Tutorial Session, Quantum Information Entropy, March 21, 2022

Quantum entanglement in topological phases and symmetry-broken phases

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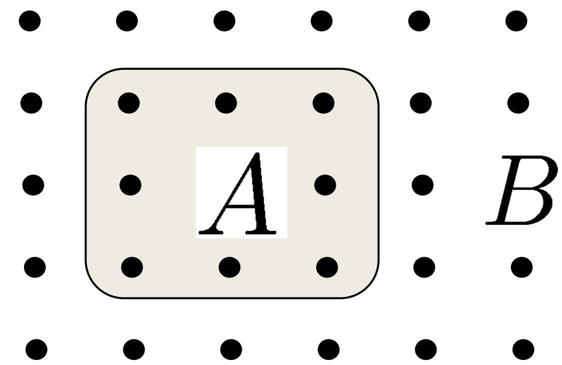
Contents

- Introduction: Entanglement entropy in condensed matter
- Quantum entanglement in topological phases
- Quantum entanglement in systems with continuous symmetry breaking

What is entanglement?

Structure of a quantum state which cannot be represented as a product form



- $|\Psi\rangle = |\Psi_A\rangle|\Psi_B\rangle$

  No entanglement between A and B

⇔ Reduced density matrix on A is a pure state.

$$\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi| = |\Psi_A\rangle\langle\Psi_A|$$

- $|\Psi\rangle \neq |\Psi_A\rangle|\Psi_B\rangle$

  There is entanglement between A and B

⇔ Reduced density matrix on A is a mixed state.

How to quantify entanglement ?

⇨ Measure how mixed the reduced density matrix is.

Entanglement entropy (von Neumann entropy)

$$S_A = -\mathrm{Tr}\, \rho_A \log \rho_A \;\; (= S_B)$$

$$= -\sum_i p_i \log p_i \qquad \{p_i\}\text{:eigenvalues of } \rho_A$$

- product state ⟶ pure state $\quad p_i = 1,\ 0$

$$|\Psi\rangle = |00\rangle \qquad \rho_A = |0\rangle\langle 0| \qquad \boxed{S_A = 0}$$

- entangled state ⟶ mixed state $\quad p_i = \frac{1}{2}, \ \frac{1}{2}$

$$|\Psi\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad \rho_A = \tfrac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$\boxed{S_A = \log 2}$$

Scaling of entanglement entropy

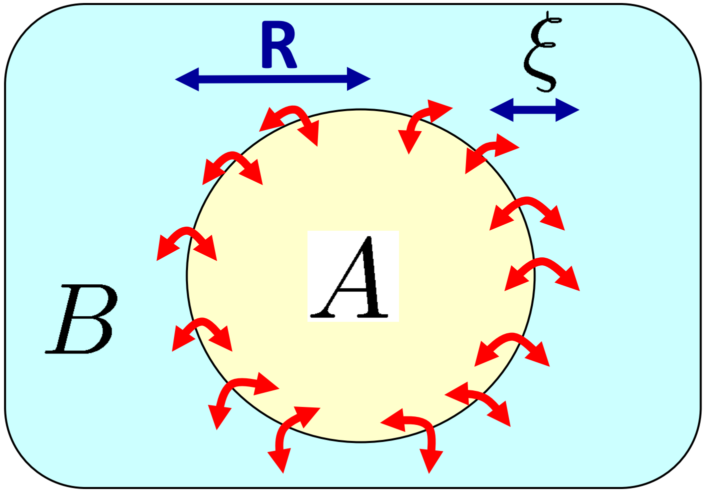
- If the system has only short-range correlations,

$$S_A \propto (\text{boundary size of A})$$

"boundary law" (or "area law")

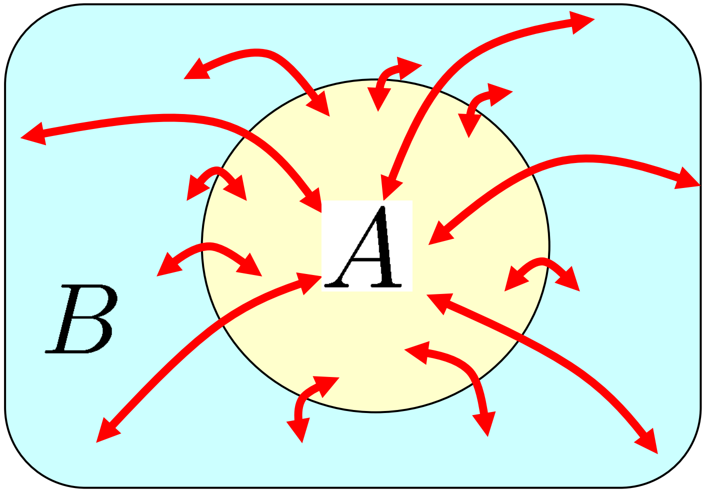
Srednicki, PRL, 1993

Wolf, Verstraete, Hastings, & Cirac, PRL, 2008



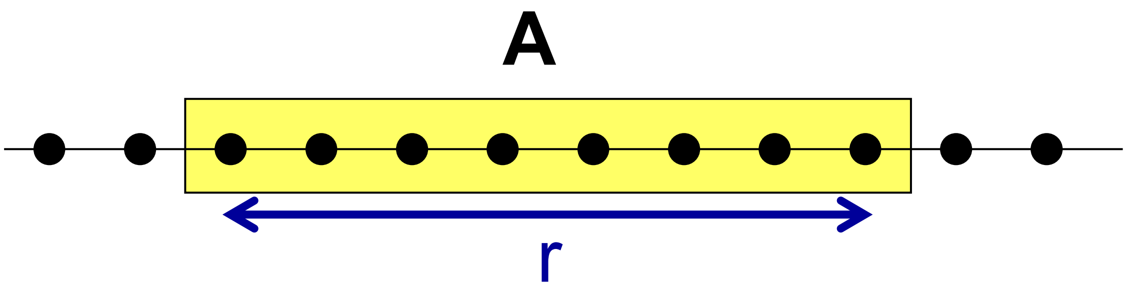
- Deviation from a boundary law (or an additional term)

  ⇨ Certain long-range or non-local correlations



Useful applications to critical phenomena, topological order, etc.

One dimension



Holzhey, Larsen, & Wilczek, Nucl.Phys.B, 1994
Vidal, Latorre, Rico, & Kitaev, PRL, 2003
Calabrese & Cardy, J. Stat. Mech., 2004

- Gapped (non-critical) system

$$S_A \to \text{const.} \ (r \to \infty)$$

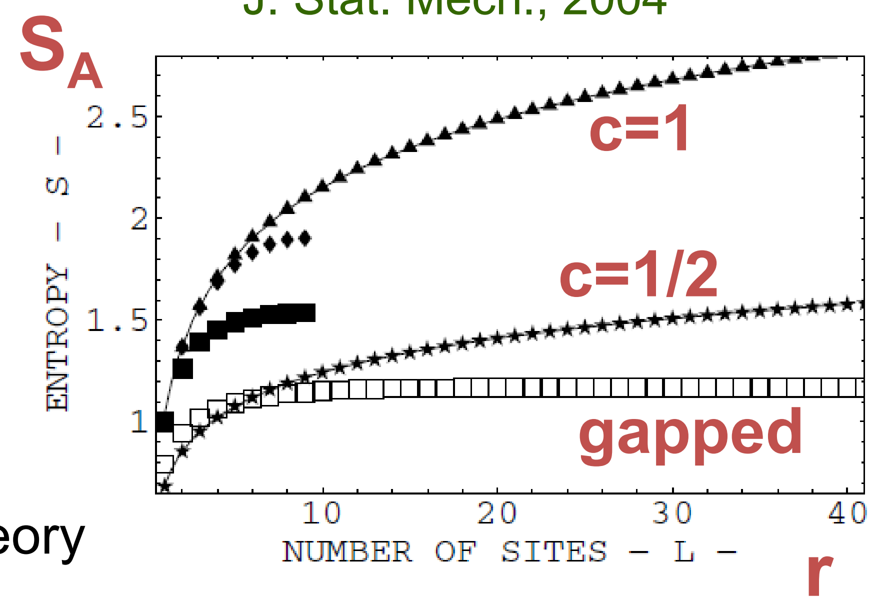
"boundary law" (or "area law")

- Critical system

$$S_A \simeq \frac{c}{3} \log r + s_1$$

c: central charge of conformal field theory

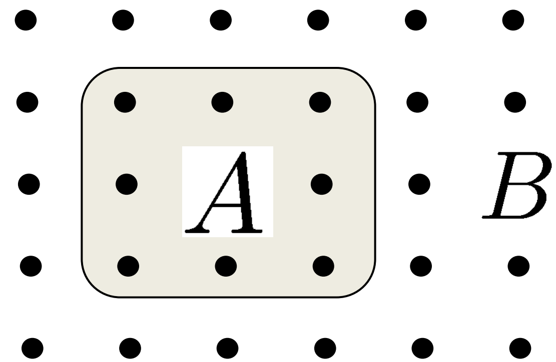
$\simeq$ number of gapless modes



Underlying field theory ⟷ generic structure of ground state

Schmidt decomposition

Express as a sum of product state:



$$|\Psi\rangle = \sum_i \lambda_i |\psi_i^A\rangle |\psi_i^B\rangle$$

$$\langle \psi_i^A | \psi_j^A \rangle = \langle \psi_i^B | \psi_j^B \rangle = \delta_{ij}$$

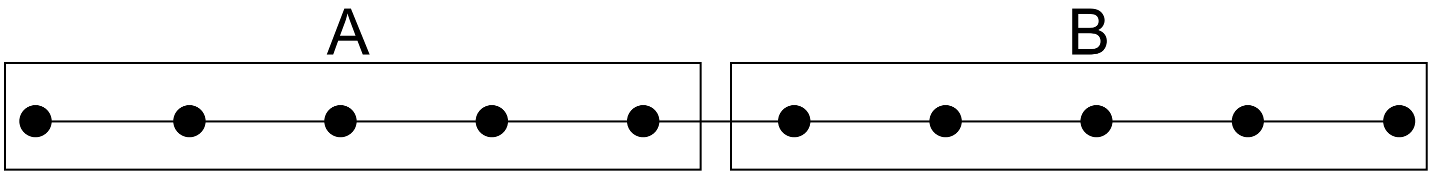
$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| = \sum_i |\psi_i^A\rangle \lambda_i^2 \langle \psi_i^A| \qquad \rho_B = \sum_i |\psi_i^B\rangle \lambda_i^2 \langle \psi_i^B|$$

$$S_A = S_B = -\sum_i \lambda_i^2 \log(\lambda_i^2)$$

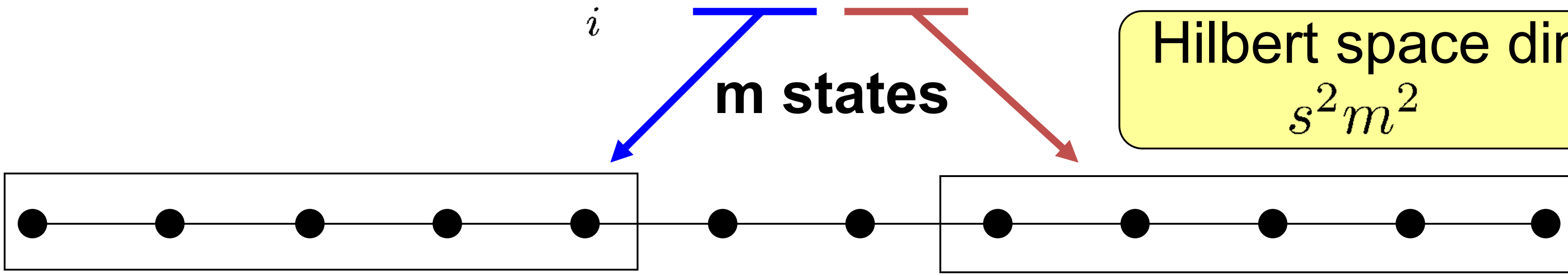
e.g. $\lambda_i^2 = \begin{cases} 1/m \quad (i = 1, 2, \ldots, m) \\ 0 \ (\text{otherwise}) \end{cases}$ $\Rightarrow$ $S_A = \log m$

(number of relevant states in decomposition) $\sim e^{S_A}$

DMRG (Density Matrix Renormalization Group)



$$|\Psi\rangle = \sum_i \lambda_i |\psi_i^A\rangle |\psi_i^B\rangle$$



Hilbert space dim: $s^2 m^2$

How many states to keep? → $m \gg e^{S_A}$

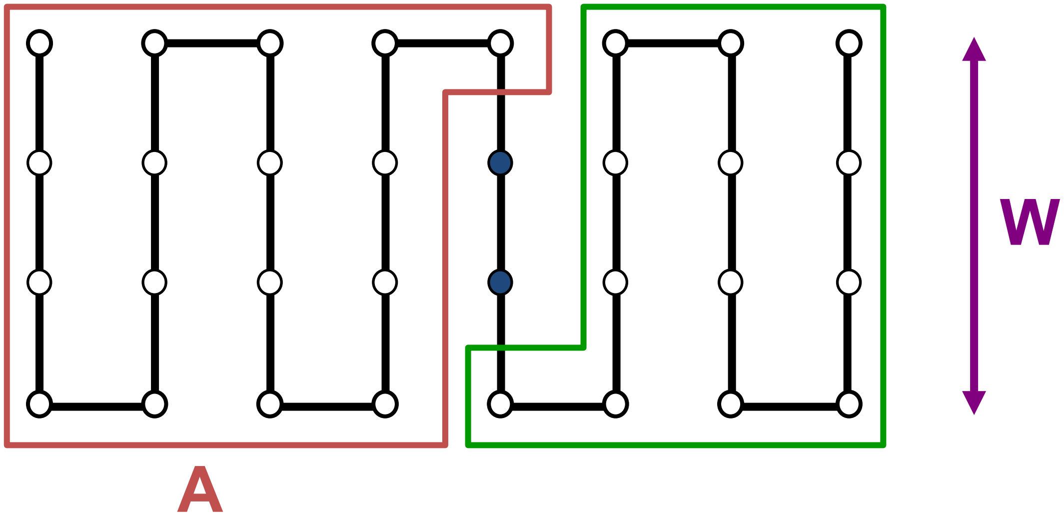
Vidal, Latorre, Rico, Kitaev, PRL, 2003

- Gapped system: $e^{S_A} \to \text{const.} \ (r \to \infty)$
- Gapless system: $S_A \simeq \frac{c}{6} \log \left[\frac{2L}{\pi} \sin \frac{\pi r}{L} \right] + \text{const.}$ (open chain)

$\xrightarrow{r = L/2}$ $e^{S_A} \simeq (\text{const.}) \times L^{c/6}$

e.g., XX chain (c=1), L=2000 → $e^{S_A} \simeq 4.7$

Why DMRG is difficult in D>1 ?



$$S_A \simeq \alpha W$$

$$\Rightarrow e^{S_A} \simeq e^{\alpha W}$$

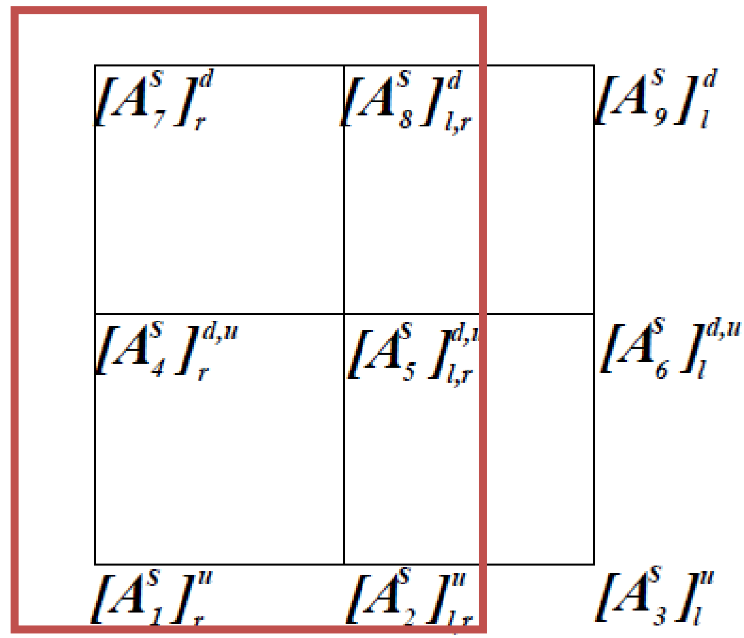
Vidal, Latorre, Rico, Kitaev, PRL, 2003

Possible solutions?

TPS: Tensor product state
(PEPS: Projected Entangled-Pair State)

Nishino *et al.*, 2000

Verstraete & Cirac, 2004



Quantum entanglement in condensed matter

Has become a huge research field…

- <u>Early seminal papers</u>

  Vidal, Latorre, Rico, and Kitaev, PRL 90, 227902 (2003)

  2651 citations

  Calabrese and Cardy, J. Stat. Mech. 2004, P06002 (2004)

  2980 citations
  (google scholar)

  …

- <u>Useful recent reviews</u>

  Laflorencie, Phys. Rep. 646, 1-59 (2016) 702 references

  Misguich, Lecture slides (2015)
  https://www.ipht.fr/Pisp/gregoire.misguich/

  104 slides

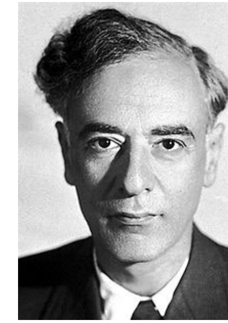
Contents

- Introduction: Entanglement entropy in condensed matter
- Quantum entanglement in topological phases
  - ✓ Topological phases / Topologically ordered phases
  - ✓ Topological entanglement entropy in topologically ordered phases

    anyonic nature of quasiparticles
  - ✓ Entanglement spectrum

    ⇨ Edge-state spectra in topological phases
- Quantum entanglement in systems with continuous symmetry breaking

Conventional paradigm of phase transitions



- Symmetry breaking and order parameter (Weiss, Landau, ...)

-Magnets: breaking of spin rotational symmetry, magnetization

- Liquid-solid transition: translational symmetry breaking, density

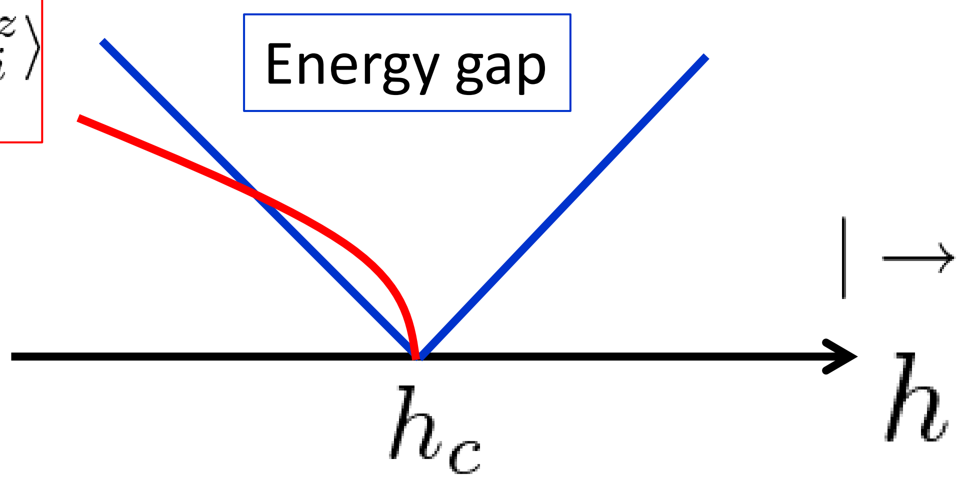
- Quantum phase transitions (still conventional)

$$H = -\sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$

$$m = \frac{1}{N} \sum_i \langle \sigma_i^z \rangle$$

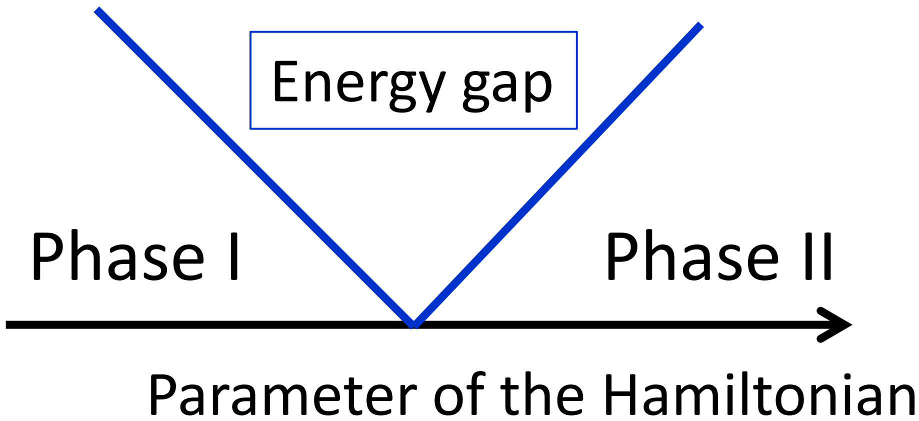
$|\uparrow\uparrow \dots\rangle$

$|\downarrow\downarrow \dots\rangle$



$|\rightarrow\rightarrow \dots\rangle$

Phase transition without symmetry breaking



- No local order parameter can distinguish the two phases.
- Different phases are distinguished by a certain topological property of the ground-state wave function
- Topological phases = Phases which are not smoothly connected to trivial phases

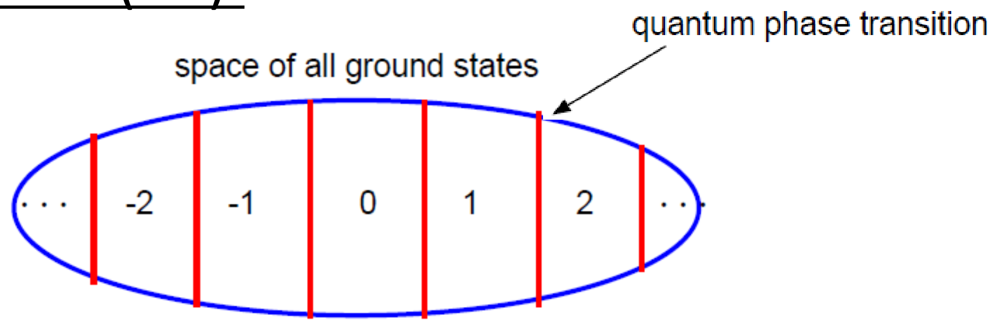
(How to define? Product state, no winding, etc.)

Topological phase transition between trivial phases is also possible:
Fuji, Pollmann, and Oshikawa, PRL **114**, 177204 (2015)

Topological phases of band insulators

➢ Time-reversal-broken case (2D):

Z classification

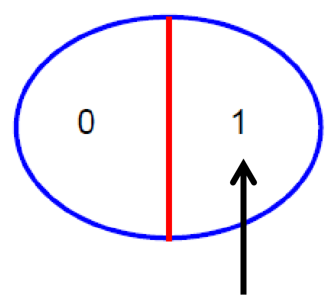


Chern number (Bloch band topology)
<-> integer quantum Hall effect

$$\sigma_{xy} = \frac{e^2}{h} \times \mathrm{Ch}$$

➢ Time-reversal-invariant case (2D & 3D):

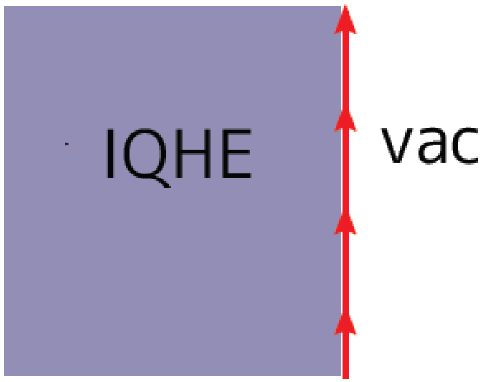
Z_2 topological insulators

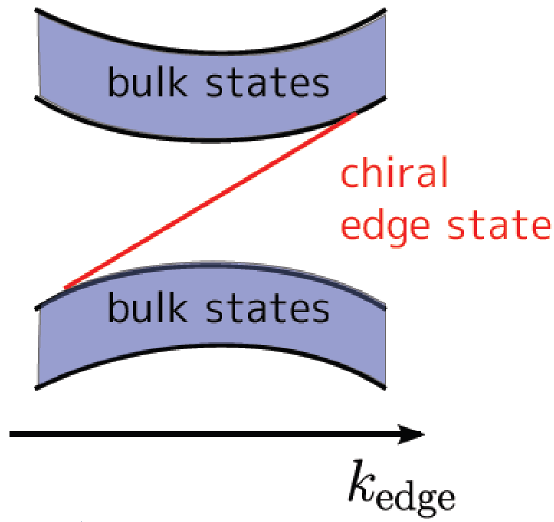


Example of a symmetry-protected topological (SPT) phase

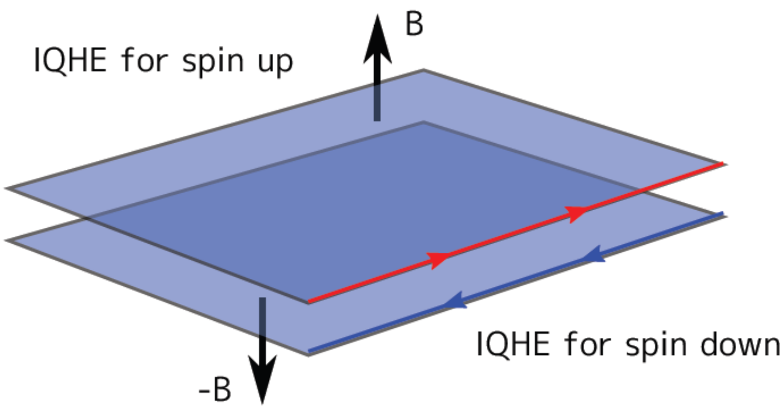
Gapless edge modes

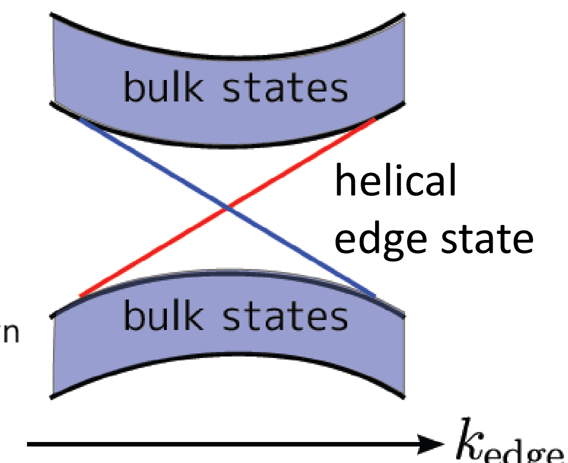
Chern number = n ⇨ n gapless chiral (unidirectional) edge modes

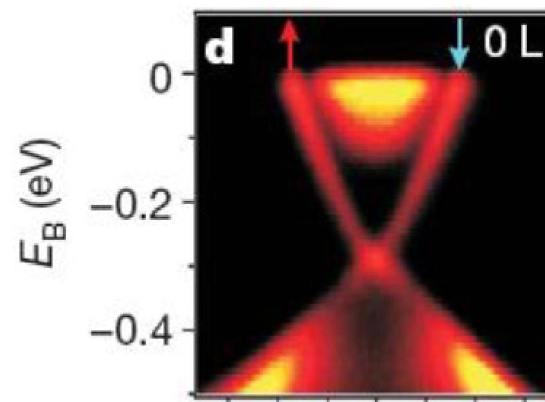




Z_2 number = 0 or 1 ⇨ even or odd gapless helical edge modes







ARPES experiment for Bi2Se3
Y. Xia et al., Nat. Phys. (2009)

Topologically ordered phases

- In strongly interacting systems, there are topological phases which cannot be understood even qualitatively by the band topology.
- Examples

  Fractional quantum Hall (FQH) states: Laughlin, Moore-Read, etc.

  Quantum spin liquids: toric code model, quantum dimer model, etc.

- Fractional excitations

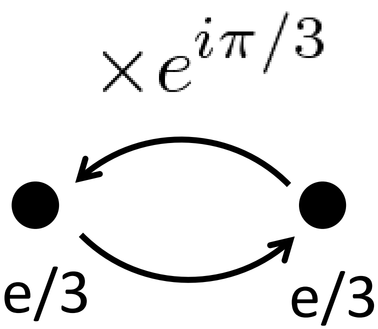
  fractional charge and statistics

  (some also show non-Abelian statistics)

- Ground-state degeneracy N that depends on the system's spatial topology

$\nu = 1/3$ FQH state

$$N = 3^g \quad \text{g: genus}$$

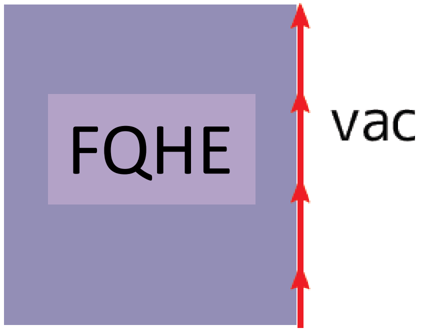


$\nu = 1/3$ FQH state

In what sense topological?

- Strongly interacting gapless edge modes (in the chiral class)

  e.g., chiral Tomonaga-Luttinger liquid



- Effective description: topological quantum field theory (TQFT)

$$S = -\int d^3x \frac{q}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} \Rightarrow \begin{cases} \text{Quantized Hall conductivity} \\ \text{Gapless edge modes} \end{cases}$$

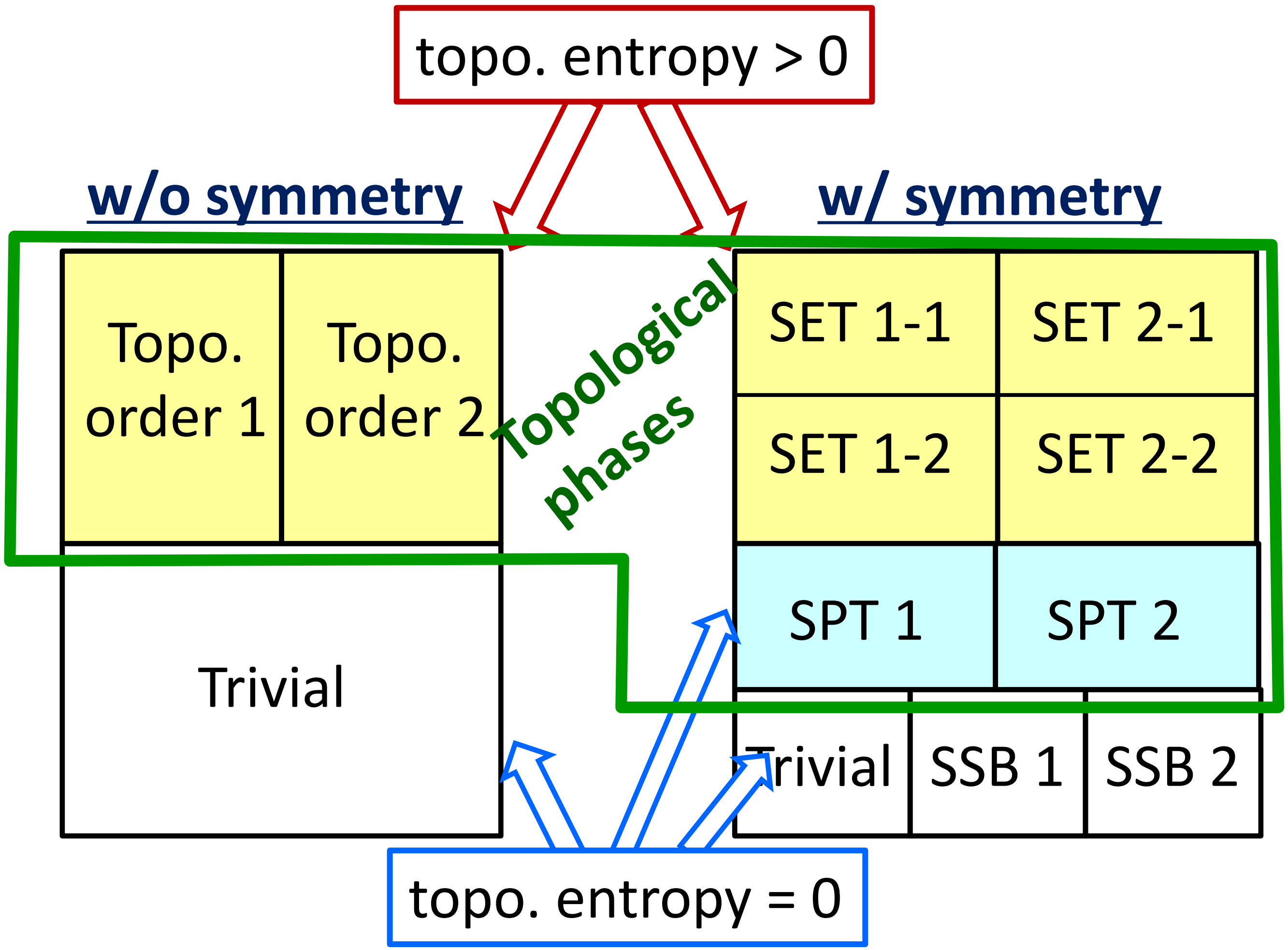
Surface of 2+1D topological field theory
⟶ 1+1D conformal field theory (Witten,1989)

- How can we identify and classify topologically ordered phases starting from a microscopic Hamiltonian?

  ⇨ Certain "long-range entanglement" in the ground state

  Topological entanglement entropy

Landscape of phases of matter a la X.-G. Wen



symmetry-enriched topological phases

symmetry-protected topological phases

spontaneous symmetry breaking

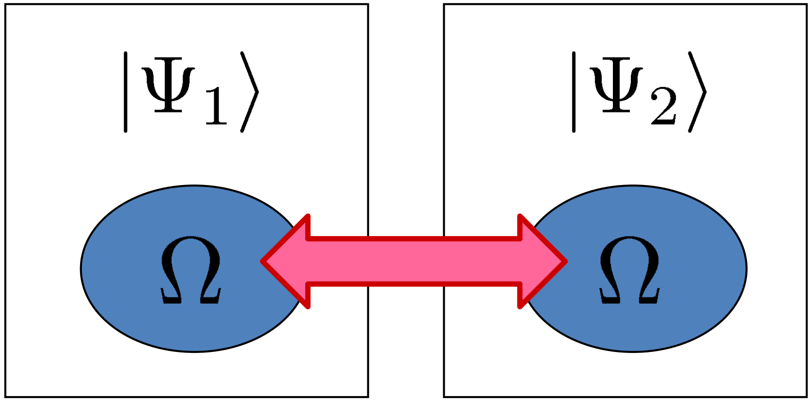
Topological entropy detects long-range entanglement which is robust against general perturbations.

Entanglement spectrum detects edge-state properties in general topological phases.

Preskill's proposal

Quantum information and physics: some future directions, J.Mod.Opt. 47 (2000) 127-137

Degenerate ground states in a topologically ordered phase



Locally indistinguishable

Distinguishable only in nonlocal regions
(Global encoding of information)

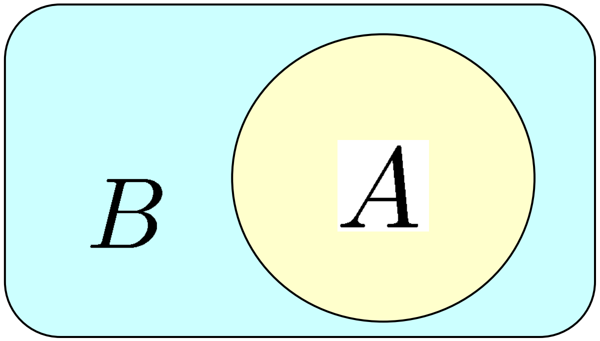
Analogous situation in quantum information
- Quantum secret sharing
- Quantum error correction

Concepts in quantum information
=>New methods for characterizing ground states in condensed matter

Note: Proposal on quantum critical phenomena

Can one interpret the c-theorem and the g-theorem in CFT in view of the information loss in the RG flow?

Entanglement entropy detects topological order!



Kitaev & Preskill, PRL 96, 110404 (2006)
Levin & Wen, PRL 96, 110405 (2006)
(Also, Hamma *et al*., PRA, 2005)

$$S_A = \alpha L_A - \gamma$$

L_A: perimeter

boundary-law contribution — universal contribution

$\gamma = \ln D_{\text{topo}}$ **: topological (entanglement) entropy**

D_{topo} **: total quantum dimension**

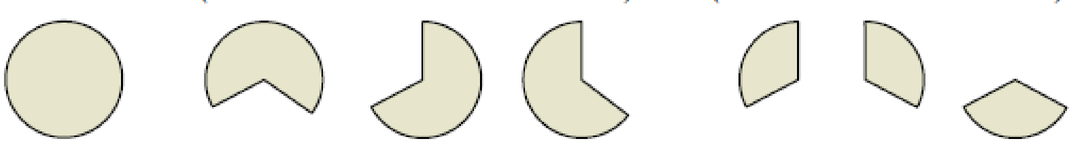
Universal constant related to the statistical properties of fractional quasiparticles

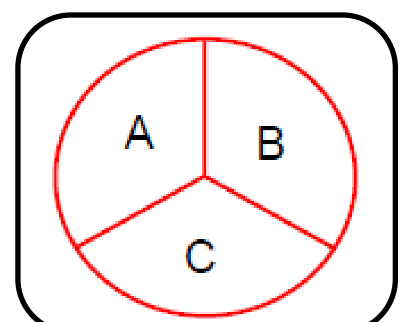
Z_2 spin liquid: $D_{\text{topo}} = 2$

$\nu = 1/q$ FQH state: $D_{\text{topo}} = \sqrt{q}$

$D_{\text{topo}} = \sqrt{\#\text{ of superselection sectors}}$ (Abelian)

$$S_{ABC} - (S_{AB} + S_{BC} + S_{AC}) + (S_A + S_B + S_C) \longrightarrow -\gamma$$

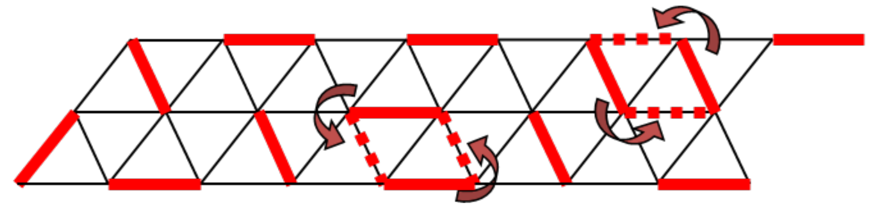




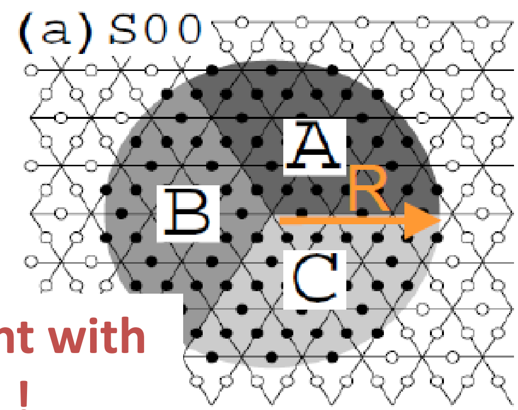
Application: Quantum dimer model on the triangular lattice

Furukawa & Misguich, Phys. Rev. B 75, 214407 (2007)

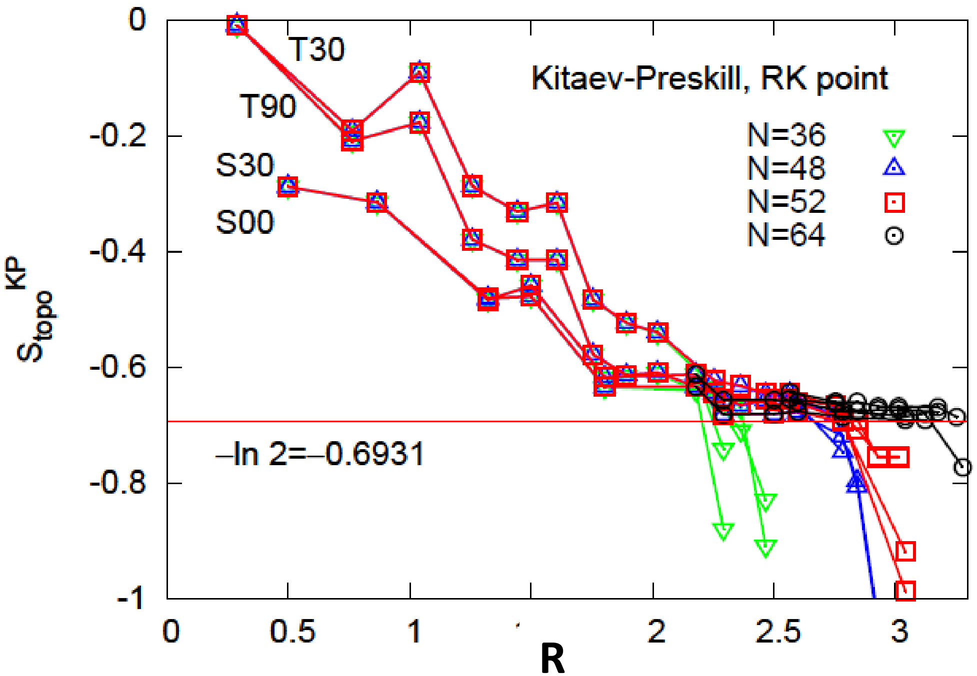
$$H = \sum_{\square}\left[-t\left(|=\rangle\langle /\!/| + h.c.\right) + v\left(|=\rangle\langle =| + |/\!/\rangle\langle /\!/|\right)\right]$$



Rokhsar-Kivelson point (t=v) $|\mathrm{RK}\rangle = \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{c\in\mathcal{E}} |c\rangle$



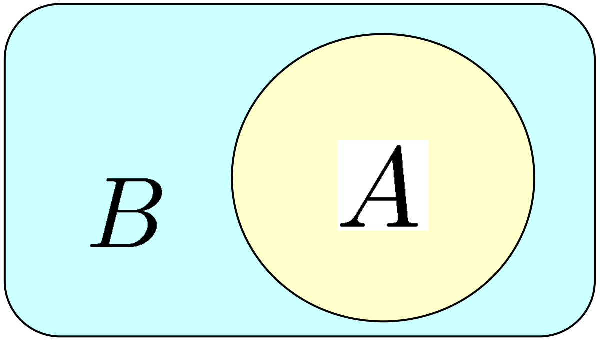
Kitaev-Preskill construction



99% agreement with the prediction !

S00 case		
Radius R	$-S_{\rm topo}^{\rm KP}/\ln 2$	
	$N=52$	$N=64$
2.18	0.9143	0.9143
2.29	0.9839	0.9835
2.50	0.9822	0.9822
2.60	0.9765	0.9760
2.78	1.0014	0.9897
3.04	1.3252	0.9967
3.12		0.9967

Entanglement spectrum



$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$$

Topological entropy does not provide a complete picture of topological phases.

In principle, the full spectrum of ρ_A contains more information than S_A.

$$\rho_A = e^{-H_{\mathrm{e}}}$$

H_{e} : **entanglement Hamiltonian**

Entanglement spectrum = full eigenvalue spectrum of H_e: $\{\xi_i\}$

Entanglement entropy

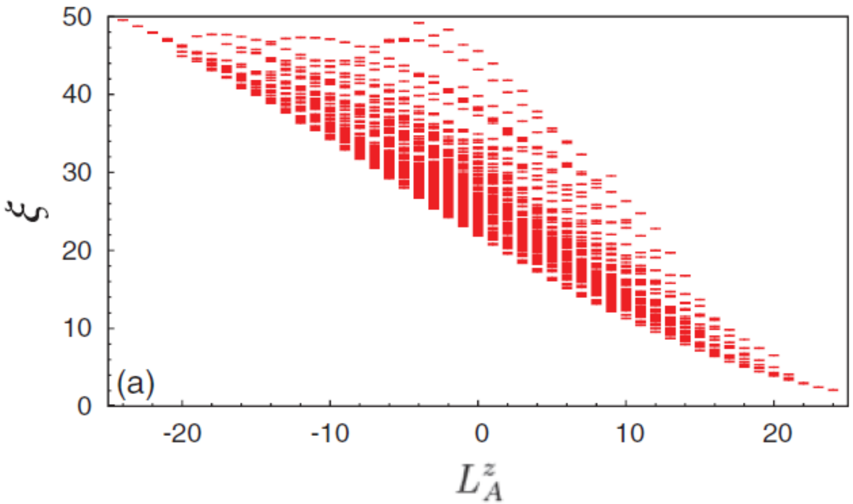
= Thermal entropy of H_e at the fictitious temperature T=1

Li and Haldane, PRL 101, 010504 (2008)

Entanglement spectra in topological phases

Remarkable correspondence with edge-state spectra

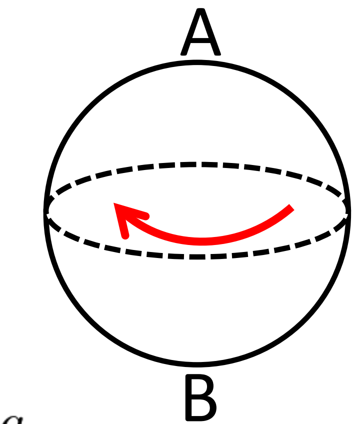
Li and Haldane, PRL 101, 010504 (2008)



$\nu = 1/3$ Laughlin state

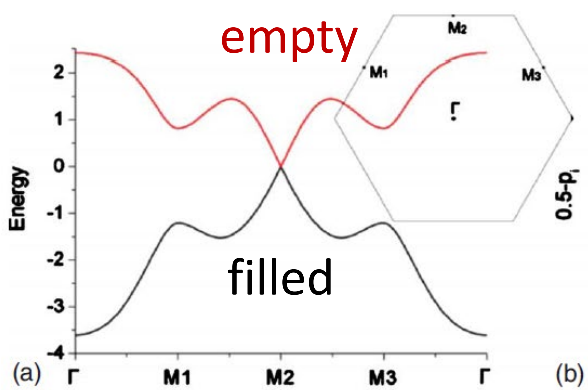
spectrum reminiscent of chiral TLL

$$H = -vP = \frac{2\pi v}{L} \sum_{n=1}^{\infty} n a_n^\dagger a_n$$

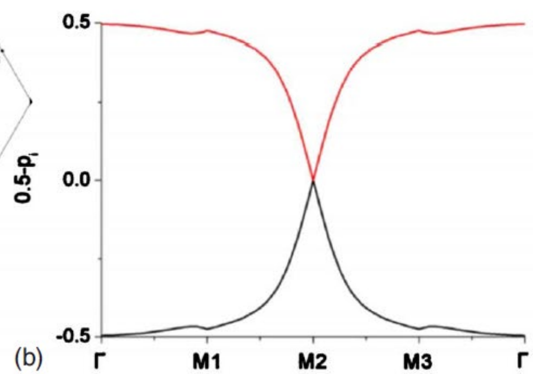


degeneracy: 1, 1, 2, 3, ... [Counting of U(1) bosons]

Sterdyniak et al., PRB 85, 125308 (2012)



Edge spectrum



entanglement spectrum

3D Z_2 topological insulator

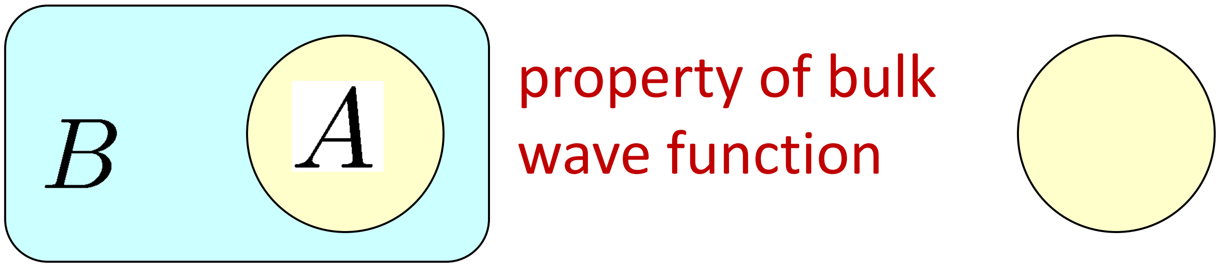
Turner, Zhang, and Vishwanath, PRB 82, 241102 (2010)

$$H_{\mathrm{e}} \propto H_{\mathrm{edge}}$$

at low energies

Implications

Correspondence between entanglement and edge-state spectra



⇨ Manifestation of the bulk-edge correspondence

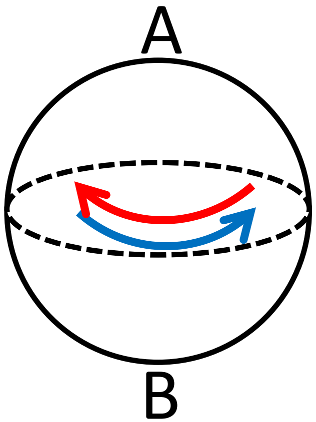
Practical aspect:

In finite-size simulations, the entanglement spectrum better reflects the bulk topological features (shows a clearer gapless structure than edge states).

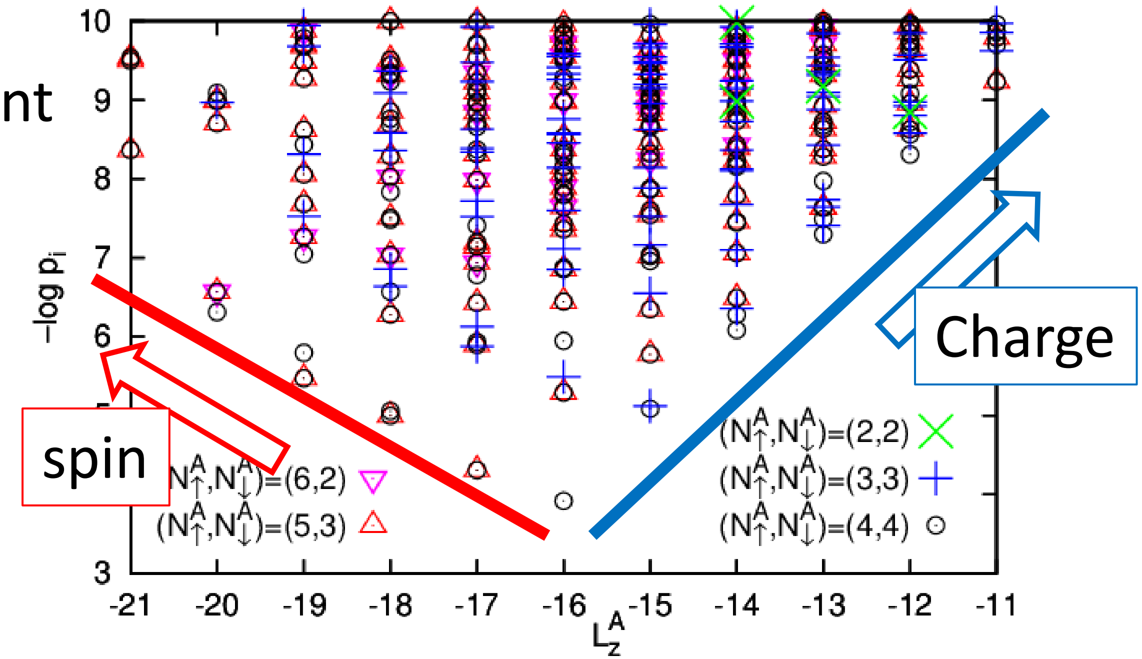
Question:

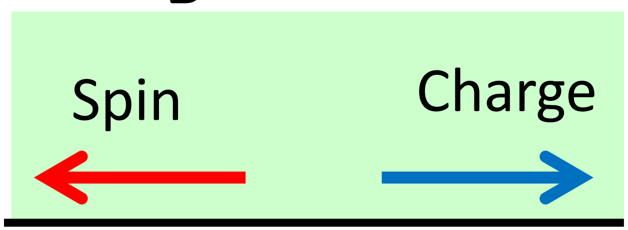
Why do we have this correspondence?

Application: bosonic integer quantum Hall state



entanglement spectrum

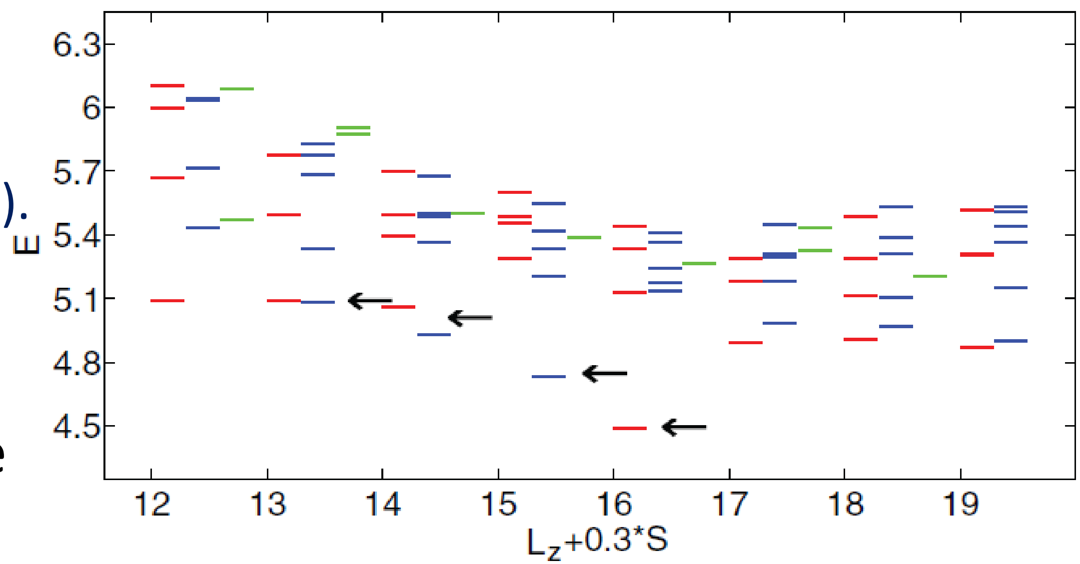




$\rho_\uparrow(x) - \rho_\downarrow(x) \quad \rho_\uparrow(x) + \rho_\downarrow(x)$

Furukawa & Ueda,
Phys. Rev. Lett. 111, 090401 (2013).
Wu & Jain,
Phys. Rev. B 87, 245123 (2013).

edge-state spectrum

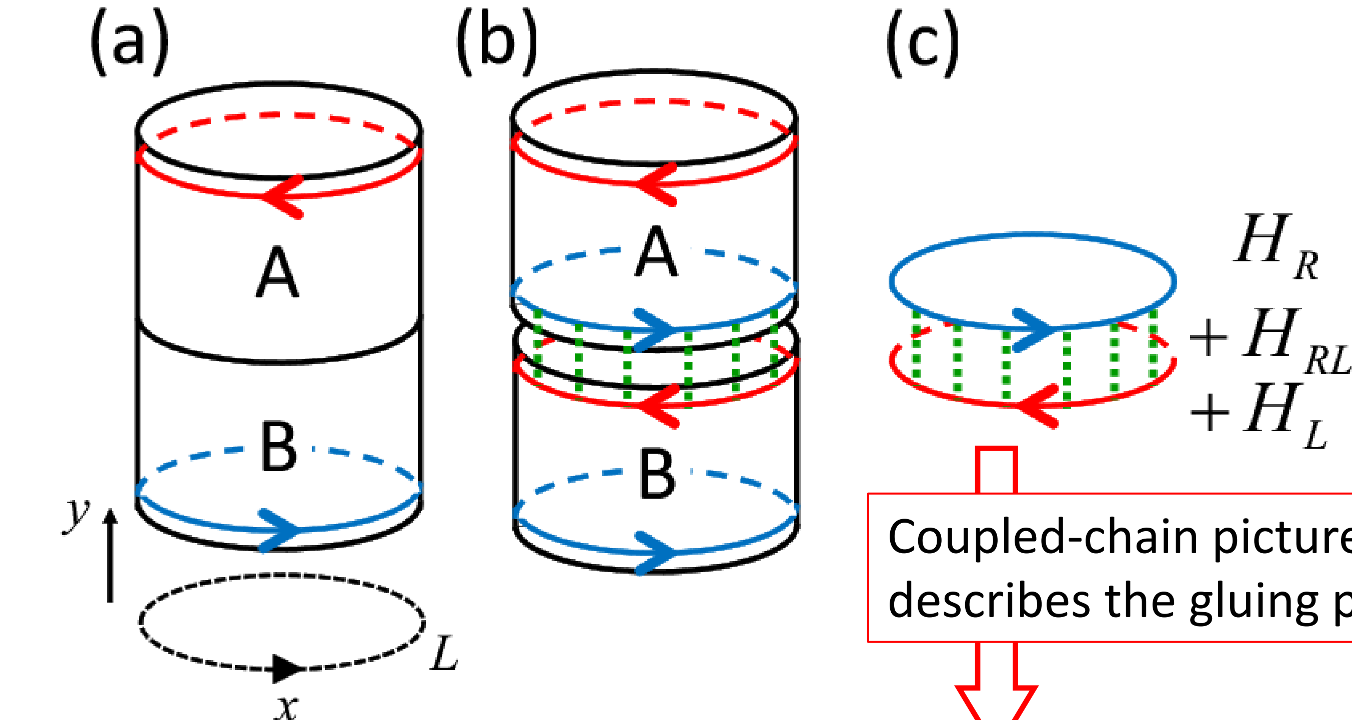


"Cut and glue" approach

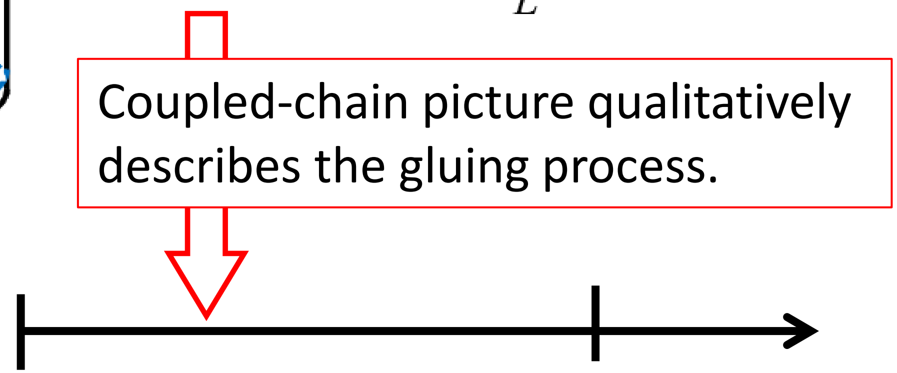
Original and general: Qi, Katsura, and Ludwig, PRL 108, 196402 (2012)

Simplified: Lundgren, Fuji, Furukawa, and Oshikawa, Phys. Rev. B 88, 245137 (2013)

$\nu = 1/q$ FQH state (q=even for bosons, odd for fermions)



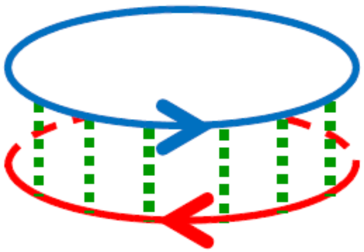
$$H(\lambda) = H_A + H_B + \lambda H_{AB}$$



$\lambda = 0$ decoupled systems

$\lambda = 1$ original system

Two coupled chiral TLLs



$H_R + H_{RL} + H_L$

Two edges of $\nu = 1/q$ FQH state

$$H_{R/L} = \int_0^L \mathrm{d}x \frac{q v_0}{4\pi} (\partial_x \phi_{R/L})^2,$$

$$\rho_{R/L} = \frac{1}{2\pi} \partial_x \phi_{R/L}$$

H_{RL} : particle tunneling and density-density interactions

$$\phi = \frac{1}{\sqrt{4\pi}} (\phi_L + \phi_R), \quad \theta = \frac{q}{\sqrt{4\pi}} (\phi_L - \phi_R)$$

Non-chiral bosonic fields

$$H \equiv H_L + H_R + H_{RL}$$

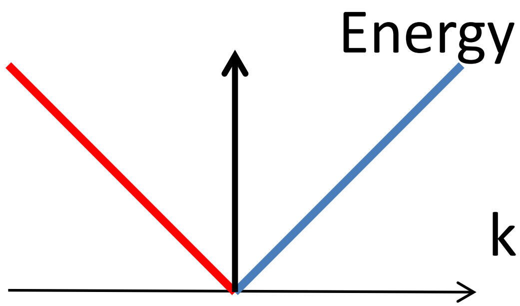
$$= \int_0^L \mathrm{d}x \left[\frac{v}{2} \left(K(\partial_x \theta)^2 + \frac{1}{K} (\partial_x \phi)^2 \right) + \frac{g}{\pi} \cos\left(\sqrt{4\pi} q \phi \right) \right]$$

sine-Gordon Hamiltonian

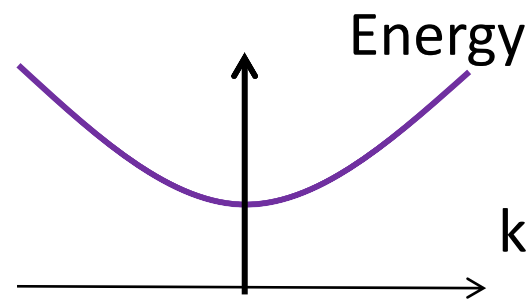
density-density intearction

particle tunneling

Free-field description of gapped phase



g decays to zero under RG.



g grows under RG.

⇨ Locking of ϕ . Energy gap opens up.

⇨ Continues to a gapped bulk state!

Very simple description of the gapped phase:

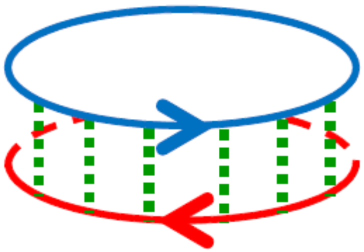
$$\frac{g}{\pi}\cos\left(\sqrt{4\pi}q\phi\right) \approx \text{const.} + \frac{vm^2}{2K}\left(\phi - \bar{\phi}_0\right)^2 + \dots,$$

⇨ H : massive Klein-Gordon model with a mass gap vm

Just with methods for free theories, we can discuss the entanglement spectrum!

cf. Qi, Katsura, and Ludwig used boundary CFT to describe the gapped phase.

Calculation of entanglement Hamiltonian



$$H_R + H_{RL} + H_L$$

Mode expansions

$$\phi_R = \sum_{k>0} \sqrt{\frac{2\pi}{qL|k|}} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx} \right)$$ + zero modes

$$\phi_L = \sum_{k<0} \sqrt{\frac{2\pi}{qL|k|}} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx} \right)$$ + zero modes

(ignored in this talk)

H : certain quadratic form of bosonic operators

⇨ Obtain the ground state. Calculate correlation functions in the 1st chain: $\langle 0 | a_k^\dagger a_k | 0 \rangle \qquad k > 0$

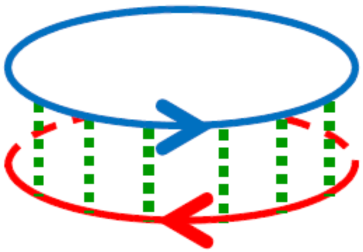
⇨ Ansatz: $\rho_A^{\mathrm{osc}} = \frac{1}{Z_{\mathrm{e}}^{\mathrm{osc}}} e^{-H_{\mathrm{e}}^{\mathrm{osc}}} \qquad H_{\mathrm{e}}^{\mathrm{osc}} = \sum_{k>0} w_k \left(a_k^\dagger a_k + \frac{1}{2} \right)$

Determine w_k so that the ground-state correlations are reproduced (Peschel's method, 2003).

$$\langle 0 | \mathcal{O}_A | 0 \rangle = \mathrm{Tr}(\mathcal{O}_A \rho_A^{osc})$$

It is sufficient to examine 2-point functions (Wick's theorem).

Result: entanglement Hamiltonian



$$H_R + H_{RL} + H_L$$

$$H_{\mathrm{e}} = v_{\mathrm{e}} \left[\frac{\pi q}{L} N_R^2 + \sum_{k>0} k a_k^\dagger a_k - \frac{\pi}{12L} \right] \propto H_R$$

with $v_{\mathrm{e}} = 4qK/m$

(at low energies)

⇨ Simple physical proof of entanglement-edge correspondence in quantum Hall states!

By calculating the thermal entropy of H_{e} , we can also obtain the topological entanglement entropy.

$$Z_{\mathrm{e}}(\beta) = \mathrm{Tr}\ e^{-\beta H_{\mathrm{e}}} = \frac{\theta_3(iq\tau_2)}{\eta(i\tau_2)} \approx \frac{1}{\sqrt{q}} e^{\pi/12\tau_2}$$

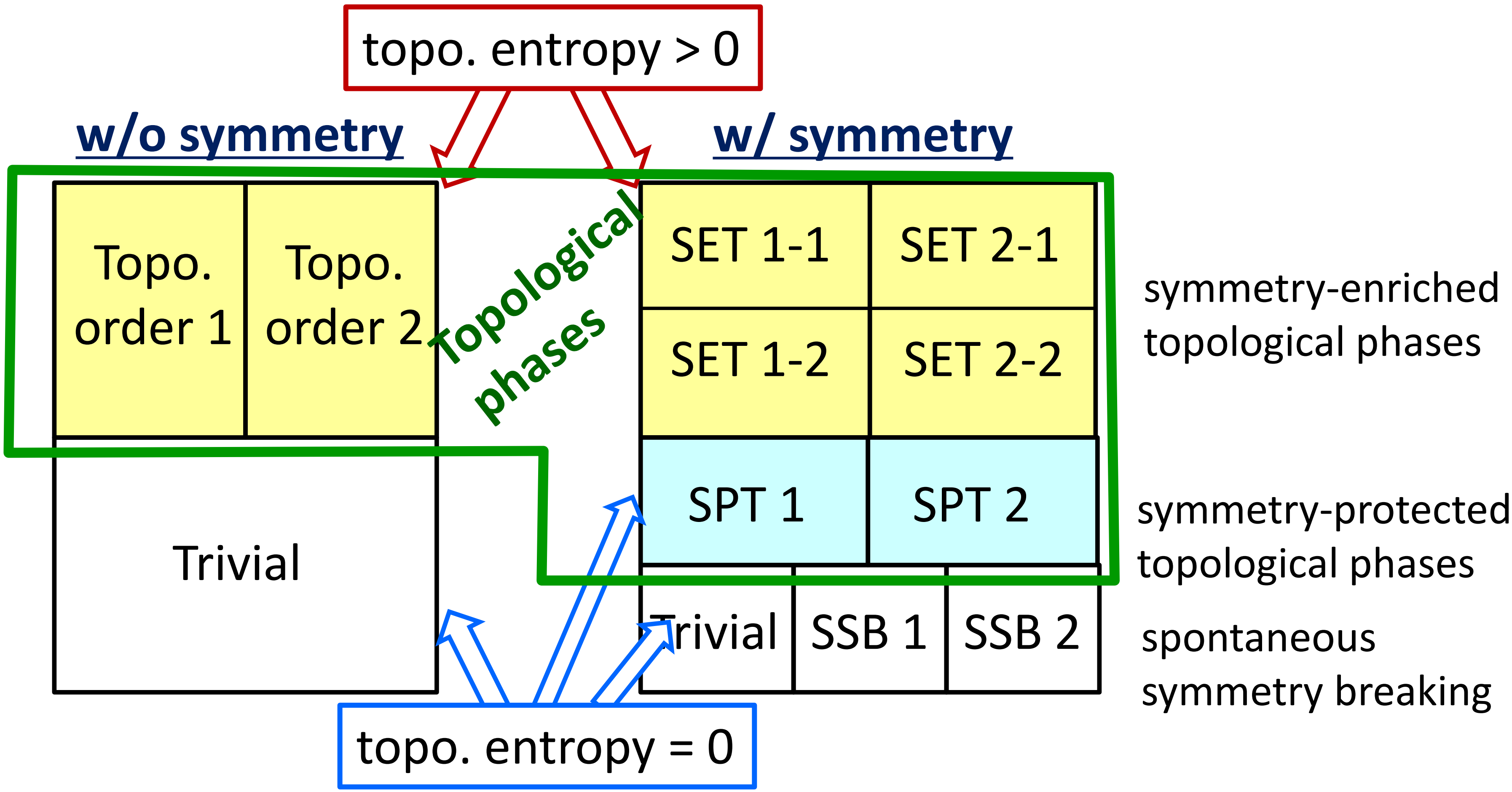
Not modular-invariant for $q \neq 1$

$$S = \left. \frac{\partial (T \ln Z_{\mathrm{e}}(\beta))}{\partial T} \right|_{T=1} \approx \frac{\pi L}{6 v_{\mathrm{e}}} - \ln \sqrt{q}$$

$$\left(\begin{array}{c} \tau = i\tau_2 = i\dfrac{\beta v_{\mathrm{e}}}{L} \\ \tau_2 \to 1/\tau_2 \\ \theta_3(i\tau_2) = \tau_2^{-1/2} \theta_3(i/\tau_2) \\ \eta(i\tau_2) = \tau_2^{-1/2} \eta(i/\tau_2) \end{array} \right)$$

Consistent with Kitaev-Preskill-Levin-Wen result!

Summary: entanglement in topological phases



Topological entropy detects long-range entanglement which is robust against general perturbations.

Breakdown of modular invariance in edge CFT

Entanglement spectrum detects edge-state properties in general topological phases.

"Cut and glue" picture

Contents

- ☐ Introduction: Entanglement entropy in condensed matter
- ☐ Quantum entanglement in topological phases
- ☐ Quantum entanglement in systems with continuous symmetry breaking

✓ Symmetry restoration in finite-size ground states

✓ Subleading logarithimic term related to the number of Nambu-Goldstone modes

Systems with continuous symmetry breaking

Metlitski & Grover, arXiv:1112.5166v2 (2015)

Brief review in Laflorencie, Phys. Rep., 2016

- O(N) nonlinear sigma model in d spatial dimensions
- SSB in the thermodynamic limit: $O(N) \to O(N-1)$

- Antiferromagnet: $SU(2) \simeq O(3) \to U(1) \simeq O(2)$
- Superfluid (BEC): $U(1) \simeq O(2) \to \text{no sym.}$

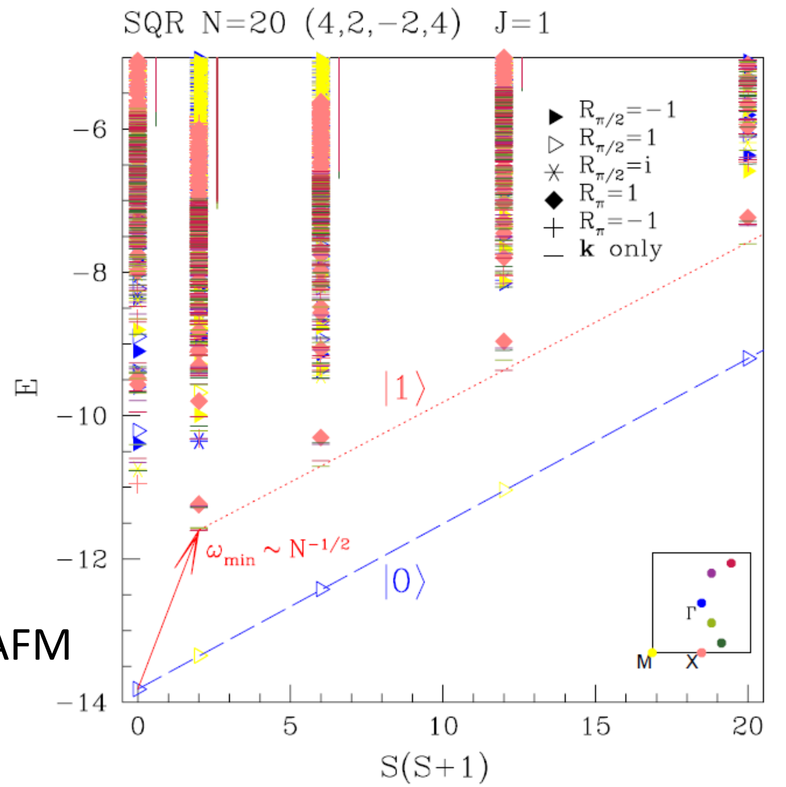
(continuous part concerned)

- Finite-size ground state in which symmetry is restored

Rotational energy of an order parameter

$$H_{\text{tower}} = \frac{c^2 \vec{S}^2}{2\rho_s V} \to E_S = \frac{c^2 S(S+1)}{2\rho_s V}$$

Anderson's tower spectrum or quasi-degenerate joint states



Spectrum of the spin-1/2 square Heisenberg AFM
C. Lhuillier, cond-mat/0502464

Systems with continuous symmetry breaking

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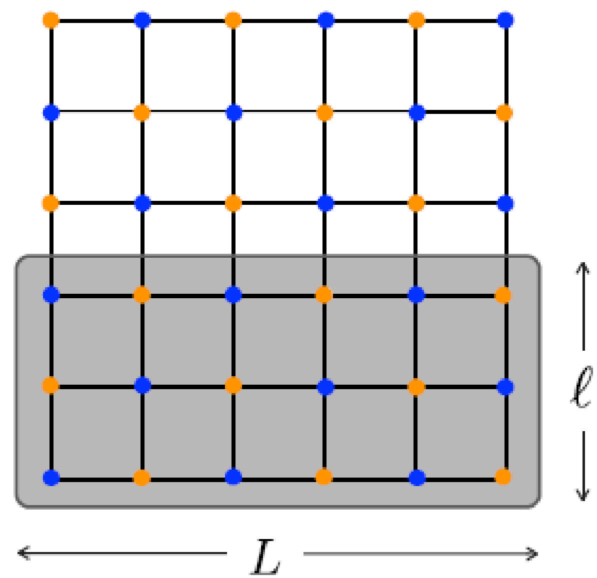
- Finite-size ground state in which symmetry is restored



- Entanglement entropy

$$S_A = \alpha L^{d-1} + \frac{N_{\rm NG}}{2} \ln \frac{\rho_s L^{d-1}}{c} + \gamma(\ell/L)$$

area law

$N_{\rm NG} = N-1$: number of Nambu-Goldstone modes

- Entanglement Hamiltonian

$$\rho_A = e^{-H_{\rm E}} \qquad H_E = \frac{\vec{S}_A^2}{2I} + \sum_\epsilon \epsilon\, a_\epsilon^\dagger a_\epsilon$$

Uniform component   oscillators

Reminiscent of the tower spectrum!

$$I^{-1} \sim (L^{d-1} \log L/a)^{-1}$$

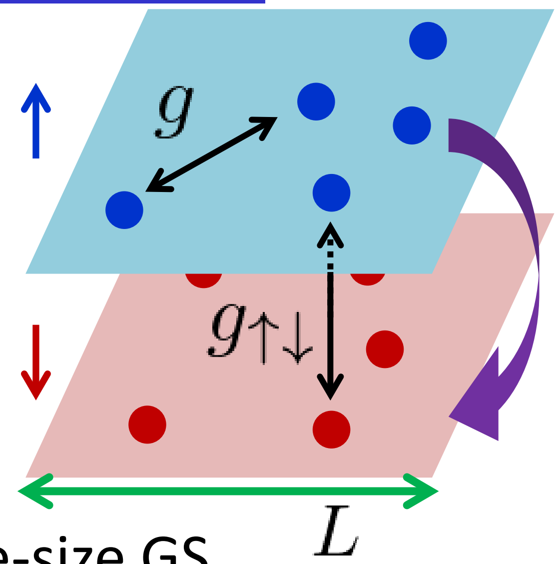
Model: binary Bose-Einstein condensates (BECs)

Yoshino, Furukawa, & Ueda, PRA, 2021

- d-dimensional binary BECs
- Binary system

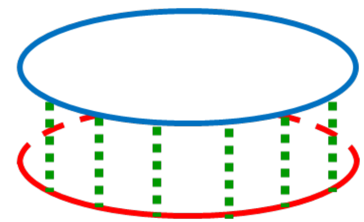
  $\simeq$ Pseudospin-1/2 system

  $\simeq$ Bilayer system (d=2), ladder (d=1), etc.



Rabi coupling (tunneling)

$$-\frac{\hbar\Omega}{2}\left(\psi_{\uparrow}^{\dagger}\psi_{\downarrow}+\mathrm{h.c.}\right)$$



- Intercomponent EE in a finite-size GS
- $d \geq 2$: SSB in the thermodynamic limit
  - ✓ $\Omega = 0$: U(1) x U(1) symmetry ⟶ no symmetry
  - ✓ $\Omega > 0$: U(1) symmetry ⟶ no symmetry

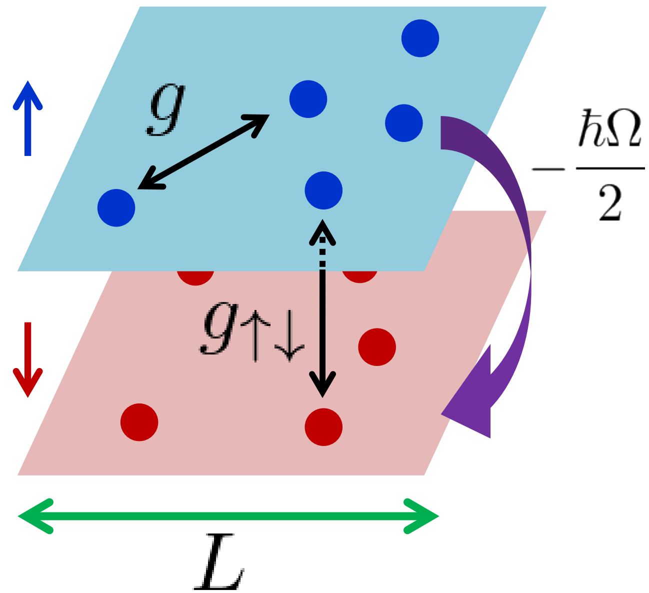
Similar to the subregion EE in the O(2) NLσ model but an extensive boundary size L^d

- $d = 1$: Equivalent to coupled Tomonaga-Luttinger liquids (TLLs)

Lundgren, Fuji, Furukawa, & Oshikawa, PRB, 2013
X. Chen & Fradkin, J. Stat. Mech., 2012
Furukawa & Y. B. Kim, PRB, 2011

Main results

Yoshino, Furukawa, & Ueda, Phys. Rev. A 103, 043321 (2021)



- $\Omega > 0$: U(1) symmetry

$$H_{\rm e} = H_{\rm e}^{\rm zero} + H_{\rm e}^{\rm osc}$$

zero mode $H_{\rm e}^{\rm zero} = \frac{G_0}{2nV}(N_\uparrow - N/2)^2$

oscillator mode $H_{\rm e}^{\rm osc} = \sum_{\mathbf{k}\neq\mathbf{0}} \xi_{\mathbf{k}} \left(\eta_{\mathbf{k}}^\dagger \eta_{\mathbf{k}} + \frac{1}{2} \right)$

$\xi_{\boldsymbol{k}} = c_{1/2} k^{1/2} + O(k^{3/2})$ fractional power!

$$S_{\rm e} = S_{\rm e}^{\rm zero} + S_{\rm e}^{\rm osc} = \frac{\sigma L^d}{c_{1/2}^{2d}} + \frac{d}{2} \ln \left[\left(\frac{2\pi e n}{G_0} \right)^{1/d} L \right] - \frac{1}{2} \ln \frac{L}{(2\pi) c_{1/2}^2} + O(1)$$

volume law — symmetry restoration — small-k oscillators

cf. Metlitski & Grover: $\frac{N_{\rm NG}(d-1)}{2} \ln L$ only from symmetry restoration

- $\Omega = 0$: U(1) x U(1) symmetry

$\xi_{\mathbf{k}} = \xi_0 + c_2 k^2 + O(k^4)$ gapped!

$S_{\rm e}^{\rm osc} = s_1 L^d - s_0$ (volume law: $s_1 L^d$)

Universal constant (s_0) determined by $g_{\uparrow\downarrow}/g$

Effective field theory

$\psi_\alpha(\boldsymbol{r}) = e^{-i\theta_\alpha(\boldsymbol{r})}\sqrt{n_\alpha(\boldsymbol{r})}$: phase-density representation

$$\mathcal{H} = \sum_{\alpha=\uparrow,\downarrow} \frac{\hbar^2}{2M}\left[n(\nabla\theta_\alpha)^2 + \frac{(\nabla n_\alpha)^2}{4n}\right] + \sum_{\alpha,\beta=\uparrow,\downarrow} \frac{g_{\alpha\beta}}{2} n_\alpha n_\beta - \hbar\Omega\sqrt{n_\uparrow n_\downarrow}\cos(\theta_\uparrow - \theta_\downarrow)$$

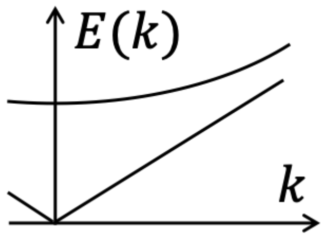
kinetic term — interactions — Rabi coupling

$g_{\uparrow\downarrow} = g_{\downarrow\downarrow} \equiv g > 0,\ g_{\uparrow\downarrow} = g_{\downarrow\uparrow}$

(a) $\Omega > 0$: Locking of the relative phase $\theta_- \equiv \theta_\uparrow - \theta_\downarrow$

Approximation: $\cos(\theta_\uparrow - \theta_\downarrow) \approx 1 - (\theta_\uparrow - \theta_\downarrow)^2/2$, $|n_\alpha(\mathbf{r},t) - n| \ll n$

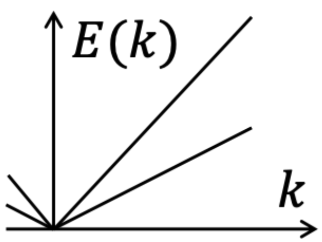
- ✓ Gapped spectrum in the θ_- sector
- ✓ Gapless spectrum in the $\theta_+ \equiv \theta_\uparrow + \theta_\downarrow$ sector



$$\epsilon_{\mathbf{k},+} \approx \hbar\sqrt{\frac{g_+ n}{M}}|\mathbf{k}| \quad \left[g_\pm = g \pm g_{\uparrow\downarrow}\right]$$ (Nambu-Goldstone mode)

(b) $\Omega = 0$: Gapless spectra in the $\theta_\pm$ sector

$$\epsilon_{\mathbf{k},\pm} \approx \hbar\sqrt{\frac{g_\pm n}{M}}|\mathbf{k}|$$



Fourier expansions

$$\theta_\pm(\boldsymbol{r}) = \theta_\uparrow(\boldsymbol{r}) \pm \theta_\downarrow(\boldsymbol{r}) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} \theta_{\boldsymbol{k},\pm} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \; n_\pm(\boldsymbol{r}) = \frac{n_\uparrow(\boldsymbol{r}) \pm n_\downarrow(\boldsymbol{r})}{2} = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} n_{\boldsymbol{k},\pm} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$H = \sum_{\mathbf{k}} \sum_{\nu=\pm} \left[\frac{n}{2}(\epsilon_{\mathbf{k}} + \hbar\Omega\delta_{\nu,-})\theta_{-\mathbf{k},\nu}\theta_{\mathbf{k},\nu} + \frac{1}{2n}(\epsilon_{\mathbf{k}} + 2g_\nu n) n_{-\mathbf{k},\nu} n_{\mathbf{k},\nu} \right]$$

We separately treat the zero mode (k=0) and the oscillator mode (k≠0).

☐ Oscillator mode (k≠0)

$$H^{\mathrm{osc}} = \sum_{\mathbf{k}\neq\mathbf{0}} \sum_{\nu=\pm} E_\nu(\mathbf{k}) \left(\gamma^\dagger_{\mathbf{k},\nu}\gamma_{\mathbf{k},\nu} + \frac{1}{2} \right) \qquad E_\nu(\mathbf{k}) := \sqrt{(\epsilon_{\mathbf{k}} + \hbar\Omega\delta_{\nu,-})(\epsilon_{\mathbf{k}} + 2g_\nu n)}.$$

$$\gamma_{\mathbf{k},\nu} = \frac{1}{\sqrt{2}} \left(\sqrt{n}\zeta_{\mathbf{k},\nu}\theta_{\mathbf{k},\nu} + \frac{i}{\sqrt{n}\zeta_{\mathbf{k},\nu}} n_{\mathbf{k},\nu} \right)$$ bogolon annihilation operator

- ✓ Ground state = bogolon vacuum: $\gamma_{\boldsymbol{k},\pm}|0^{\mathrm{osc}}\rangle = 0 \; (\boldsymbol{k} \neq \boldsymbol{0})$
  - ➡ Correlation fn's in the up component $\langle 0^{\mathrm{osc}}|\theta_{-\mathbf{k},\uparrow}\theta_{\mathbf{k},\uparrow}|0^{\mathrm{osc}}\rangle$ etc.
- ✓ Peschel's method for the reduced density matrix

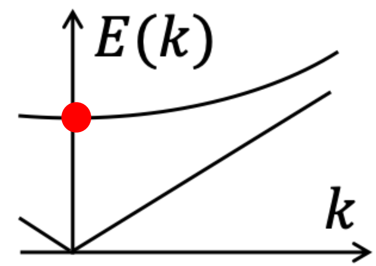
I. Peschel, J. Phys. A: Math. Gen. **36** L205 (2003)

$$\rho^{\mathrm{osc}}_\uparrow = \frac{1}{Z^{\mathrm{osc}}_{\mathrm{e}}} \mathrm{e}^{-H^{\mathrm{osc}}_{\mathrm{e}}}, \quad H^{\mathrm{osc}}_{\mathrm{e}} = \frac{1}{2} \sum_{\mathbf{k}\neq\mathbf{0}} \left(nF_{\mathbf{k}}\theta_{-\mathbf{k},\uparrow}\theta_{\mathbf{k},\uparrow} + \frac{G_{\mathbf{k}}}{n} n_{-\mathbf{k},\uparrow} n_{\mathbf{k},\uparrow} \right)$$

Treatment of the zero mode

- Zero mode (k=0)

$\Omega > 0$ case



$$H^{\mathrm{zero}} = \frac{g_+}{4V}N^2 + E_{\mathbf{0},-}\left(\gamma^\dagger_{\mathbf{0},-}\gamma_{\mathbf{0},-} + \frac{1}{2}\right)$$

tower spectrum — gapped mode at k=0

$$\gamma_{\mathbf{0},-} = \frac{1}{\sqrt{2}}\left(\sqrt{n}\zeta_{\mathbf{0},-}\theta_{\mathbf{0},-} + \frac{i}{\sqrt{n}\zeta_{\mathbf{0},-}}n_{\mathbf{0},-}\right)$$

$$|0^{\mathrm{zero}}\rangle = \sum_{\delta N}\frac{1}{\sqrt{z}}\exp\left(-\frac{\delta N^2}{N\zeta^2_{\mathbf{0},-}}\right)\left|N_\uparrow = \frac{N}{2}+\delta N\right\rangle\left|N_\downarrow = \frac{N}{2}-\delta N\right\rangle$$

Ground state: Schmidt-decomposed!

$$\rho^{\mathrm{zero}}_\uparrow = \sum_{\delta N}\frac{1}{z}\exp\left(-\frac{2\delta N^2}{N\zeta^2_{\mathbf{0},-}}\right)\left|N_\uparrow = \frac{N}{2}+\delta N\right\rangle\left\langle N_\uparrow = \frac{N}{2}+\delta N\right|$$

reduced density matrix

$$H^{\mathrm{zero}}_{\mathrm{e}} = \frac{2\delta N^2}{N\zeta^2_{\mathbf{0},-}} = \frac{G_0}{2nV}(N_\uparrow - N/2)^2$$

entanglement Hamiltonian $G_0 = 2\left(\frac{2g_-n}{\hbar\Omega}\right)^{1/2}$

$$Z^{\mathrm{zero}}_{\mathrm{e}} = \sum_{\delta N=-\infty}^{\infty}\exp\left[-\frac{G_0}{2nVT}(\delta N)^2\right] \approx \sqrt{\frac{2\pi nVT}{G_0}}$$

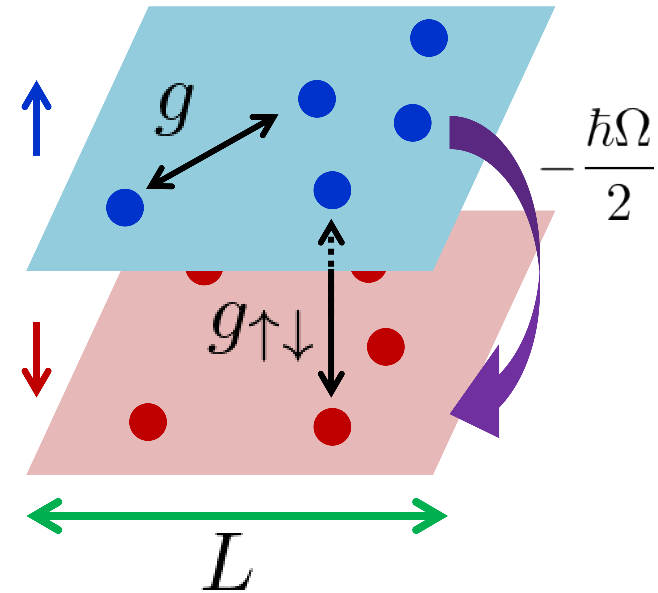
Partition function at a fictitious temperature T

$$S^{\mathrm{zero}}_{\mathrm{e}} = \frac{\partial}{\partial T}\left(T\ln Z^{\mathrm{zero}}_{\mathrm{e}}\right)\Big|_{T=1} = \frac{1}{2}\ln\frac{2\pi enV}{G_0} = \frac{d}{2}\ln\left[\left(\frac{2\pi en}{G_0}\right)^{1/d}L\right]$$

Entropy at T=1

Summary

Yoshino, Furukawa, & Ueda, Phys. Rev. A 103, 043321 (2021)



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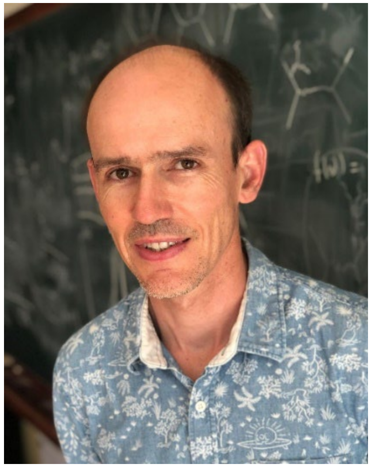
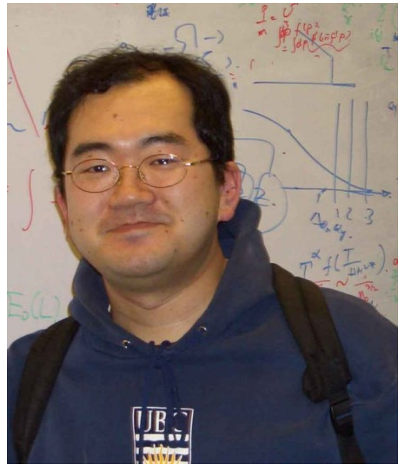
Universal constant determined by $g_{\uparrow\downarrow}/g$

Acknowledgements



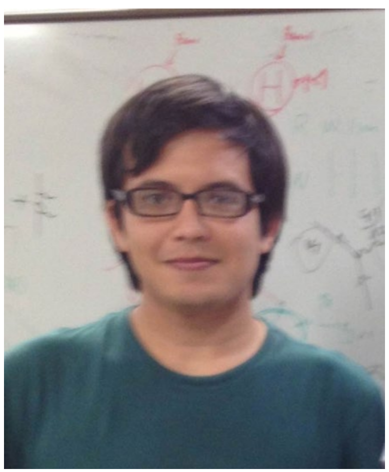
Gregoire Misguich (Saclay)



Masaki Oshikawa (ISSP)

Masahito Ueda Takumi Yoshino

(Univ. of Tokyo)



Rex Lundgren (Univ. of Maryland)

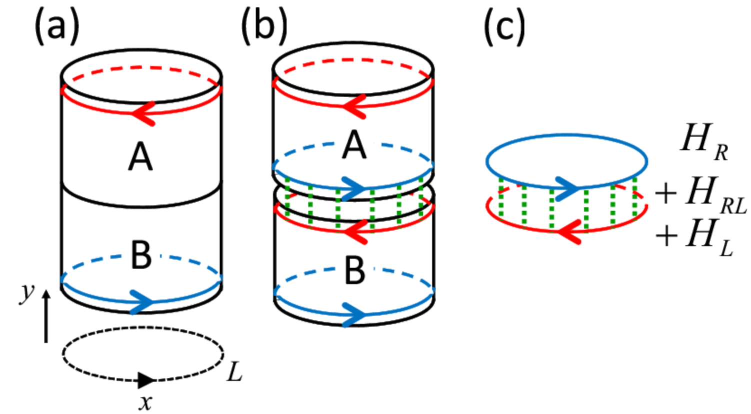
Yohei Fuji (Univ. of Tokyo)

Summary

☐ Quantum entanglement in topological phases

- ✓ Topological entanglement entropy for topologically ordered phases
- ✓ Entanglement-edge correspondence in topological phases

Modular non-invariance of the edge CFT



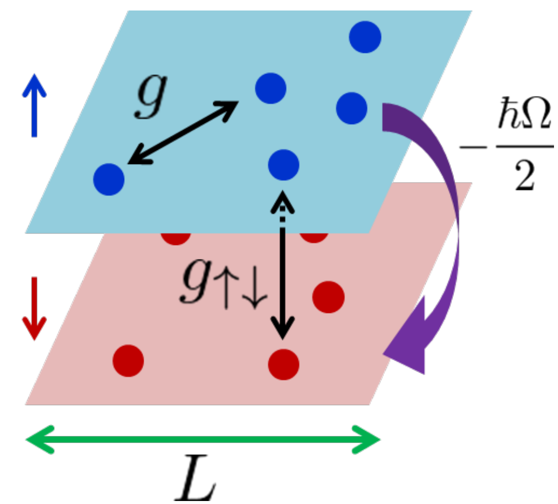
"cut and glue" approach

☐ Systems with continuous symmetry breaking

- ✓ Subleading logarithmic term related to the number of Nambu-Goldstone modes
- ✓ Simple setup: binary Bose-Einstein condensates

$$S_e = \frac{\sigma L^d}{c_{1/2}^{2d}} + \frac{d}{2}\ln\left[\left(\frac{2\pi e n}{G_0}\right)^{1/d} L\right] - \frac{1}{2}\ln\frac{L}{(2\pi)c_{1/2}^2} + O(1)$$

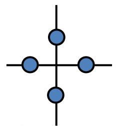
volume law | symmetry restoration | small-k oscillators

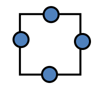


Toric code model

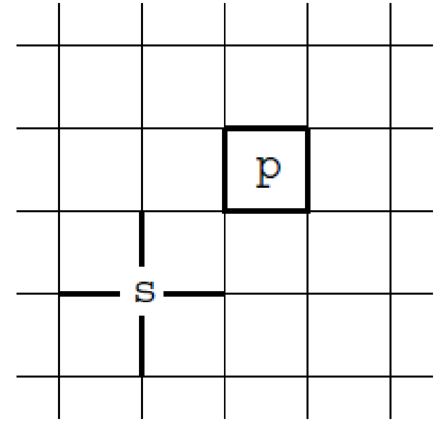
Kitaev, quant-ph,1997; Annal.Phys,2003

$$H = -J_A \underbrace{\sum \prod \sigma_i^x}_{A_s} - J_B \underbrace{\sum \prod \sigma_i^z}_{B_p}$$





$J_A, \; J_B > 0$

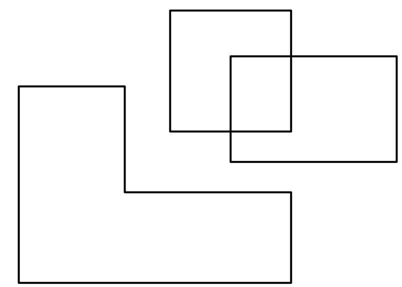


All terms commute.

Ground state: $A_s = +1, \; B_p = +1 \;$ for all $s, \; p$

$J_B = 0$ → Degenerate manifold $\mathcal{E}$

Loop configs.



$\sigma_i^x = -1$

$J_B > 0$ → Resonace between loop configs.

→ Ground state: $|\Psi\rangle = \sum_{c \in \mathcal{E}} |c\rangle$

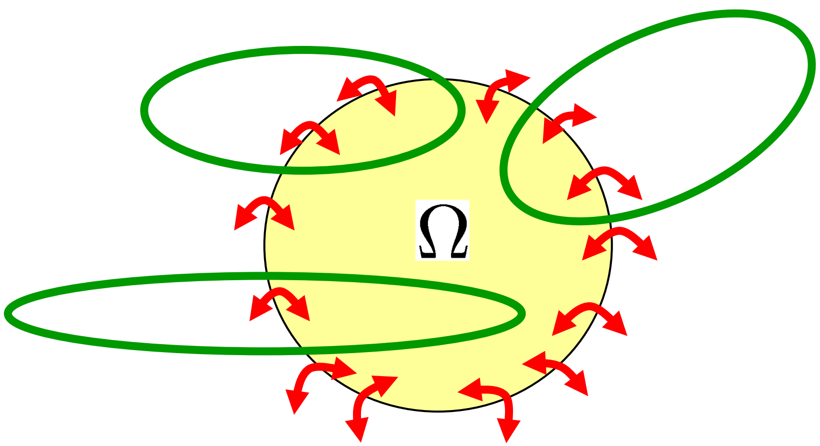
String correlations

$$\langle W^x(C)\rangle = \langle \prod_{i\in C} \sigma_i^x \rangle = 1$$

$$\langle W^z(C')\rangle = \langle \prod_{i\in C'} \sigma_i^z \rangle = 1$$

C, C': closed loop

Hastings & Wen, PRB, 2005



String correlations

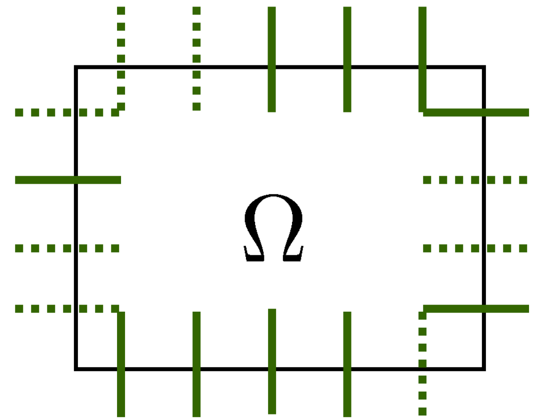
⇨ topological entropy

Entanglement entropy in toric code model

Decomposition in terms of boundary confings. Hamma *et al.*, PRA, 2005

$$|\Psi\rangle = \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{c\in\mathcal{E}} |c\rangle$$

$$= \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{\alpha} \sum_{c_\Omega \in \mathcal{E}_\Omega^\alpha} |c_\Omega\rangle \sum_{c_{\bar\Omega} \in \mathcal{E}_{\bar\Omega}^\alpha} |c_{\bar\Omega}\rangle$$



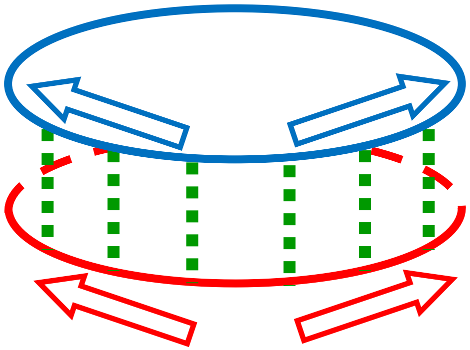
$$= \sum_\alpha \underbrace{\left(\frac{|\mathcal{E}_\Omega^\alpha|\cdot|\mathcal{E}_{\bar\Omega}^\alpha|}{|\mathcal{E}|}\right)}_{p_\alpha = \frac{1}{2^{L-1}}}{}^{1/2} \underbrace{\left(\frac{1}{\sqrt{|\mathcal{E}_\Omega^\alpha|}} \sum_{c_\Omega\in\mathcal{E}_\Omega^\alpha} |c_\Omega\rangle\right)}_{|\psi_\Omega^\alpha\rangle} \underbrace{\left(\frac{1}{\sqrt{|\mathcal{E}_{\bar\Omega}^\alpha|}} \sum_{c_{\bar\Omega}\in\mathcal{E}_{\bar\Omega}^\alpha} |c_{\bar\Omega}\rangle\right)}_{|\psi_{\bar\Omega}^\alpha\rangle}$$

``Schmidt decomposition''

$$S_\Omega = -\sum_\alpha p_\alpha \ln p_\alpha = L\ln 2 - \ln 2$$

← **string correlation**

Two coupled non-chiral TLLs

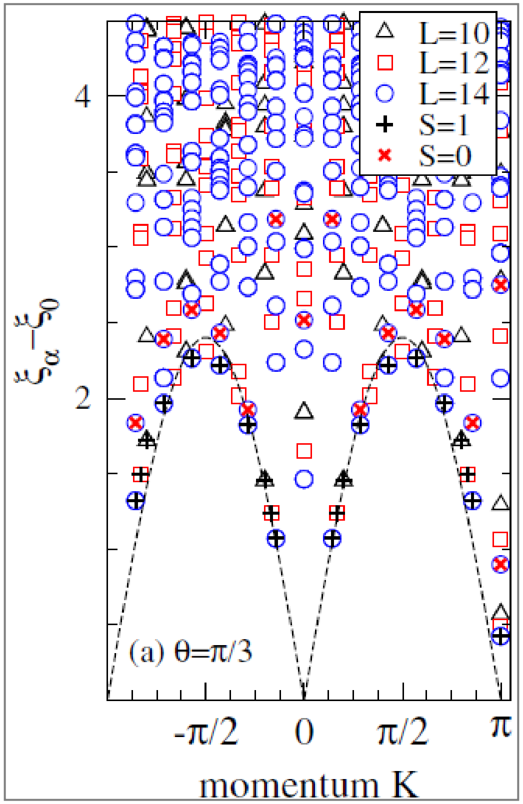


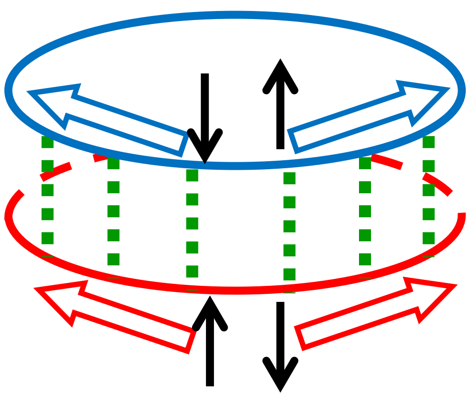
Direct applications to ladder systems and Hubbard chains

cf. Numerical study on spin ladders

Entanglement spectrum remarkably resembles the single-chain spectrum.

Poilblanc, PRL 105, 077202 (2010)



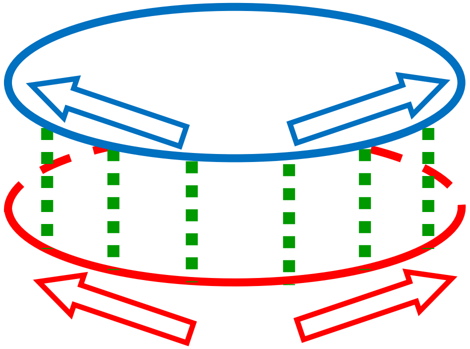


helical TLL

helical TLL

Application to time-reversal–invariant topological insulators

Hamiltonian



$$H_\nu = \int \mathrm{d}x \frac{v_0}{2} \left[K \left(\partial_x \theta_\nu\right)^2 + \frac{1}{K} \left(\partial_x \phi_\nu\right)^2 \right], \quad \nu = 1, 2.$$

K: TLL parameter
- K<1: repulsive interaction
- K>1: attractive interaction

Coupled system: separation into symmetric and antisymmetric channels

$$\phi_\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2), \quad \theta_\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm \theta_2)$$

$$H_+ = \int \mathrm{d}x \left\{ \frac{v_+}{2} \left[K_+ \left(\partial_x \theta_+\right)^2 + \frac{1}{K_+} \left(\partial_x \phi_+\right)^2 \right] + g_+ \cos\left(\sqrt{2}\phi_+ / r\right) \right\}$$

$$H_- = \int \mathrm{d}x \left\{ \frac{v_-}{2} \left[K_- \left(\partial_x \theta_-\right)^2 + \frac{1}{K_-} \left(\partial_x \phi_-\right)^2 \right] + g_- \cos\left(\sqrt{2}\theta_- / \tilde{r}\right) \right\}$$

g+ and g- grow under RG ⇨
- gapped phases of spin ladders
- topological insulators

Result for a fully gapped phase

Entanglement Hamiltonian

$$H_{\mathrm{e}} = \int \mathrm{d}x \frac{v_{\mathrm{e}}}{2} \left[K_{\mathrm{e}} (\partial_x \theta_1)^2 + \frac{1}{K_{\mathrm{e}}} (\partial_x \phi_1)^2 \right]$$

$$v_{\mathrm{e}} = 4\sqrt{\frac{K_+}{K_- m_+ m_-}}, \quad K_{\mathrm{e}} = \sqrt{\frac{K_+ K_- m_-}{m_+}}.$$

$v_+ m_+, \ v_- m_-$: mass gap in "+" and "-" channels

- This resembles H_1, but has a modified TLL parameter.

  The entanglement-edge correspondence is slightly violated!

- In some special cases, symmetry enforces K_e=K.
  - Non-interacting topological insulators: K_e=K=1
  - SU(2)-symmetric spin ladders: K_e=K=1/2

cf. In XXZ spin ladders, H_e has been found to be given by an XXZ chain with an anisotropy modified from the physical chain.

Peschel and Chung, EPL, 2011; Lauchli and Schliemann, PRB, 2012

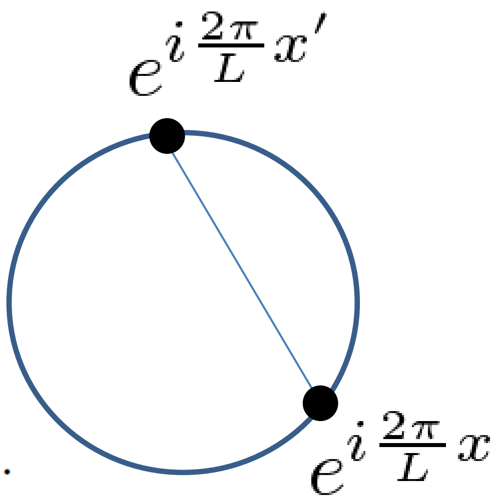
Results for gapless phases

More remarkable violations of the entanglement-edge corresdpondence are found in gapless phases of ladder systems

- Partially gapless case (gap only in "-" channel)

$$H_{\mathrm{e}}^{\mathrm{osc}} = \sum_{k\neq 0} w_k \left(b_{k,1}^{\dagger} b_{k,1} + \frac{1}{2} \right) \qquad w_k \approx 2\sqrt{v_{\mathrm{e}}|k|}$$

$$H_{\mathrm{e}} = \int \mathrm{d}x \frac{v_{\mathrm{e}}}{2} \left[K_{+} (\partial_x \theta_1)^2 + \frac{1}{K_{+}} (\partial_x \phi_1)^2 \right]$$
$$- \frac{2K_{+}}{\pi} \iint dx dx' \partial_x \theta_1(x) \partial_{x'} \theta_1(x') \ln \left| e^{i\frac{2\pi}{L}x} - e^{i\frac{2\pi}{L}x'} \right| .$$



Emergence of a long-range interaction <-> critical correlations

- Fully gapless case

$$H_{\mathrm{e}} = w \sum_{k\neq 0} \left(b_{k,1}^{\dagger} b_{k,1} + \frac{1}{2} \right)$$

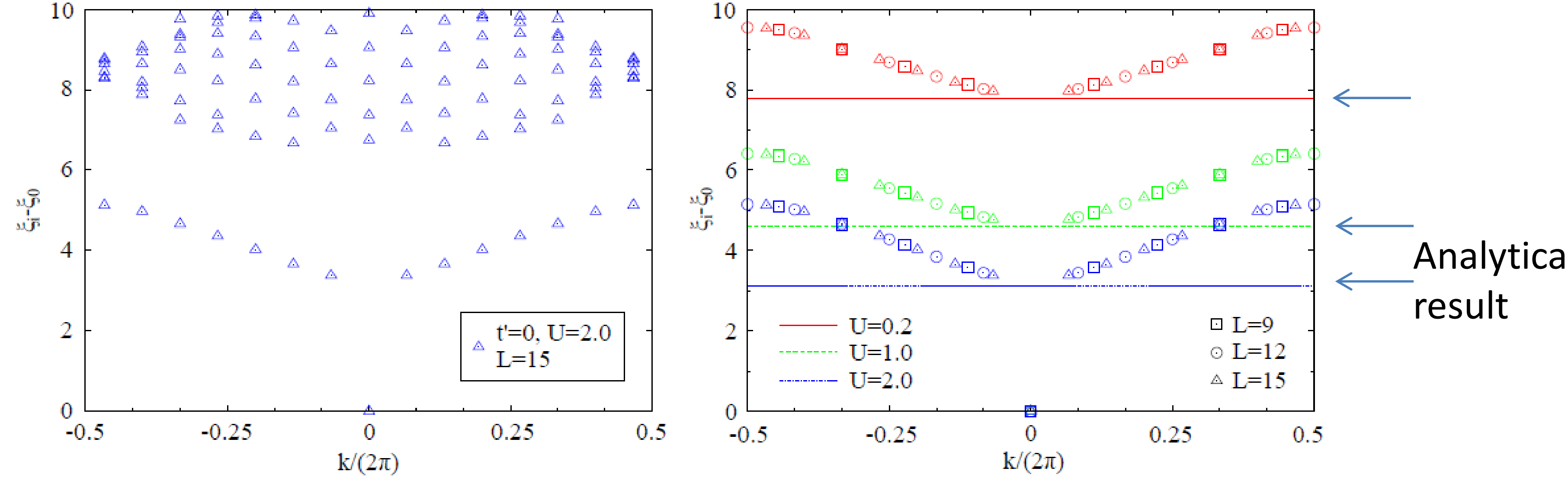
w is determined by K+ and K-

Bosonic modes with a flat dispersion

cf. Chen & Fradkin, PRB, 2013

Numerical demonstrations (hard-core bosons on a ladder)

Fully gapless case



Partially gapless case