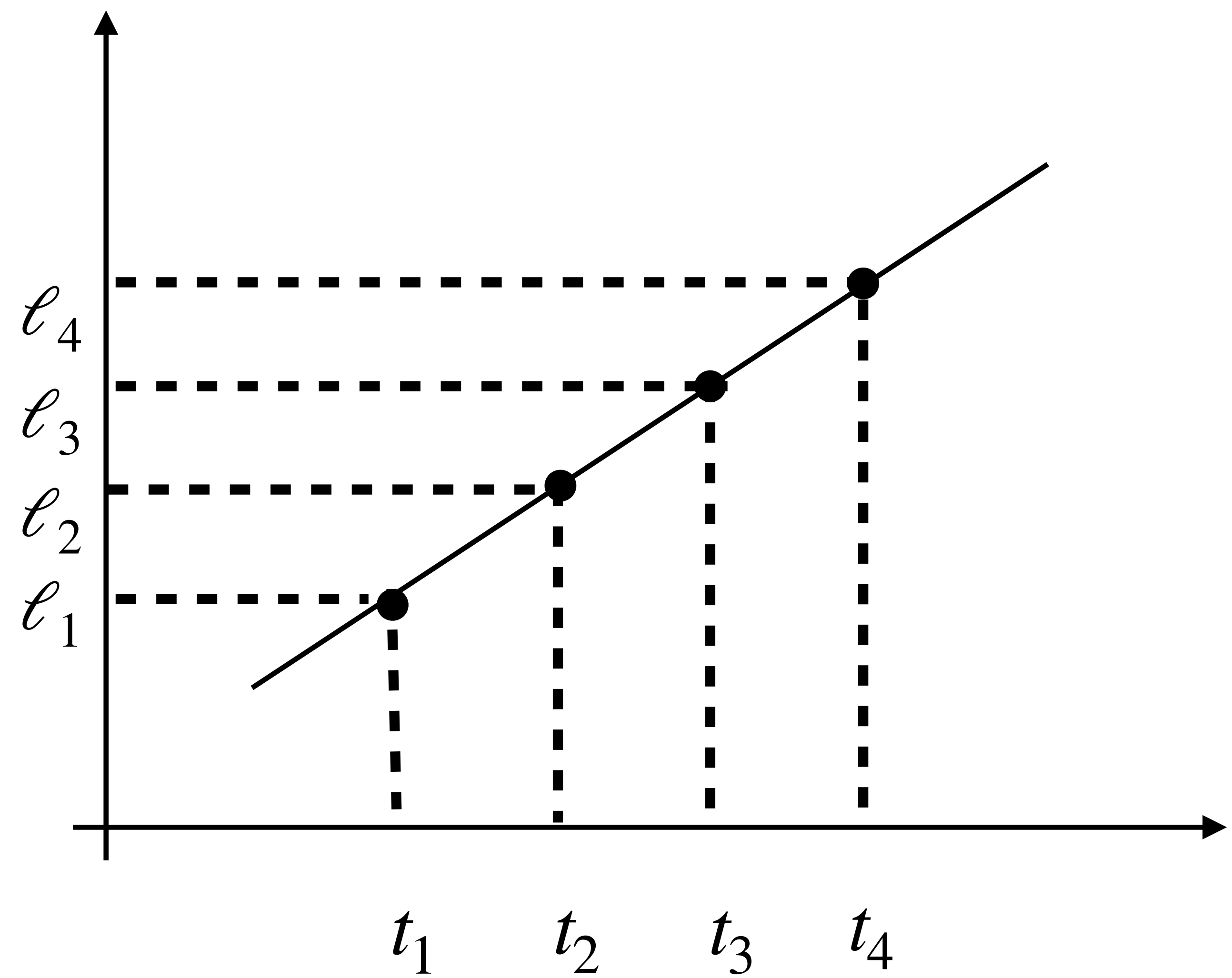



EPR

$S_{\mathrm{R}}$s for Case I and II exhibit deviation when the influence of the entanglement reaches the subsystem R

# Propagation of Entanglement




Slope ~ propagation speed of entanglement

# Outline

- Introduction: Thermalization of isolated systems, transport of entanglement
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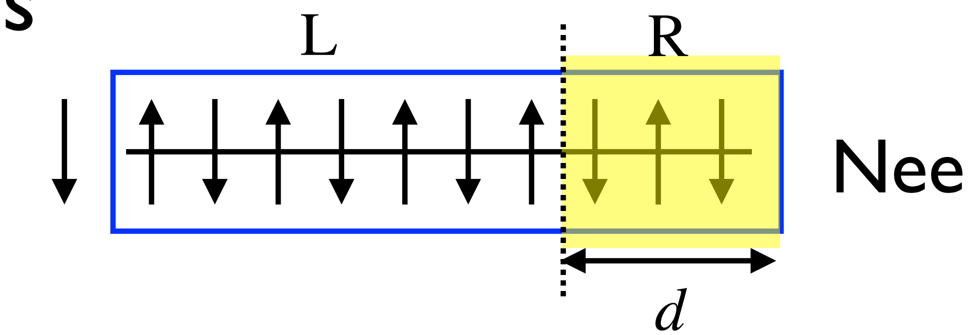
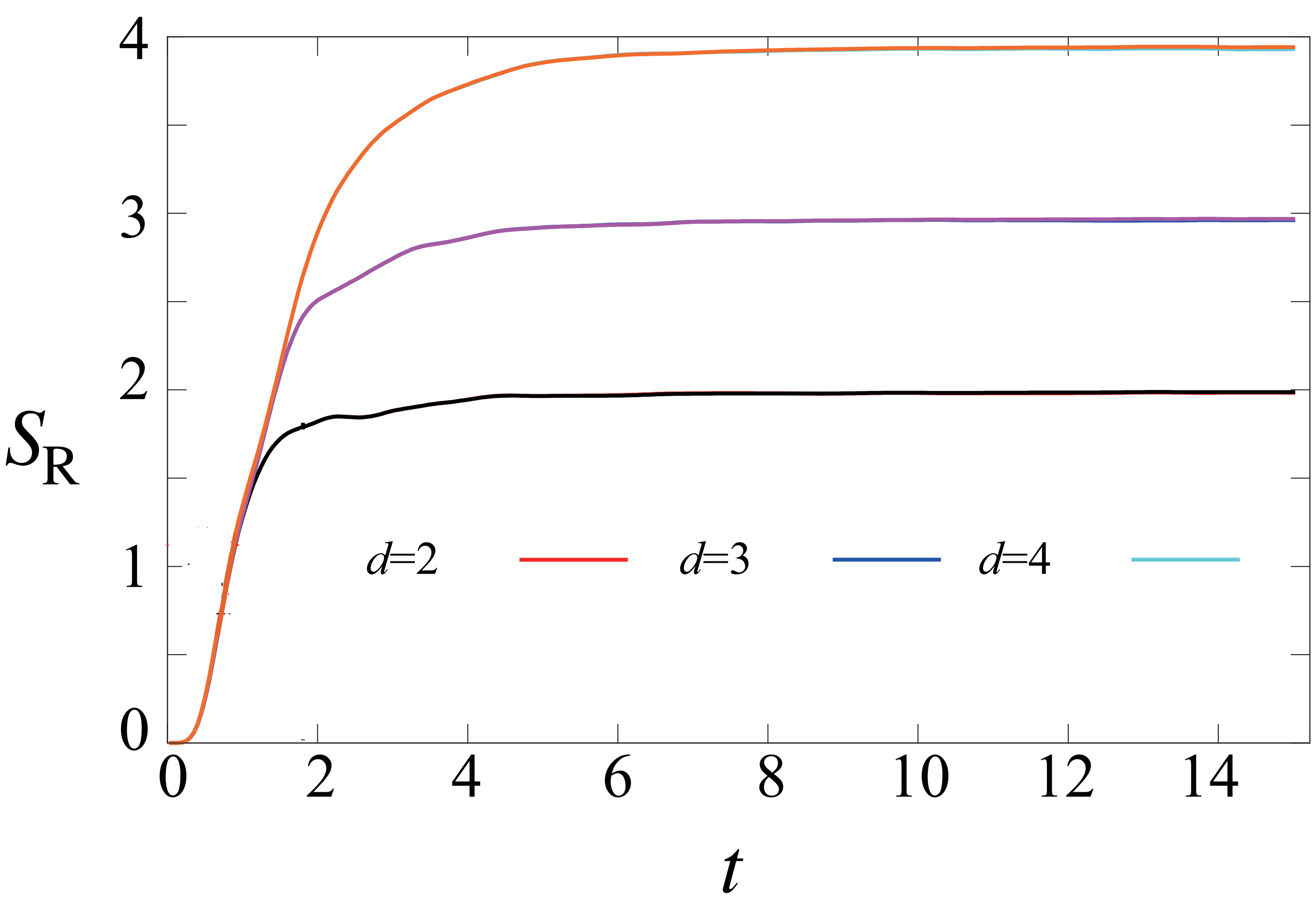
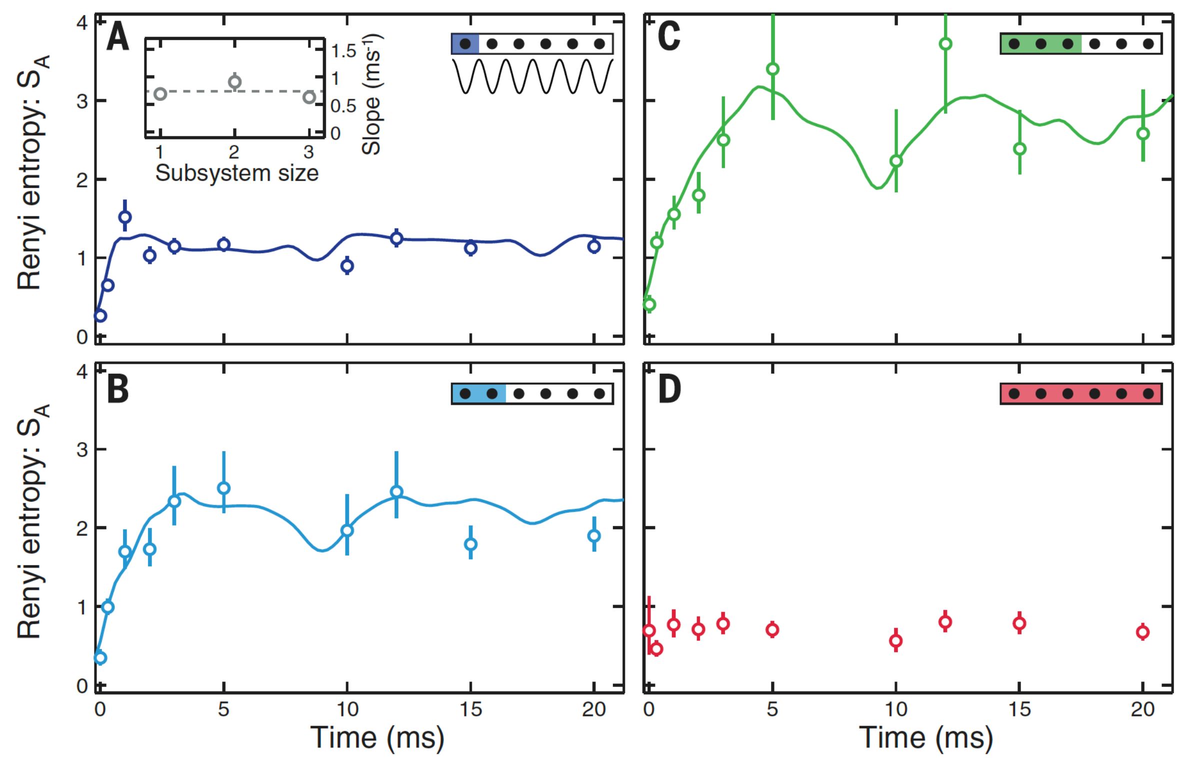
# Thermalization: Extended chaotic Ising model

extended chaotic Ising model, 14+1 sites



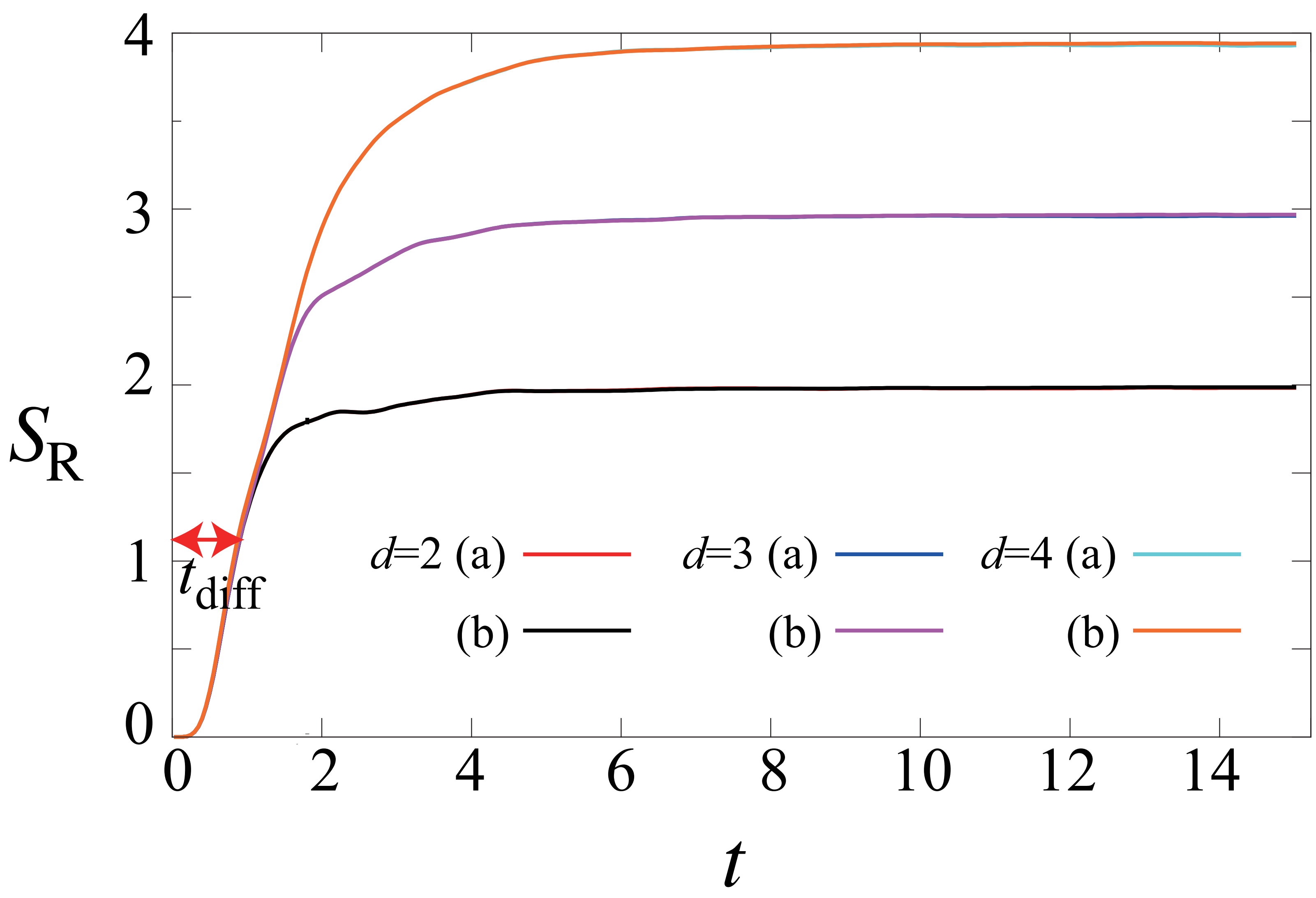



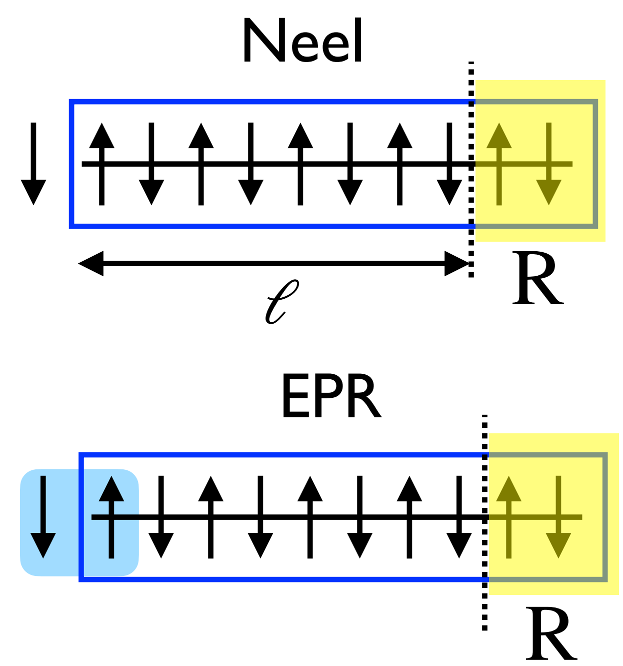




- Consistency with experimentally measured Renyi entropy
- Saturation value of $S_R \propto$ size of subsystem: "volume law"

# Entanglement entropy: Extended chaotic Ising model

extended chaotic Ising model, 14+1 sites



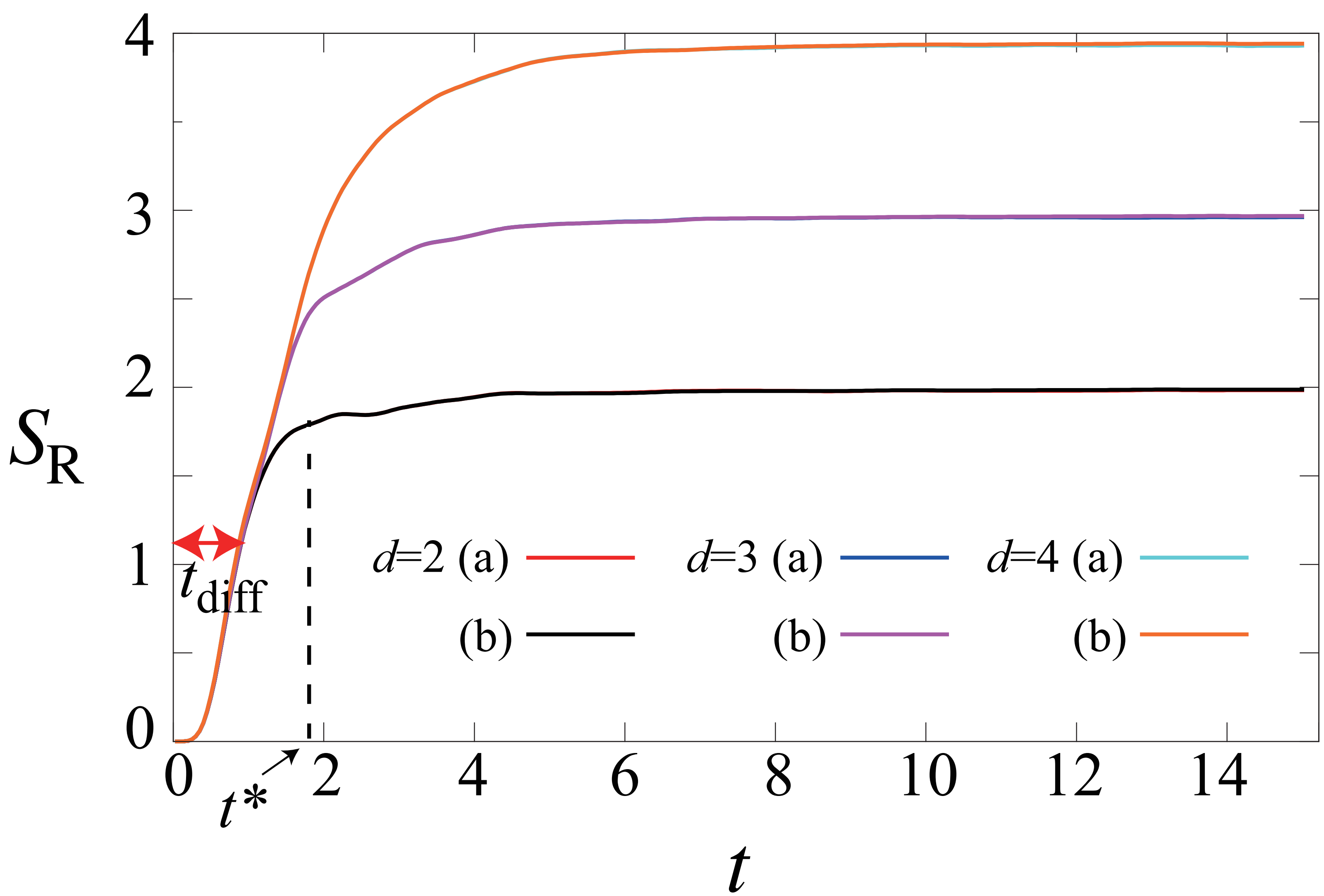



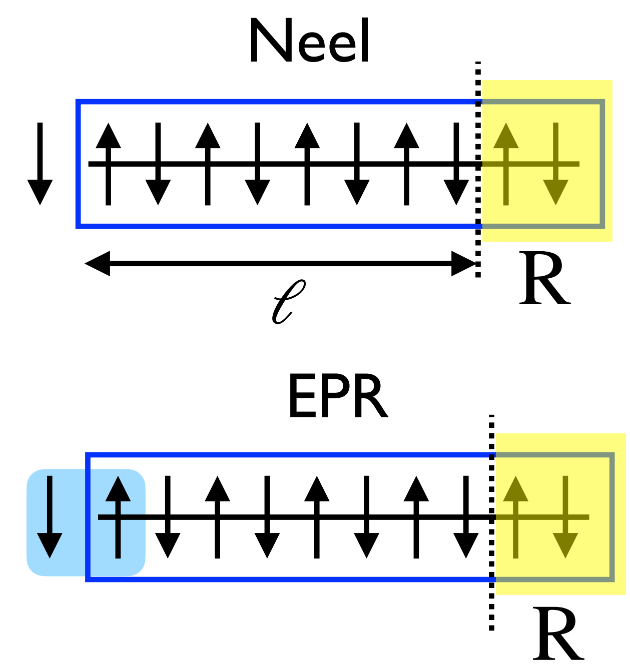

$S_R$s for Neel and EPR exhibit deviation at $t_{dif}$

➡ influence of entanglement reaches the subsystem R at $t_{dif}$

## Entanglement entropy: Extended chaotic Ising model

extended chaotic Ising model, 14+1 sites



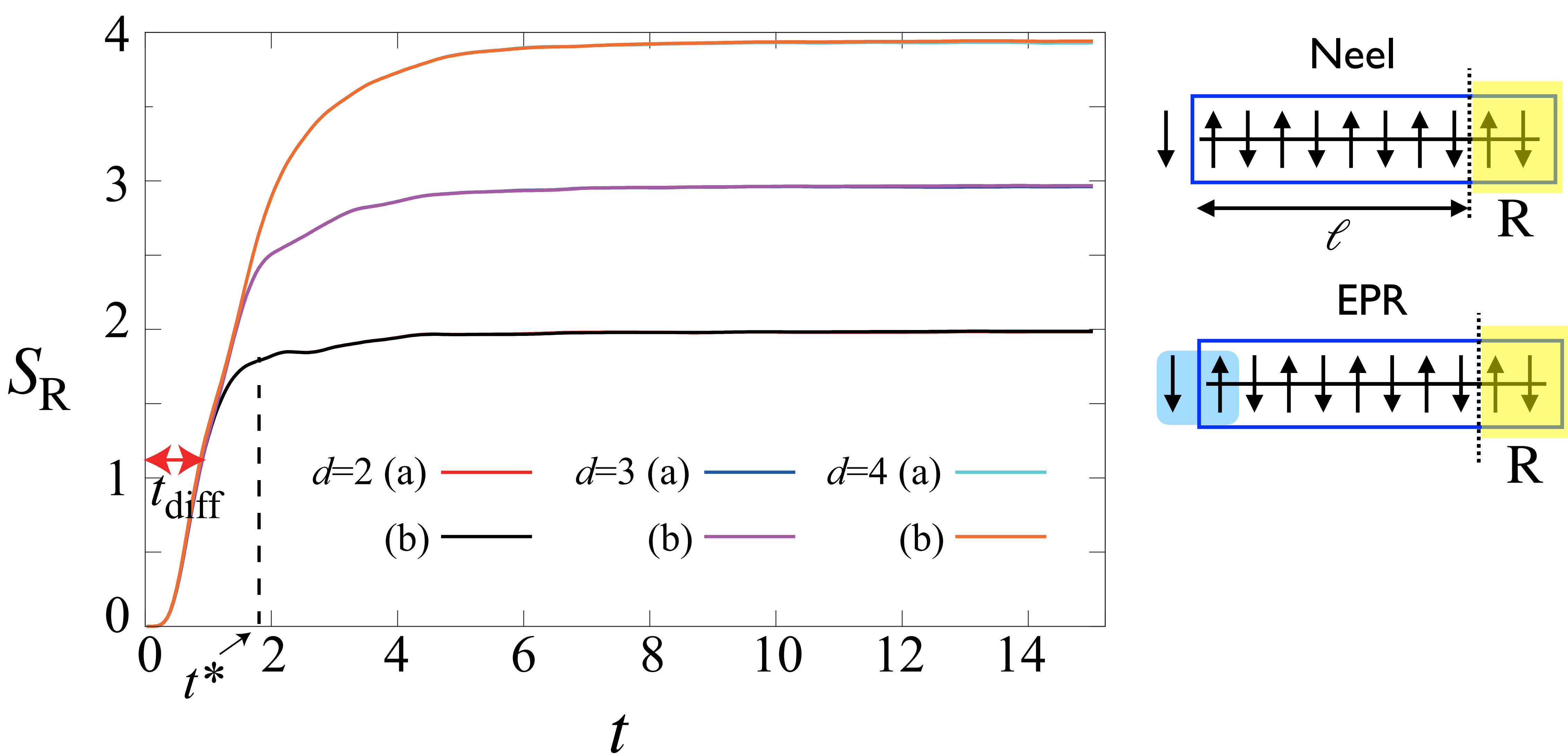

Two time scales

$t*$ : $S_R(t)$ deviates from linear growth and saturates

$t_{dif}$ : influence of entanglement reaches the subsystem R

# Entanglement entropy: Extended chaotic Ising model

extended chaotic Ising model, 14+1 sites

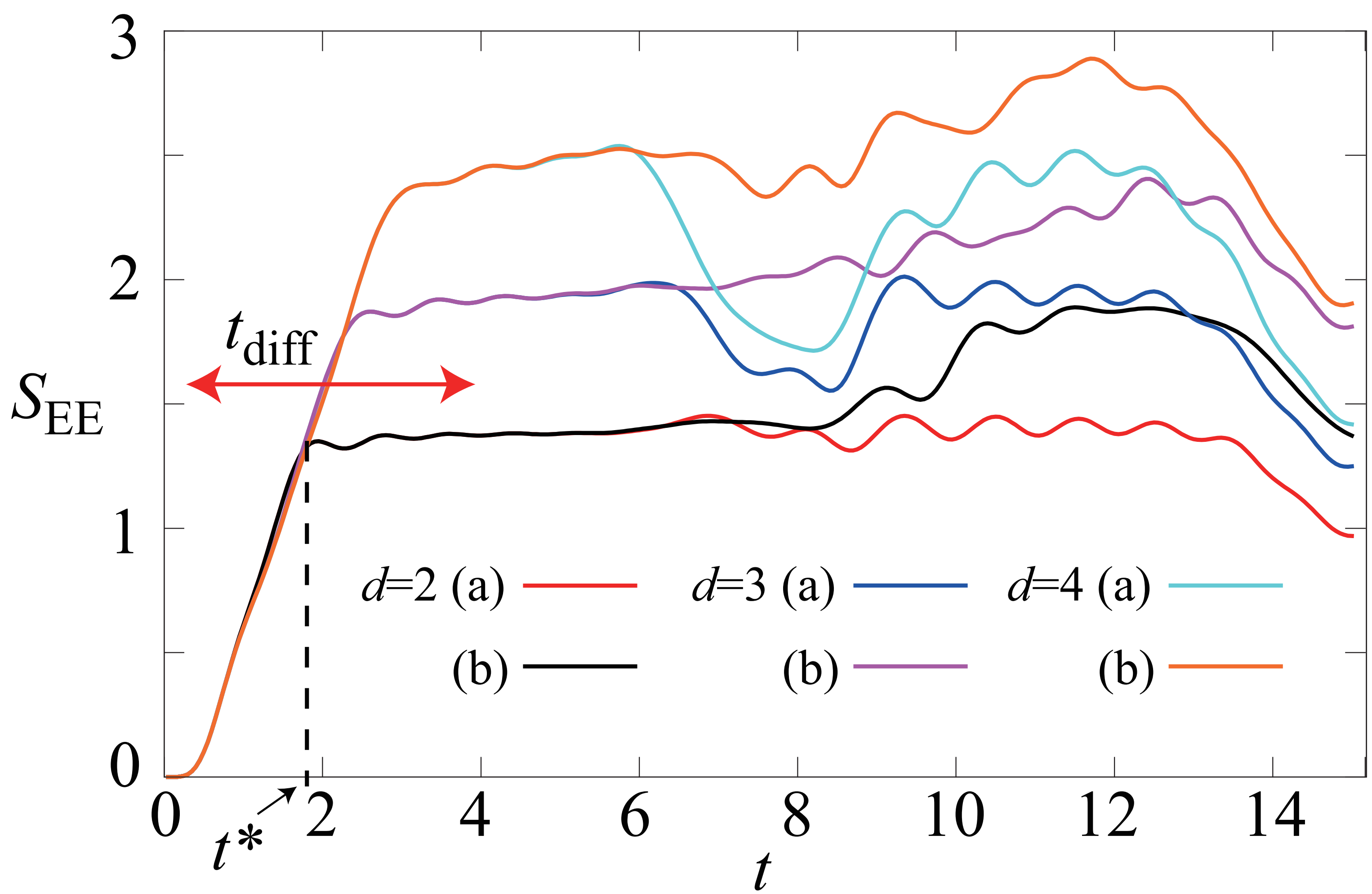


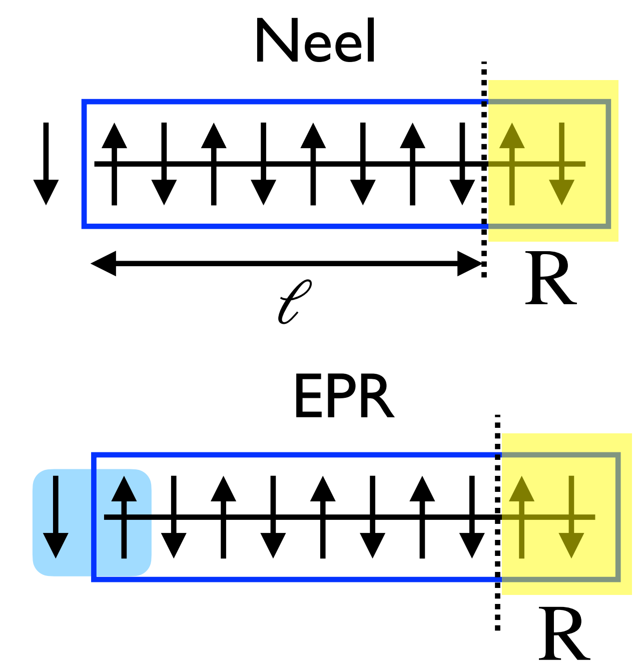

$t^* > t_{dif}$

system thermalizes after the entanglement of the left edge at the initial moment spreads over the system

# Entanglement entropy: Transverse Ising model

Transverse Ising model, 14+1 sites




- large fluctuation of EE ⟺ system does not thermalize

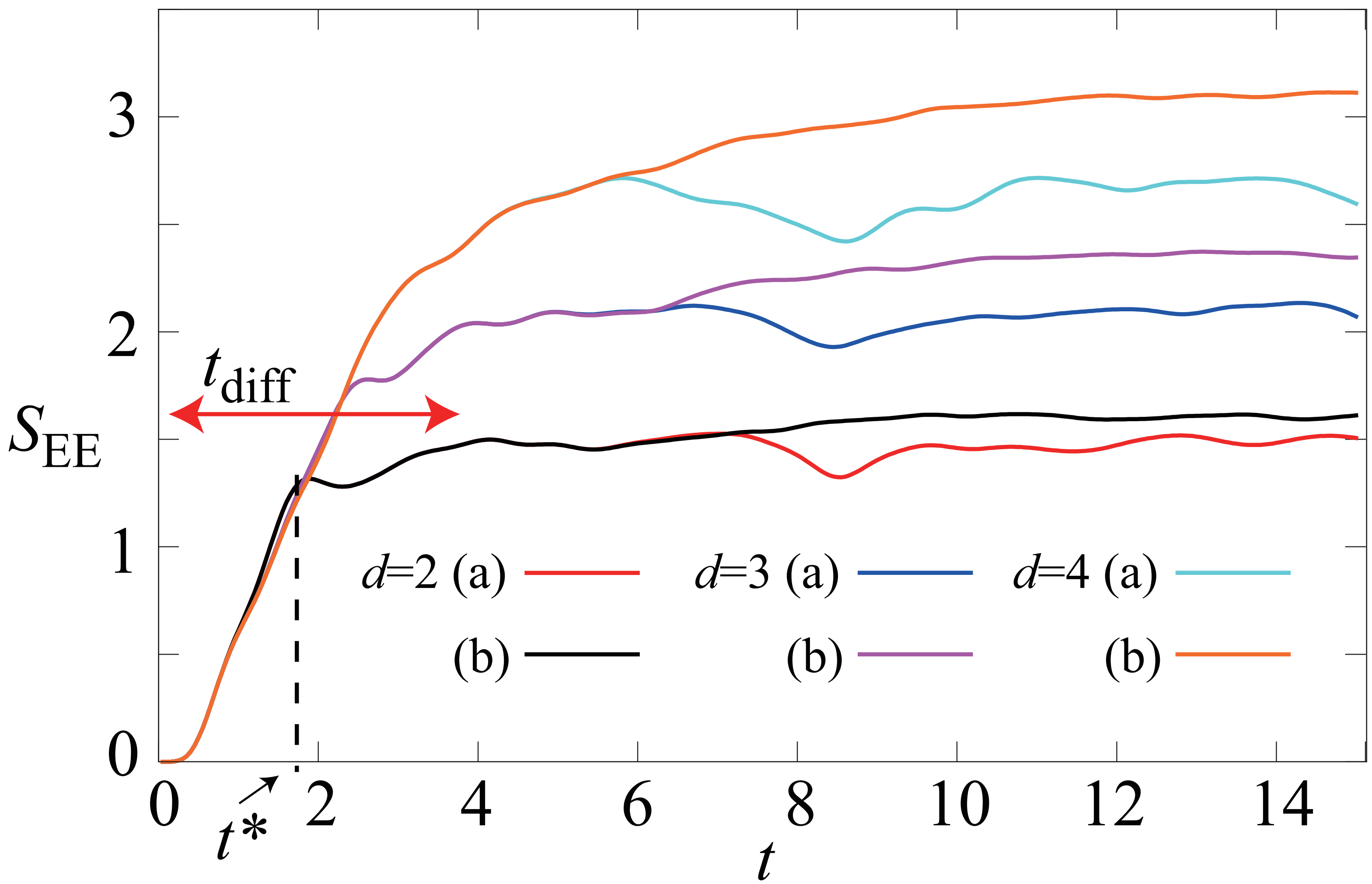
- entanglement reaches the subsystem R after the saturation: $t^* < t_{dif}$

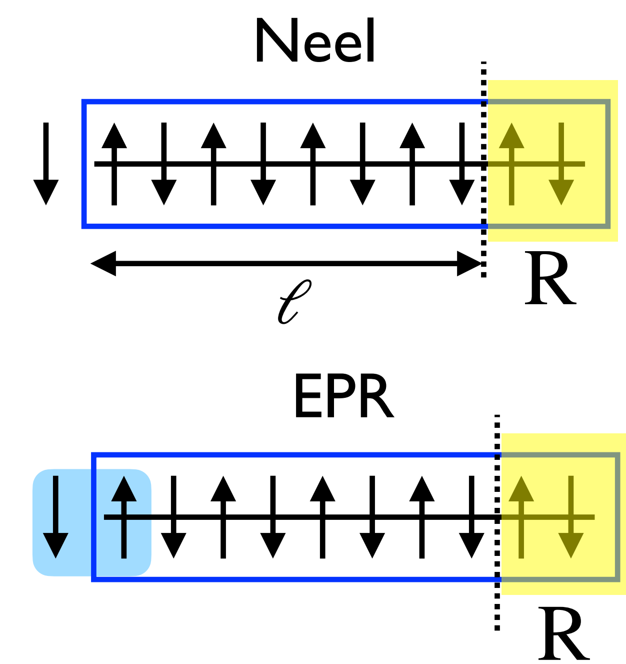
# How about Chaotic Ising Model?

- Non-integrable
- Thermalize or not is controversial

# Entanglement entropy: Chaotic Ising model

chaotic Ising model, 14+1 sites



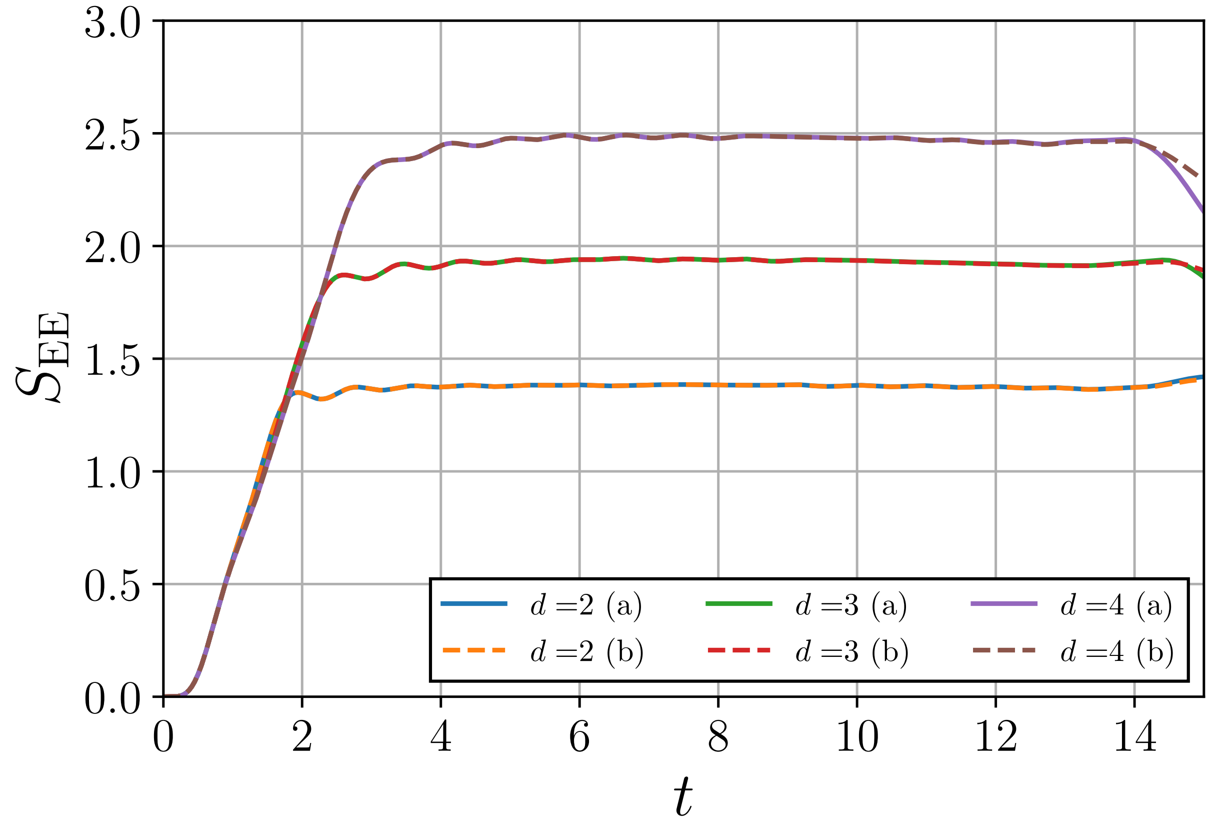

- large fluctuation of EE → system seems not thermalized

- entanglement reaches the subsystem R after the saturation: $t^* < t_{dif}$

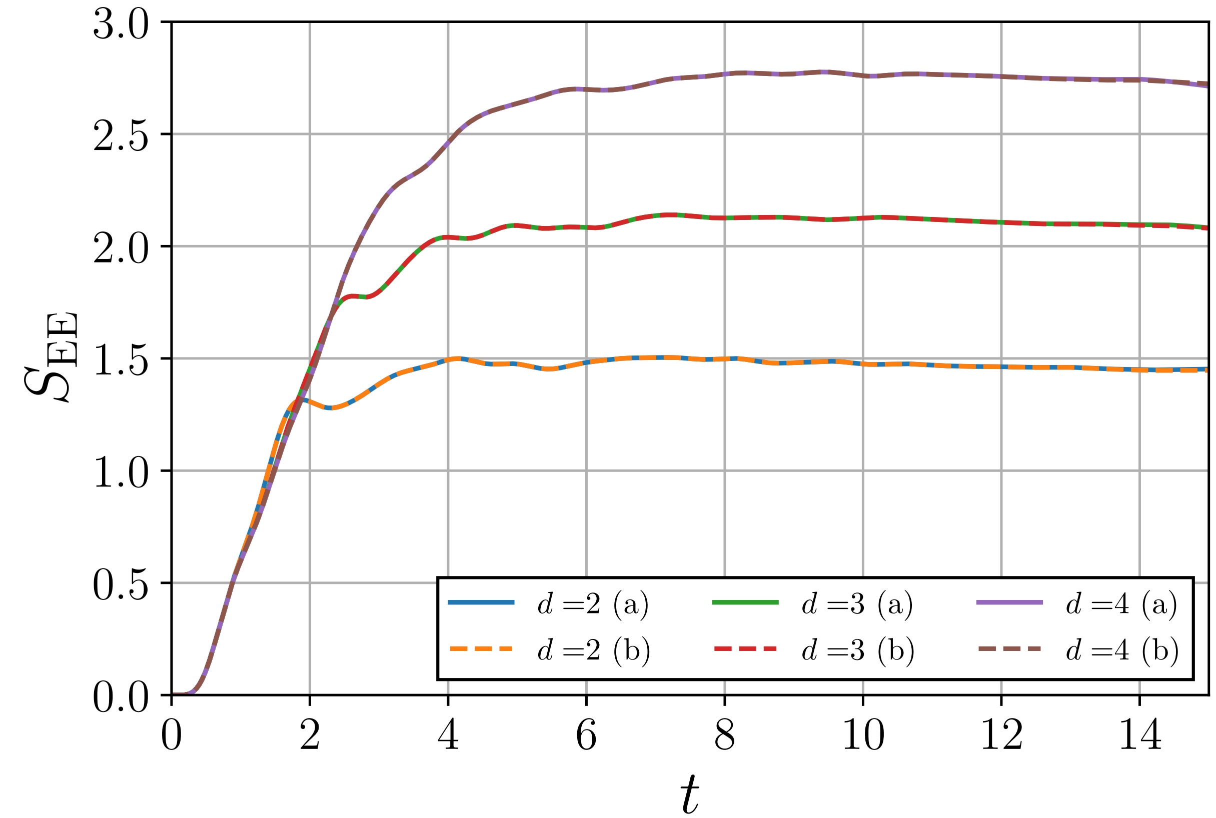
# Larger system size

(Time-evolving block decimation algorithm)

Transverse Ising model, 30+1 sites




Chaotic Ising model, 30+1 sites




Smaller fluctuation compared to N=14

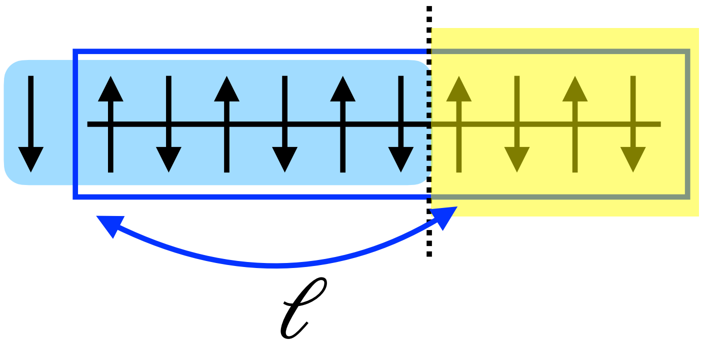
due to suppression of finite size effect

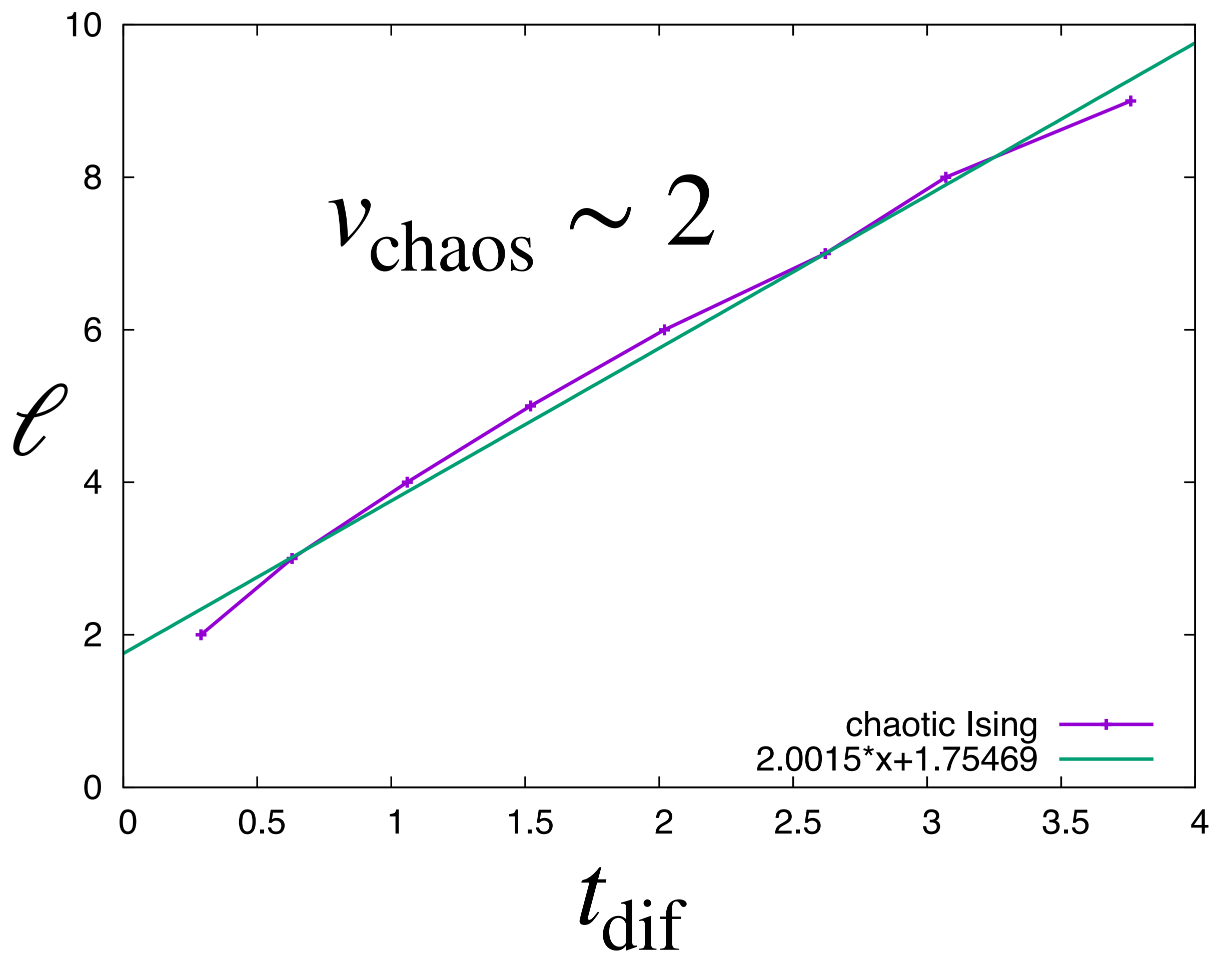
# How about the propagation speed?

- Difference between three model?
- Parameter dependence?

# Entanglement entropy: Chaotic Ising model

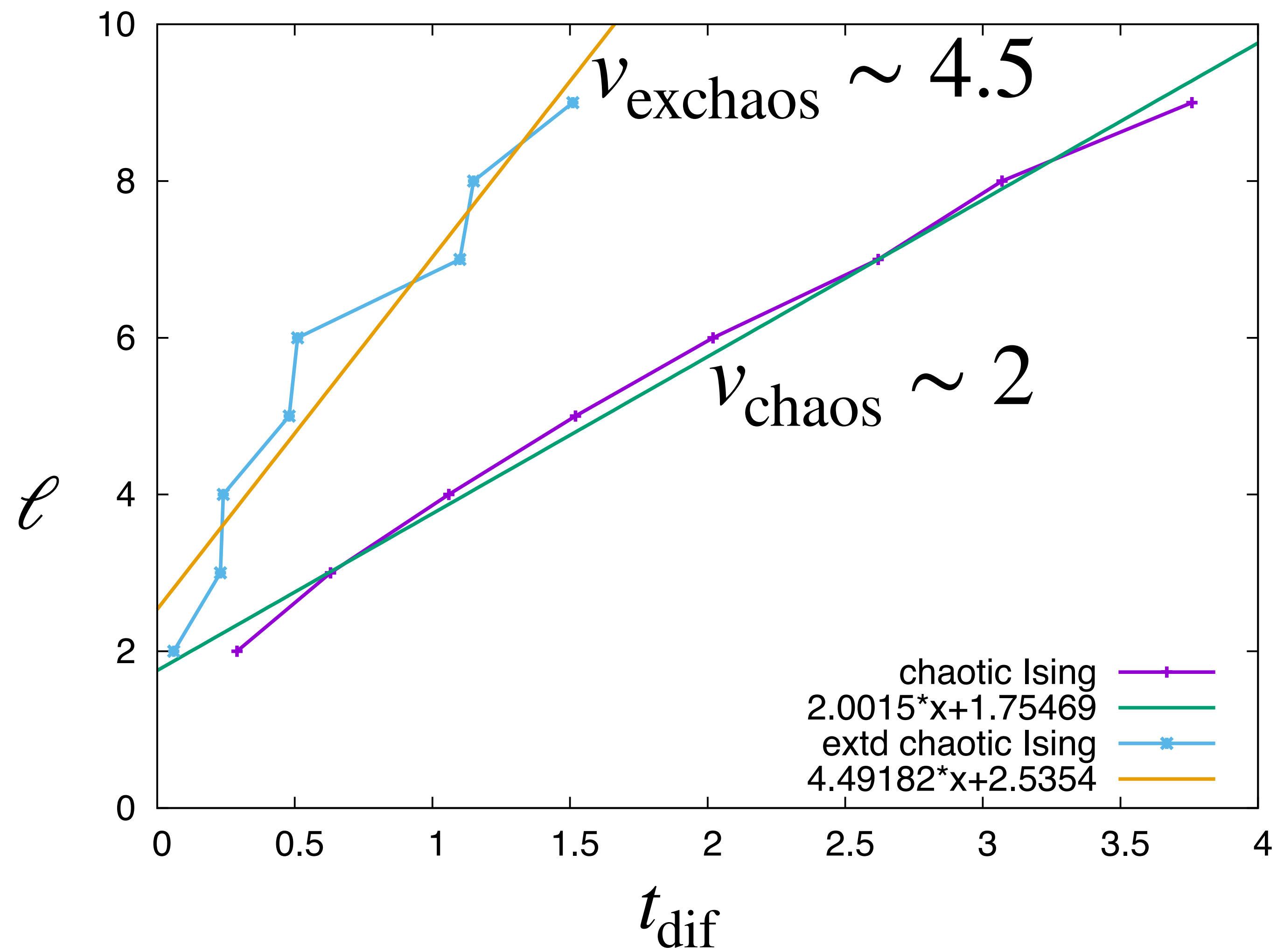
chaotic Ising model




- Entanglement ballistically propagates with a constant velocity irrespective to the integrability

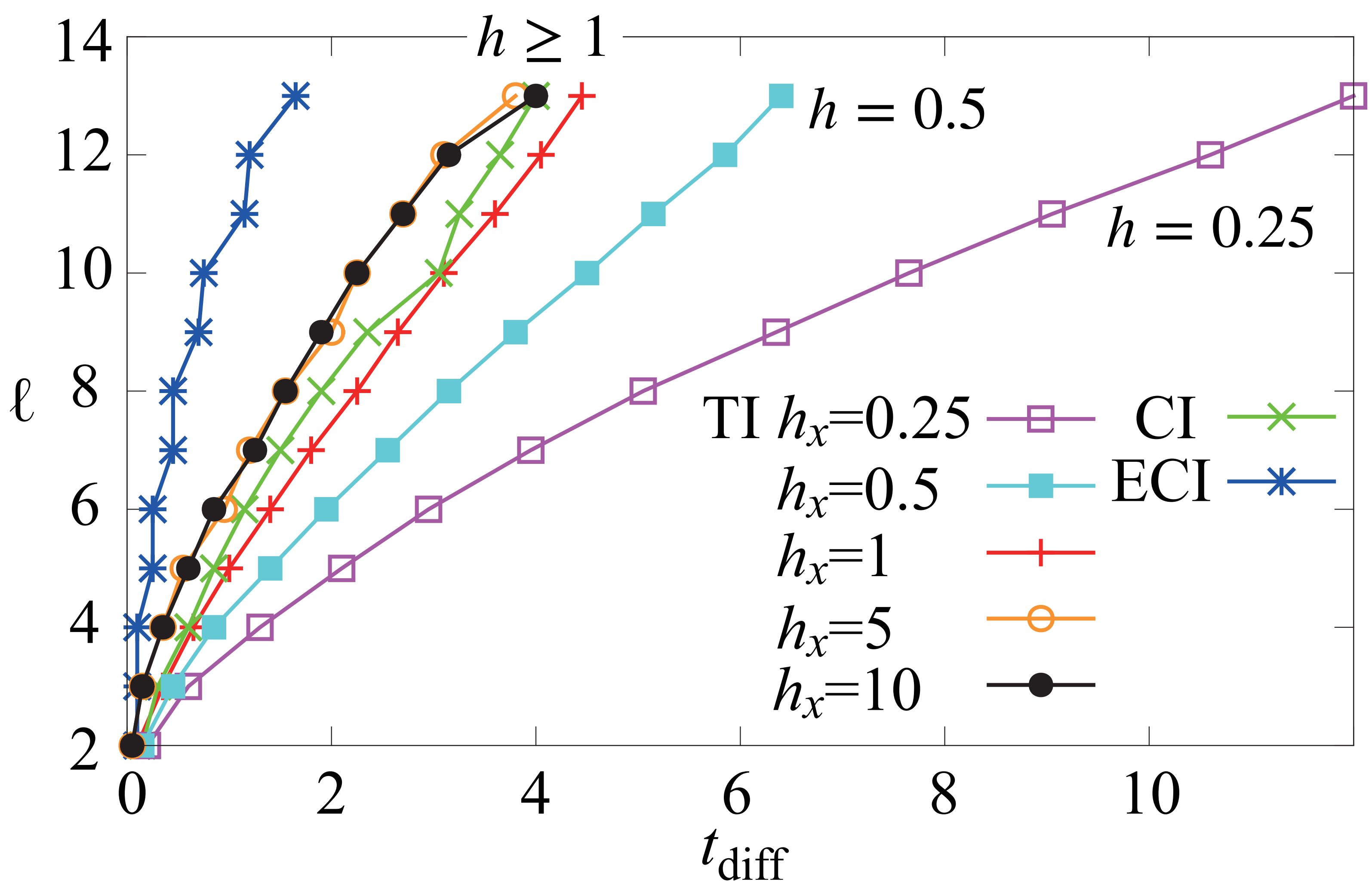
# Propagation speed of entanglement

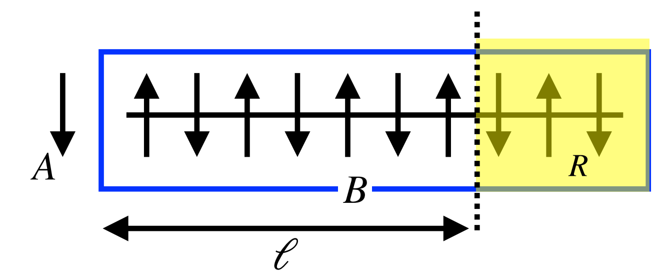



- $v_{\rm exchaos} \gg v_{\rm chaos}$ is related to thermalization?
- $v_{\rm chaos}$ and $v_{\rm exchaos}$ are much less than the Lieb-Robinson bound for the Ising model $v_{\rm LR} = 12eJ$.

# Propagation speed of entanglement

transverse Ising model $H = -J\sum_i \sigma_i^z \sigma_{i+1}^z - h\sum_i \sigma_i^x \quad J = 1$




$$v_{\rm EE} \simeq \begin{cases} 2h & (h < 1) \\ 2 & (h \geq 1) \end{cases}$$

## Propagation speed of entanglement

transverse Ising model $$H = -J\sum_i \sigma_i^z \sigma_{i+1}^z - h\sum_i \sigma_i^x$$

Jordan-Wigner transformation

$$H = \sum_k \varepsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2}\right)$$

$$\varepsilon_k = 2\left(J^2 + h^2 - 2hJ\cos k\right)$$

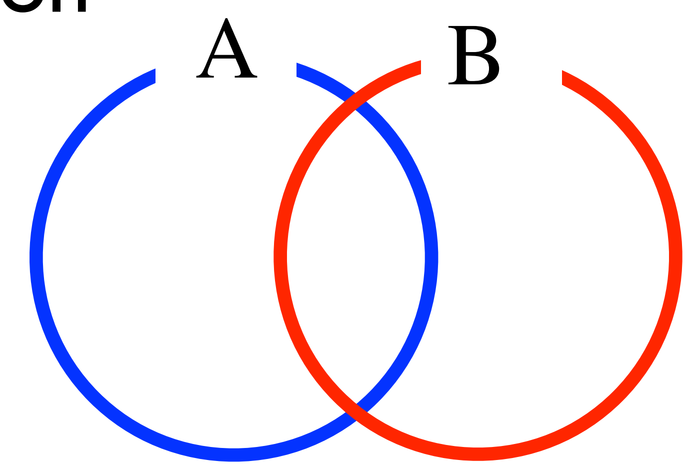
$$v_{\mathrm{max}} = \max\left(\frac{d\varepsilon_k}{dk}\right) = \begin{cases} 2h & (h/J < 1) \\ 2J & (h/J \geq 1) \end{cases}$$

the propagation speed of entanglement coincides with the **maximum group velocity** of quasiparticles

entanglement is carried by quasiparticles

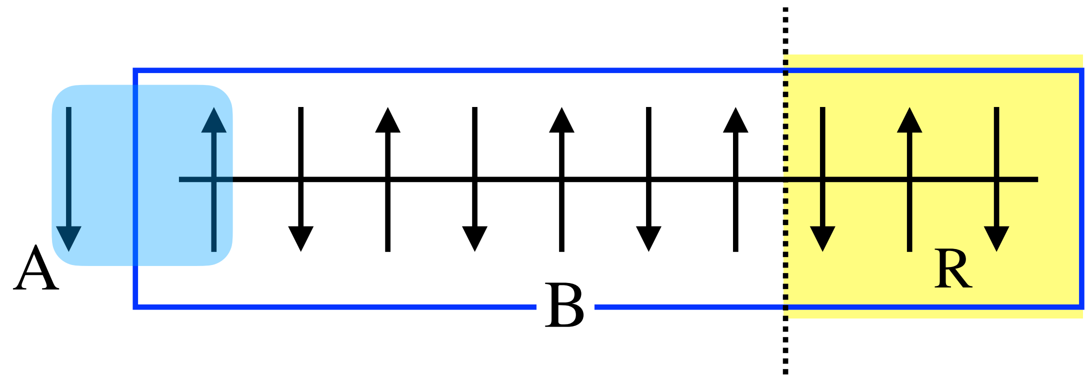
# Bipartite mutual information

mutual information measures correlation
between A and B



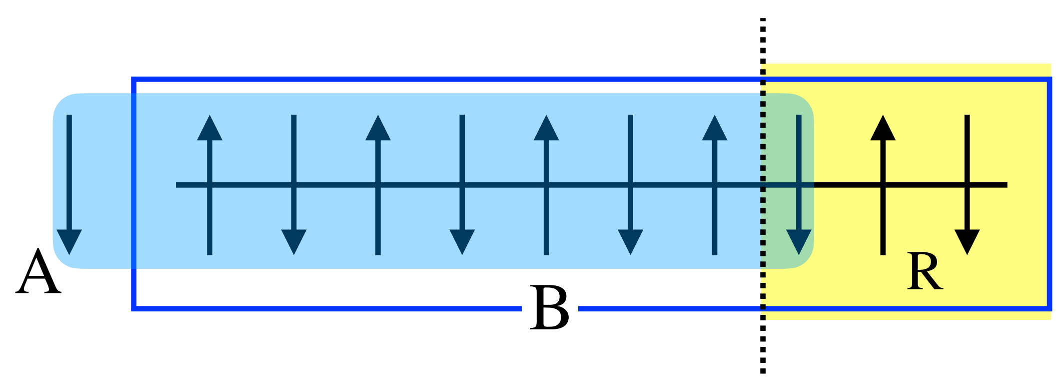

$$I(\mathrm{A}\,|\,\mathrm{B}) = S_{\mathrm{A}} + S_{\mathrm{B}} - S_{\mathrm{AB}}$$

$t = 0$




$S_{\mathrm{A}} = 1,\ S_{\mathrm{R}} = 0,\ S_{\mathrm{AR}} = 1$ $\quad I(\mathrm{A}\,|\,\mathrm{R}) = 0$

$t = t_{\mathrm{MI}}$

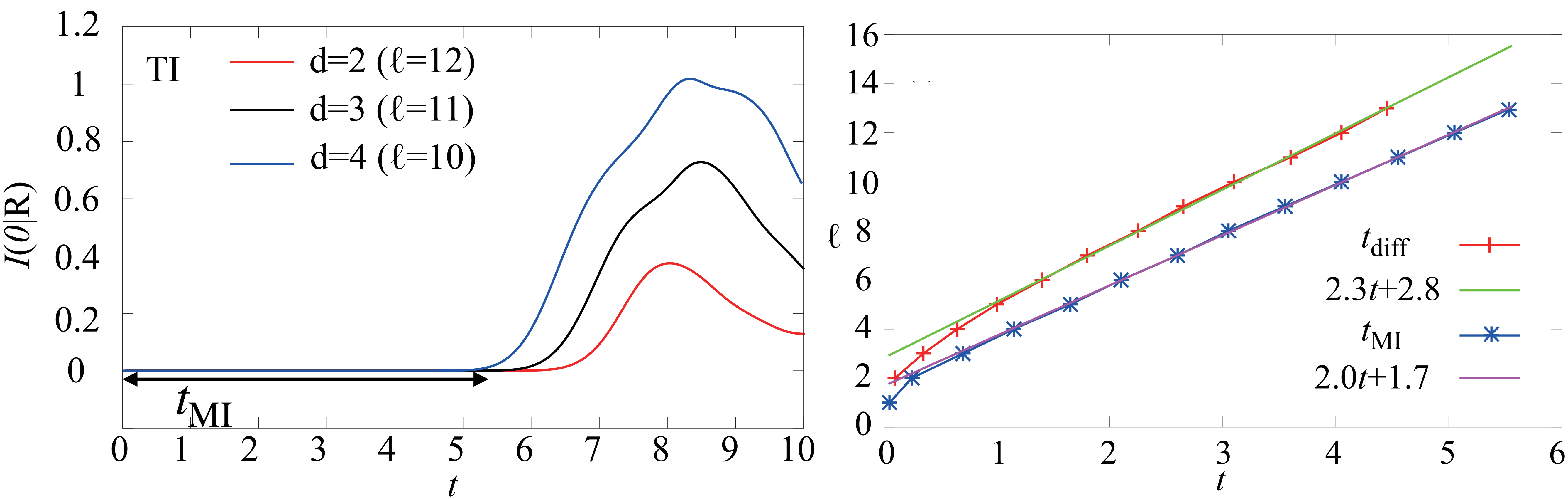



entanglement reaches the subsystem R at $t_{\mathrm{MI}}$

correlation between A and R $\quad I(\mathrm{A}\,|\,\mathrm{R}) \neq 0$

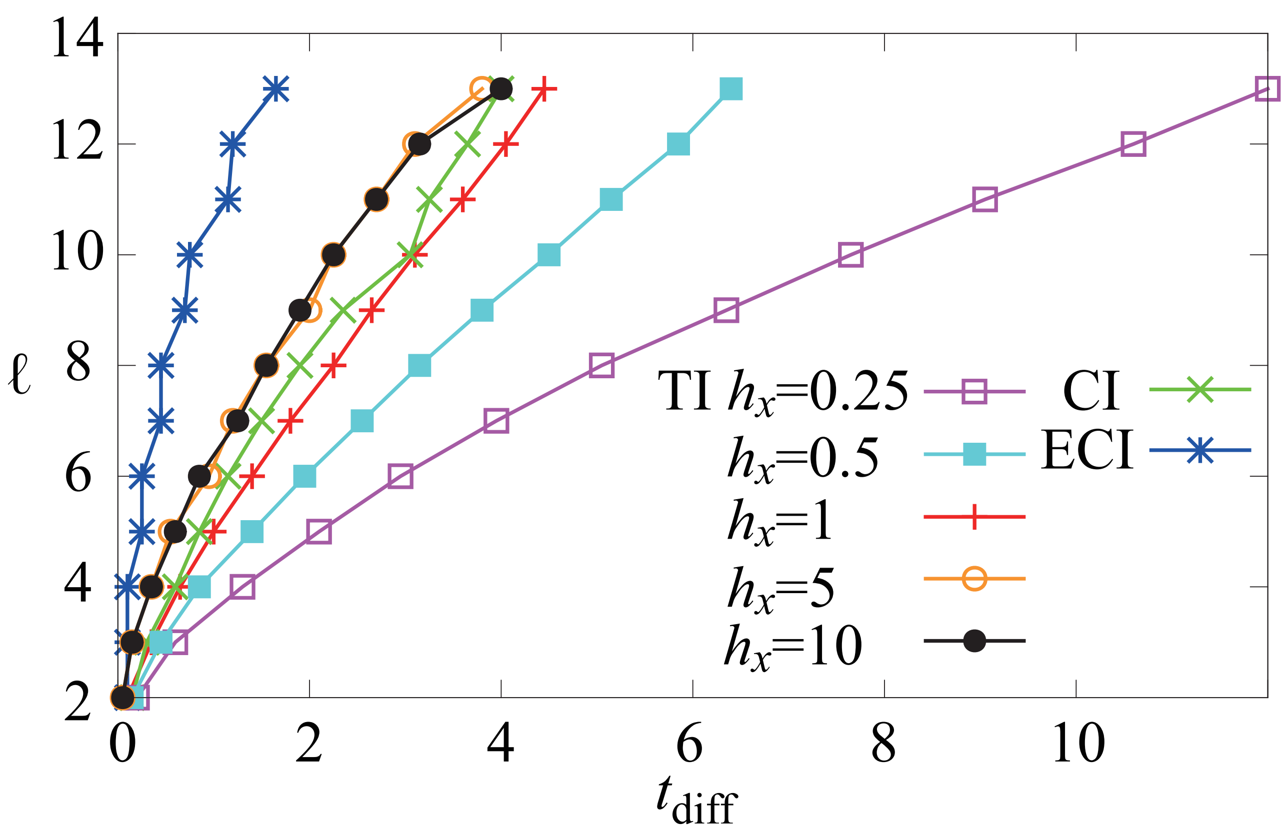
# Bipartite mutual information

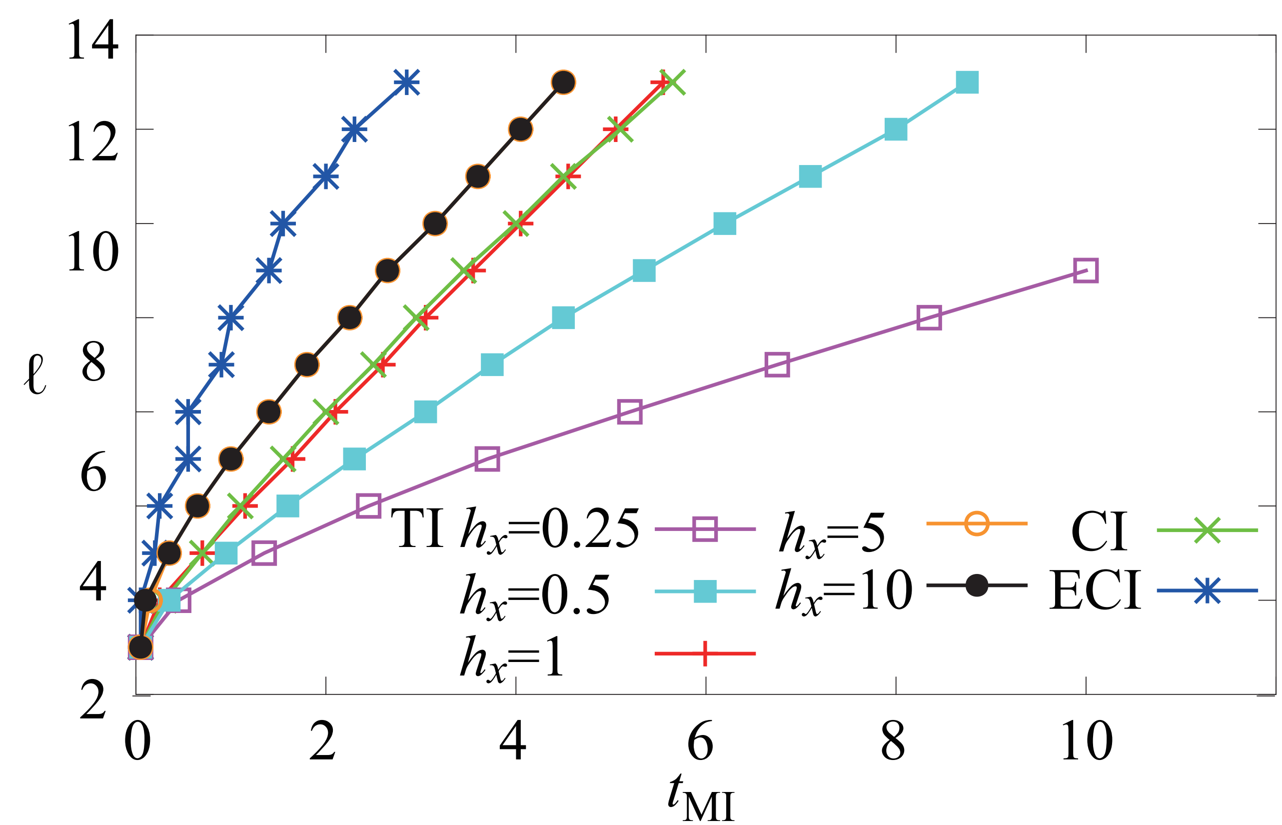
Propagation speed of entanglement evaluated by two quantities




- $t_{\rm MI} \sim t_{\rm dif}$ means that A and R starts to have correlation when the influence of the entanglement reaches the subsystem.
- Two velocities measured by entanglement entropy and mutual information agree well: $v_{\rm MI} \sim v_{\rm EE}$ .
- Bipartite information is also carried by quasiparticles.

# Consistency between two method

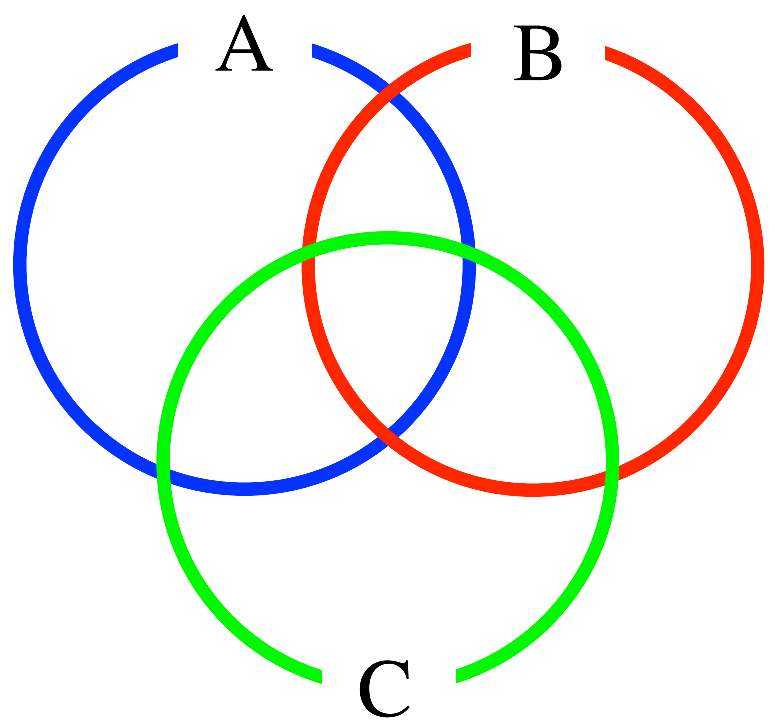


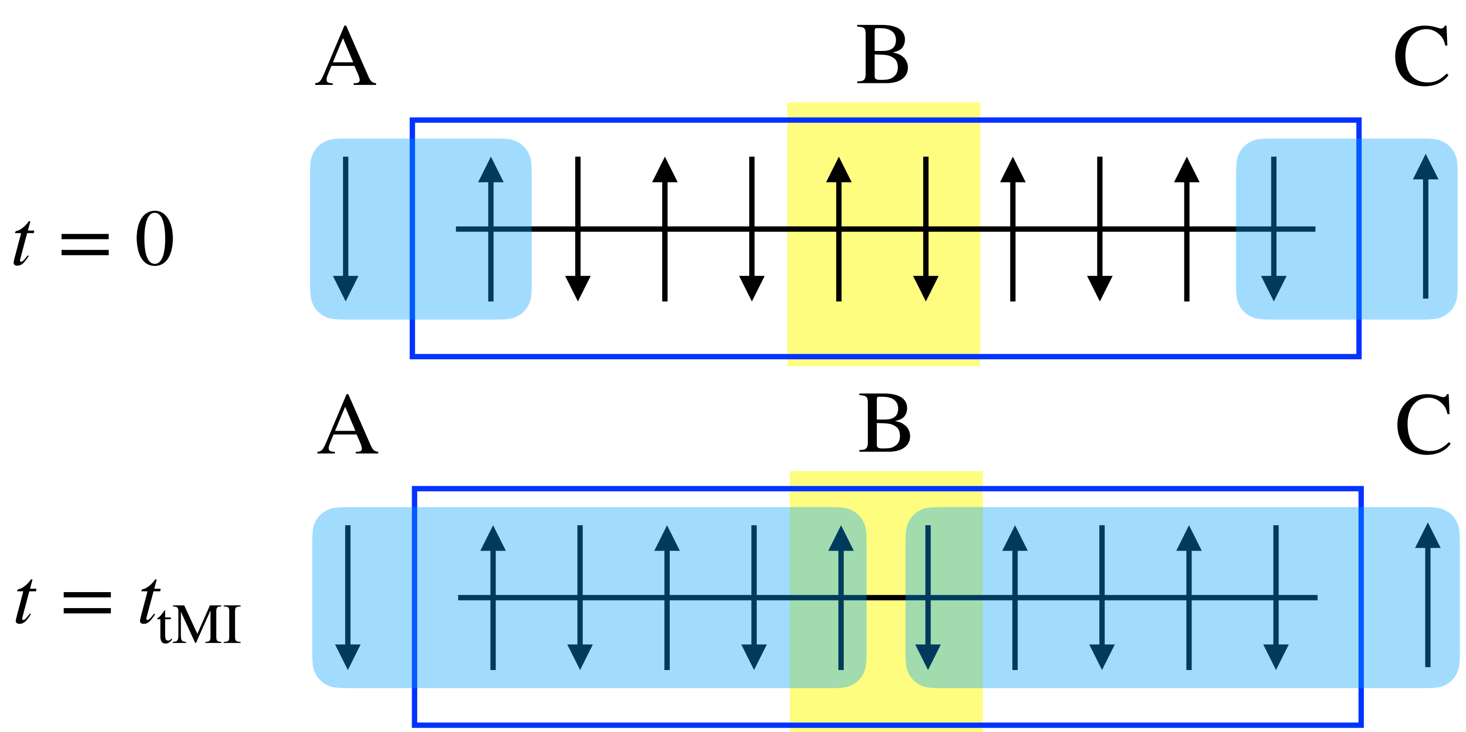




Qualitatively the same behavior is checked

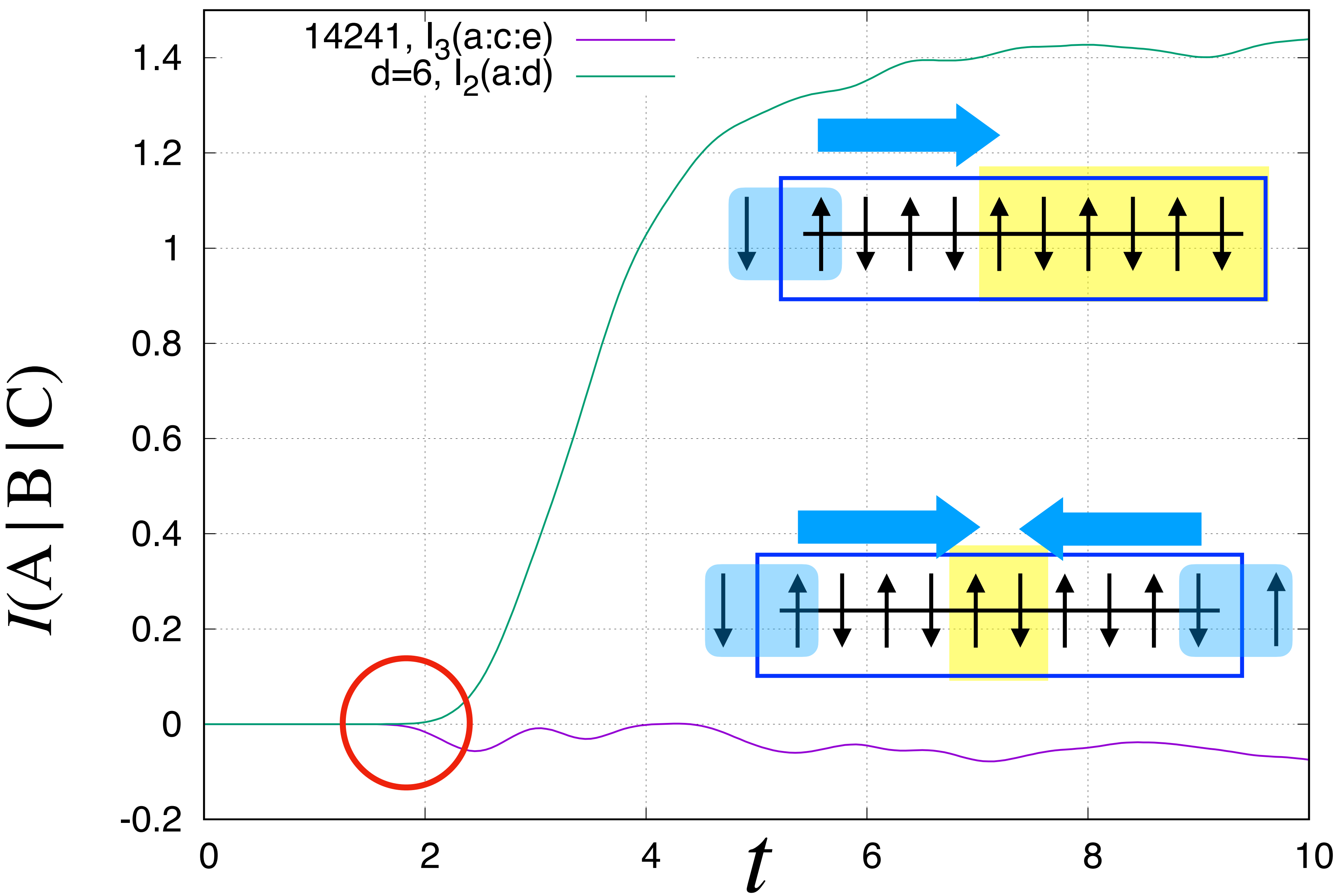
# Tripartite mutual information

$$I(\mathrm{A}\,|\,\mathrm{B}\,|\,\mathrm{C}) = S_{\mathrm{A}} + S_{\mathrm{B}} + S_{\mathrm{C}} - S_{\mathrm{AB}} - S_{\mathrm{BC}} - S_{\mathrm{CA}} + S_{\mathrm{ABC}}$$



# Tripartite mutual information



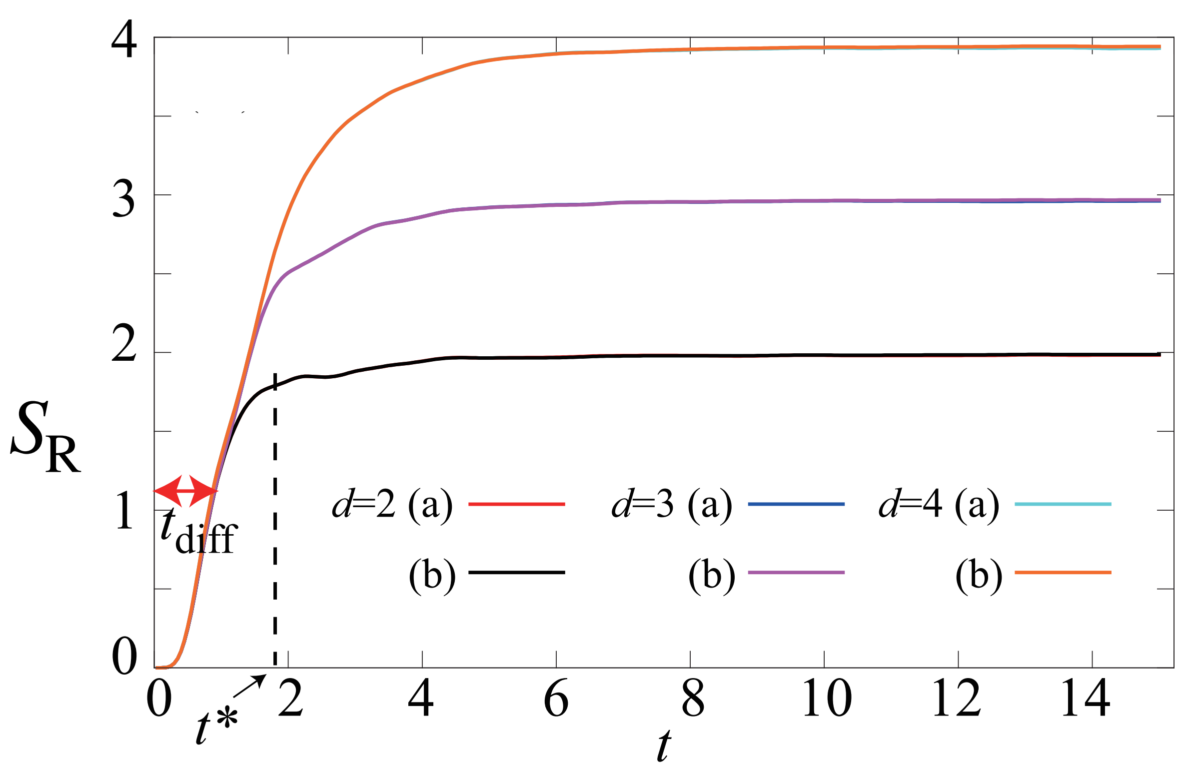

- Tripartite bipartite correlations propagate with the same speed!
- Negative $I_3(\mathrm{A}\,|\,\mathrm{B}\,|\,\mathrm{C})$ implies that scrambling takes place.

# Outline

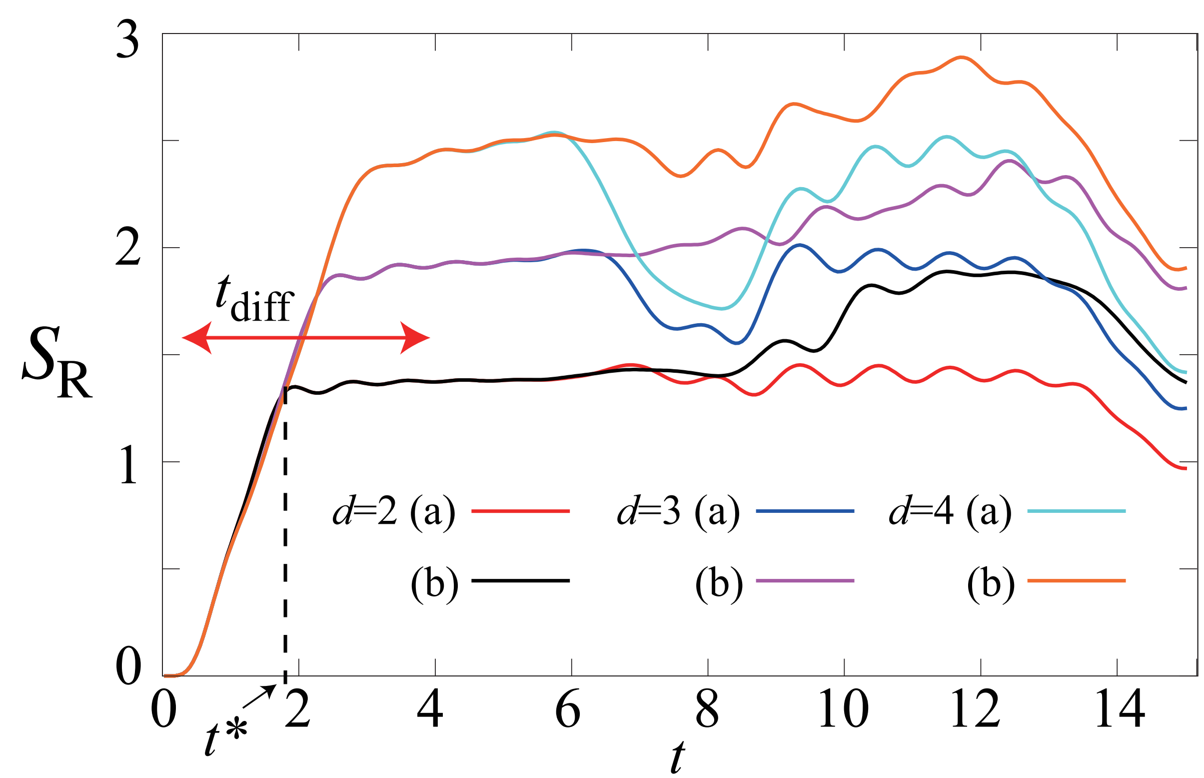
- Introduction: Thermalization of isolated systems, transport of entanglement
- Model
- Time-evolution of entanglement entropy
- Discussion on transport of entanglement and thermalization
- Conjecture for the condition of thermalization
- Summary

# Discussion on two time scales

extended chaotic Ising

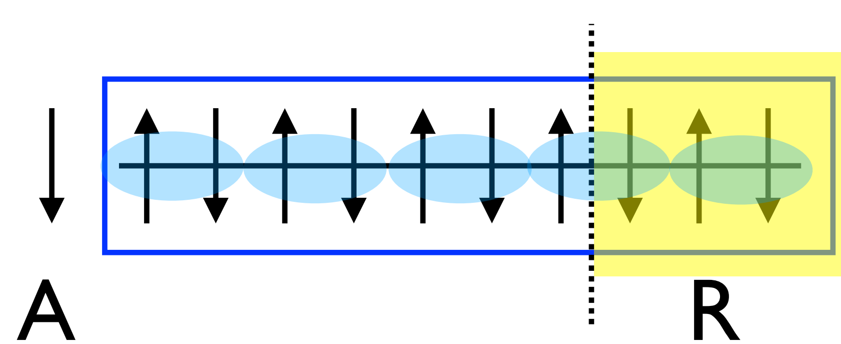



Transverse Ising




$S_R$ increases and saturates even before influence of A gets to R

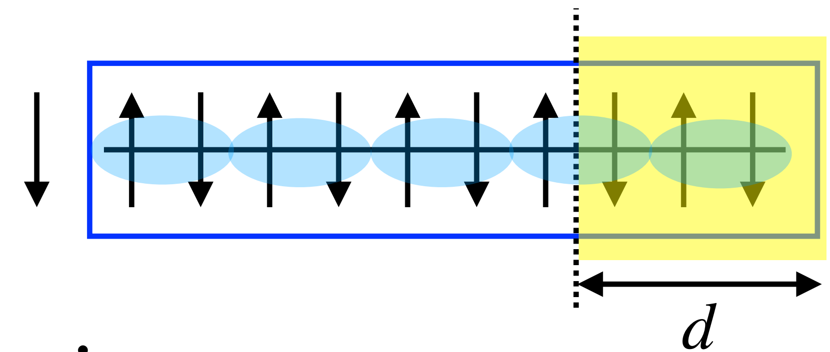
$t^*$ is determined by locally created entanglement




$t^*$ might not appropriate to characterize propagation speed of entanglement

# Discussion on local creation of entanglement

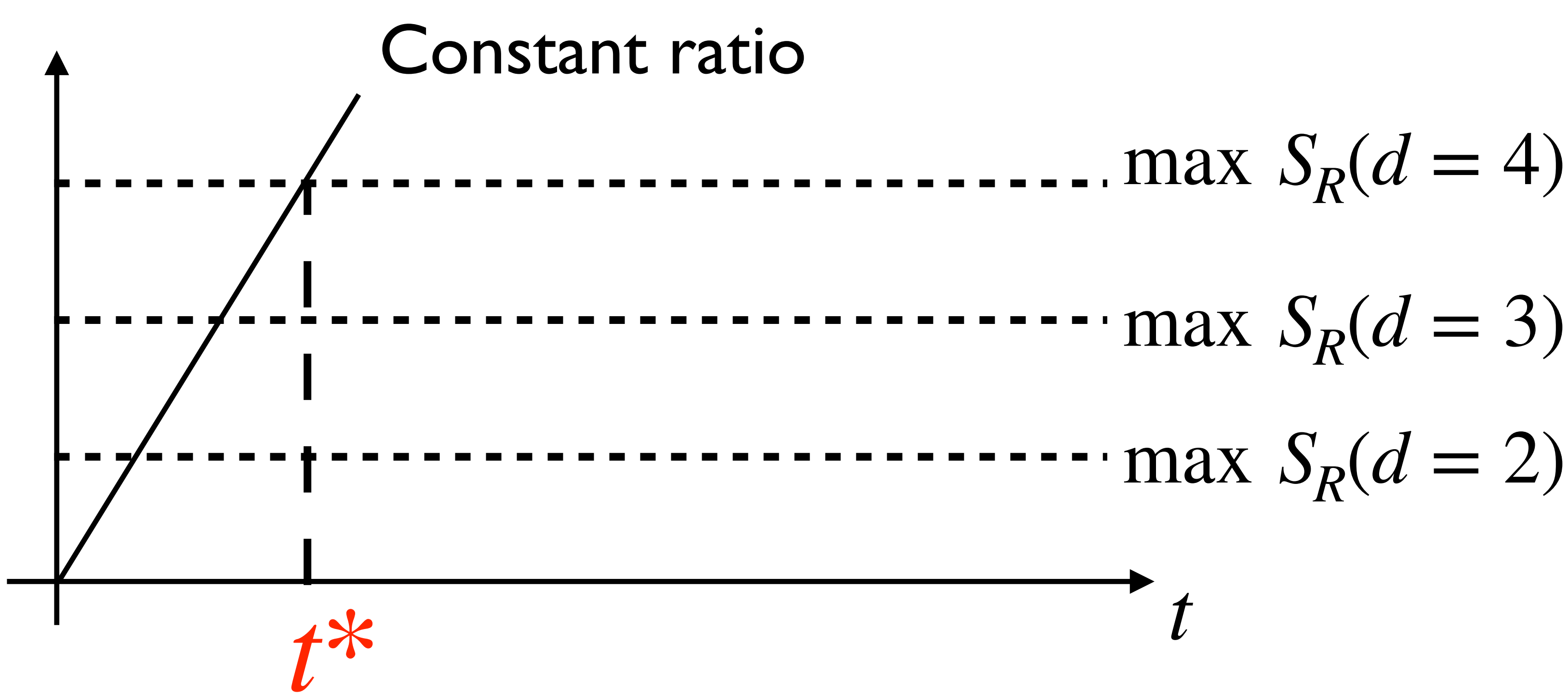
$S_R$ increases with a constant ratio




➡ EE is locally created with a constant ratio

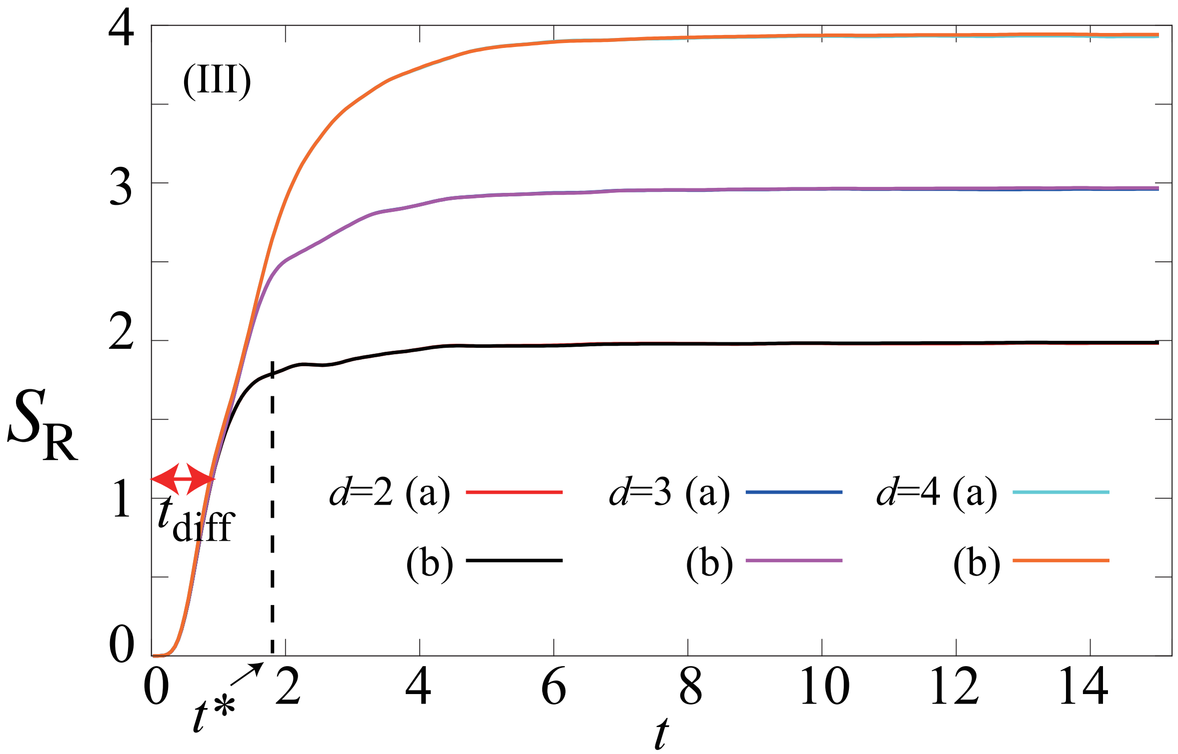
Size of the Hilbert space for the subsystem: $2^d$

➡ $\max S_R \propto d$ : volume law



# Discussion on entanglement transport and thermalization

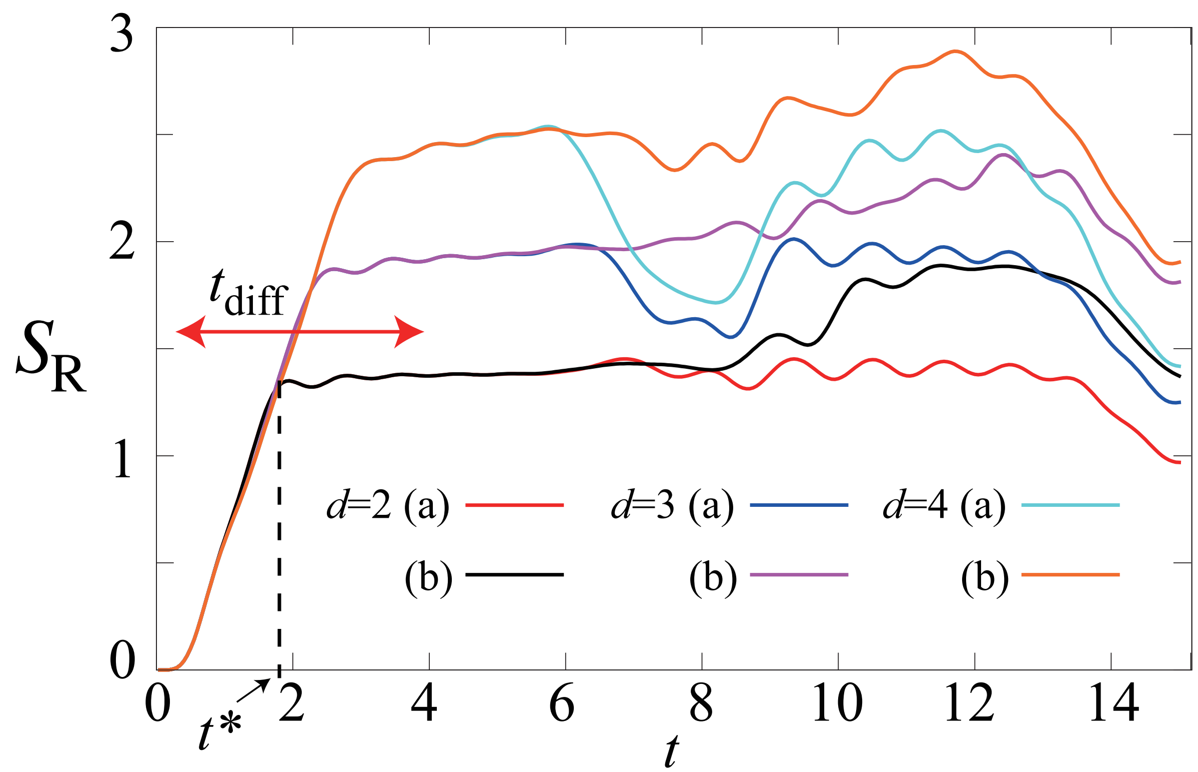
extended chaotic Ising




$t_{dif} \ll t^*$

thermalizes

Transverse Ising




$t_{dif} > t^*$

not thermalizes

$\boxed{\text{Condition for thermalization: } t_{dif} \ll t^*}$

$t_{dif} \ll t_*$ ➡ entanglement spreads very fast over the system

➡ system is “scrambled" enough ➡ Thermalization

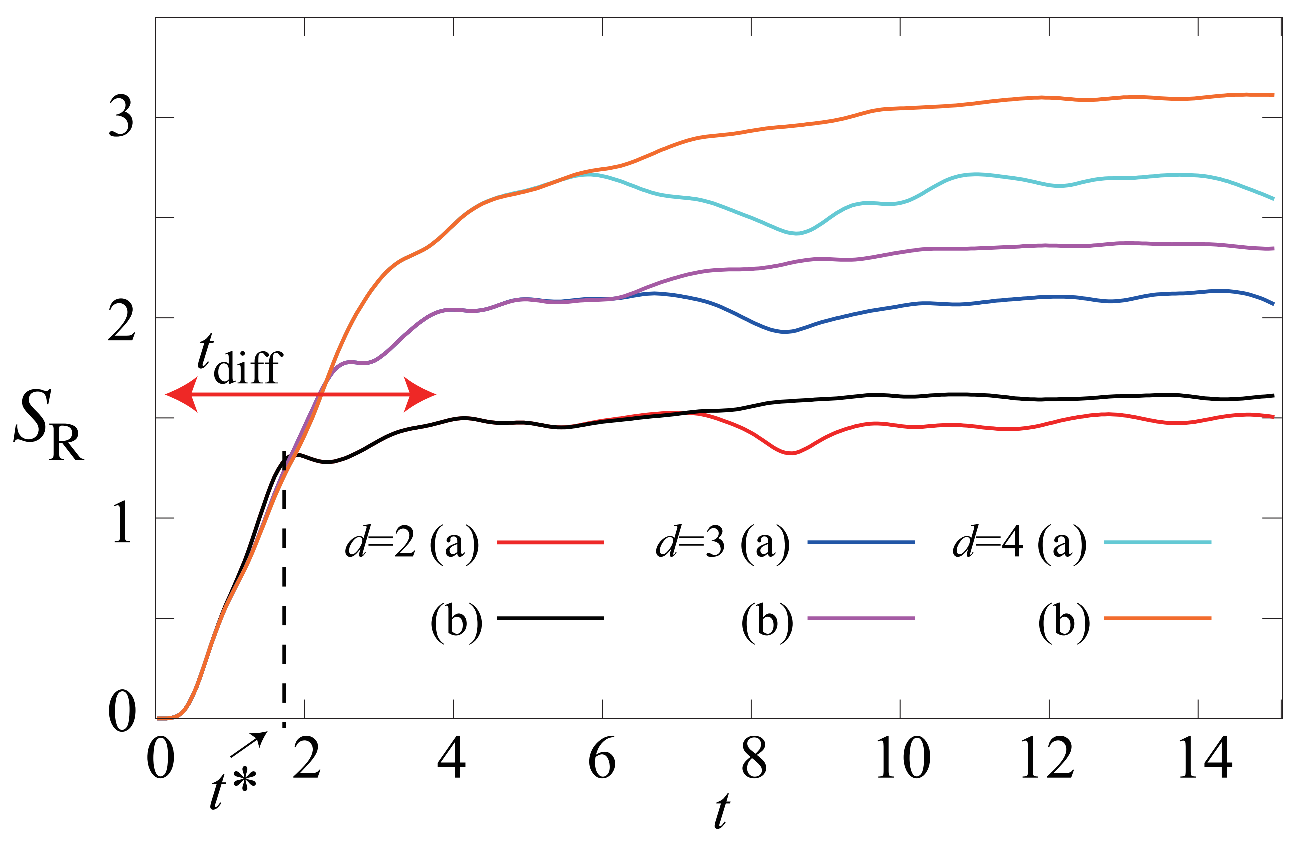
# Speculation on Chaotic Ising model

Our conjecture

Condition for thermalization: $t_{\rm dif} \ll t^*$

$t_{\rm dif} \ll t^*$ thermalizes

$t_{\rm dif} > t^*$ not thermalizes



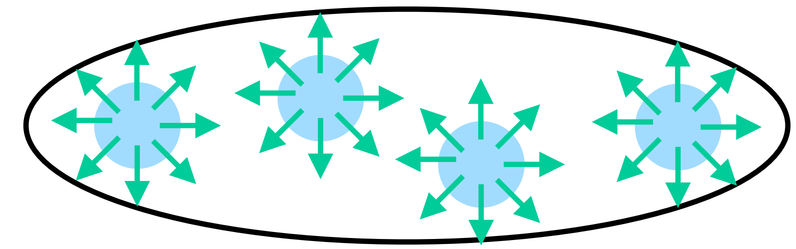

$t_{\rm dif} > t^*$ not thermalizes

# Conjecture

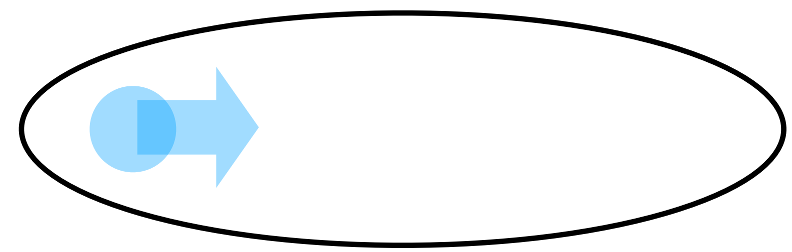
## Thermalization, EE production and EE transport

- Locally thermalize due to local EE production toward maximum value of EE

$t \sim t^*$

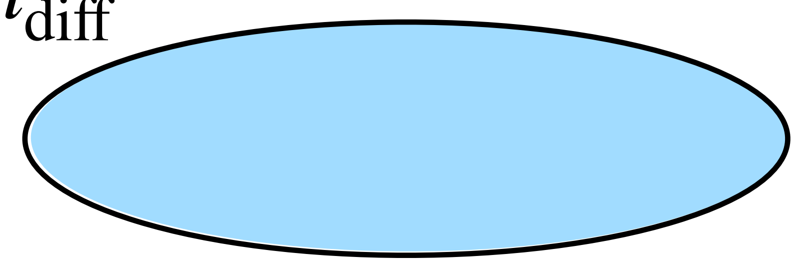


- Locally produced EE propagates

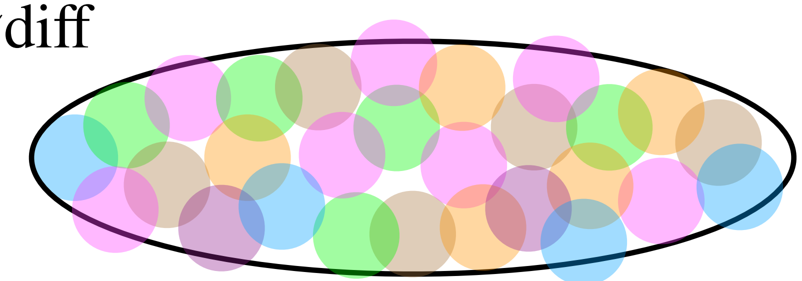


$t \sim t_{\mathrm{diff}}$

$t^* \gg t_{\mathrm{diff}}$



$t^* \ll t_{\mathrm{diff}}$

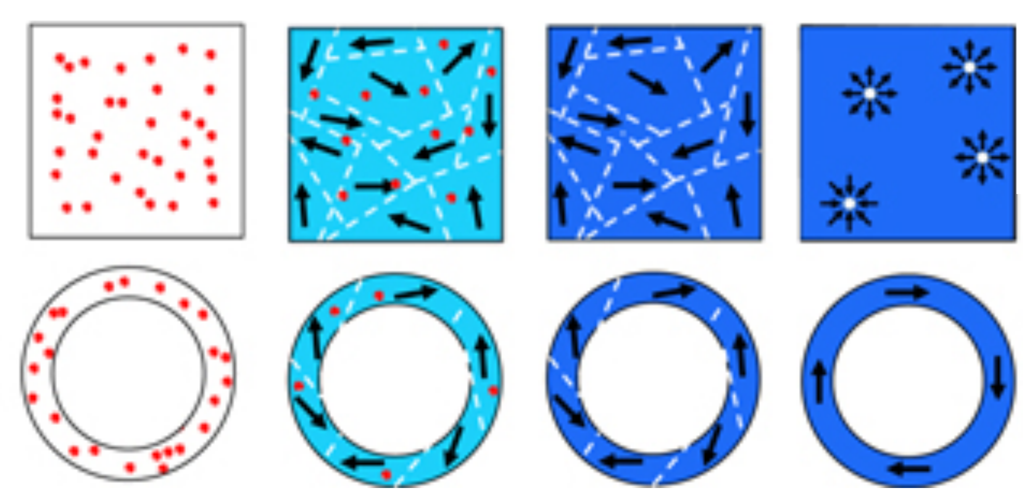


Scrambled enough for thermalization

Global thermalization is not achieved

“Kibble-Zurek” like mechanism

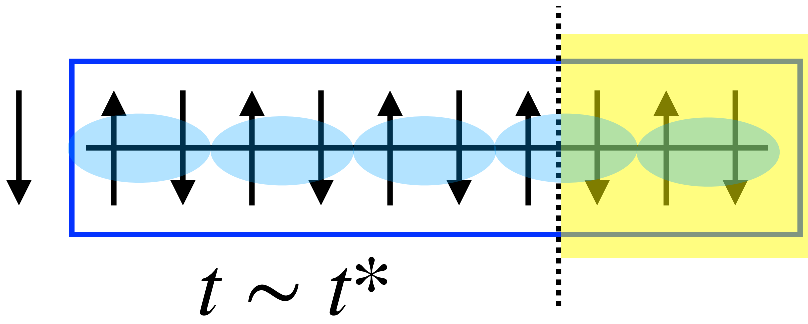
global phase coherence of condensate is achieved after locally formed condensates are Josephson coupled

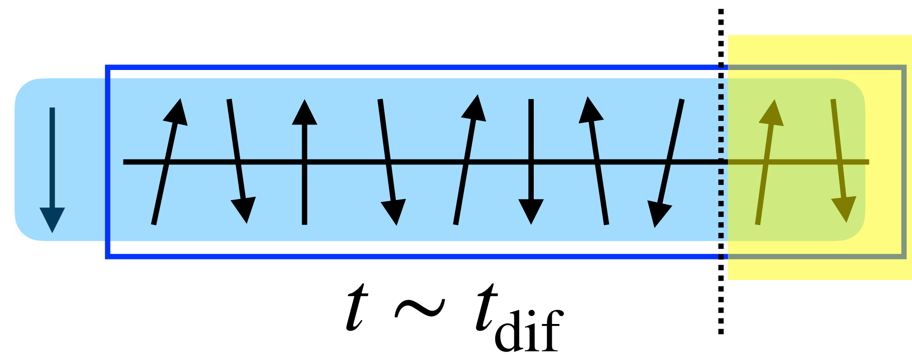


# Summary

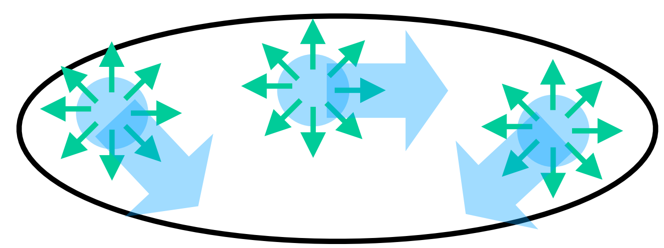
- Two time scales characterizing time evolution of $S_R$

  - Transport of entanglement: $t_{\rm dif}$
  - Local creation of entanglement: $t^*$




- Entanglement propagates with a constant velocity
- Condition for thermalization: $t_{\rm dif} \ll t^*$
- Kibble-Zurek" like mechanism



Future perspectives

- Relation between thermalization and the two time scales
- Tripartite mutual information and OTOC