Multipartitioning topological phases and quantum entanglement

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Based on arXiv:2110:11980 and forthcoming

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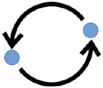
Topologically-ordered phases in (2+1)D

- Phases that are not described by the symmetry-breaking paradigm (no local order parameter).
  E.g., fractional quantum Hall states

- Support anyons; neither bosons nor fermions but have non-trivial exchange (braiding) statistics

$e^{i\theta}$ 

- Characterized by the properties of anyons (fusion, braiding, etc.)

- Bulk-boundary correspondence

- How can we extract/measure topological data?
  Direct observations of abelian braiding statistics [Nakamura et al (20), Bartolomei et al (20)] , central charge [Banerjee et al (18), Kasahara et al (18)]

- Topological data can be captured by topological entanglement entropy [Levin-Wen, Kitaev-Preskill (05)]

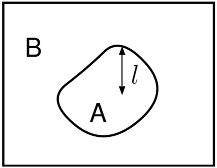
- This talk: go beyond bipartition and study reflected entropy and entanglement negativity

Entanglement entropy

- The von-Neumann entanglement entropy:

$$S_A := -\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

for the reduced density matrix $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$

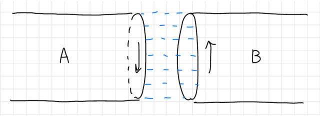


- For topologically-ordered ground states in (2+1)D [Levin-Wen, Kitaev-Preskill (05)]

$$S_A = const. \times \ell - \ln \mathcal{D}$$

"Topological entanglement entropy" $\gamma = \ln \mathcal{D}$ carries universal data

Edge theory approach to entanglement



- The ground state $|GS\rangle$ *near the entangling surface* is well approximated by a conformal boundary state $|B\rangle$: [Qi-Katsura-Ludwig (12)]

$$[T(\sigma) - \bar{T}(\sigma)]\,|B\rangle = 0 \quad (0 \leq \sigma < 2\pi)$$

  where T $(\bar{T})$ is the stress tensor for the edge state of A (B).

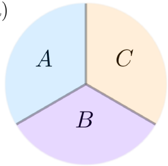
- More precisely, near the entangling boundary, $|GS\rangle \sim e^{-\epsilon H_{edge}}|B\rangle$ where $\epsilon \sim 1/$(bulk gap).

- The reduced density matrix is

$$\rho_A \propto \mathrm{Tr}_B \left[e^{-\epsilon H_{edge}} |B\rangle\langle B| e^{-\epsilon H_{edge}} \right]$$

- Go beyond bipartition, and study entanglement quantities

(a)



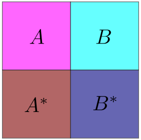
- Multipartite entanglement?

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle\right], \quad |\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\left[|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\right]$$

- Reflected entropy and entanglement negativity

Reflected Entropy

[Dutta-Faulkner (19)]



- The von-Neumann entropy of a canonical purification:

$$\rho_{AB} = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|_{AB}$$

$$\to \left|\sqrt{\rho}_{AB}\right\rangle_{AA^*BB^*} \equiv \sum_i \sqrt{p_i} \left|\psi_i\right\rangle_{AB} \left|\psi_i^*\right\rangle_{A^*B^*} .$$

By tracing out BB^*, we define:

$$R_{A:B} := S_{vN}(A \cup A^*)$$

- Satisfies $h_{A:B} \equiv R_{A:B} - I_{A:B} \geq 0$. $h_{A:B}$ is sometimes called Markov gap. [Hayden-Parrikar-Sorce (21)]

- Admits holographic dual $R_{A:B} = 2E_W$ [Dutta-Faulkner (19)]

Reflected entropy

- Reflected entropy can capture tripartite entanglement [Akers-Rath (19)]

$$h_{\mathrm{GHZ}} = 0 \quad h_{\mathrm{W}} = 1.49 \ln 2 - 0.92 \ln 2 > 0.$$

- More genetically, $h_{A:B} = 0$ if and only if a state $|\psi\rangle \in \mathcal{H}_{ABC}$ is a sum of "triangle state": [Zou-Siva-Soejima-Mong-Zaletel (20)]

$$|\psi\rangle = \sum_j \sqrt{p_j} |\psi_j\rangle_{A^j_R B^j_L} |\psi_j\rangle_{B^j_R C^j_L} |\psi_j\rangle_{C^j_R A^j_L}$$

where $\mathcal{H}_\alpha = \oplus_j \mathcal{H}_{\alpha^j_L} \otimes \mathcal{H}_{\alpha^j_R}$. E.g., $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}} [| \uparrow\uparrow\uparrow\rangle + | \downarrow\downarrow\downarrow\rangle]$

Reflected entropy

- For the ground state of (1+1)D CFT defined on a circle;

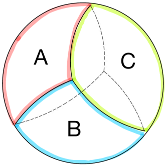
$$h_{A:B} = (c/3)\ln 2$$

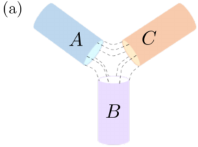
(for any N_A/N and N_B/N in the limit $N \to \infty$) [Zou-Siva-Soejima-Mong-Zaletel (20)]

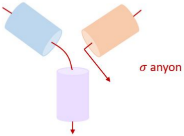
- (1+1)D system is gapped if and only if $h = 0$. [Zou-Siva-Soejima-Mong-Zaletel (20)]
- Holography: $h_{A:B} \geq \frac{\ell_{AdS}}{2G_N}\ln 2$ [Hayden-Parrikar-Sorce (21)]
When $h_{A:B} > 0$ there is no Markov recovery channel for ρ_{ABB^*} or ρ_{BAA^*} ($A \to B \to B^*$ or $B \to A \to A^*$ are not quantum Markov chains); $h_{A:B}$ gives the optimal fidelity of a recovery process of canonical purification.

Reflected entropy (and negativity) in tripartition setup

[Liu-Sohal-Kudler-Flam-SR (21)]







- Studied chiral p-wave superconductor (the Ising TQFT) with $c = 1/2$ and the integer quantum Hal state with $c = 1$.

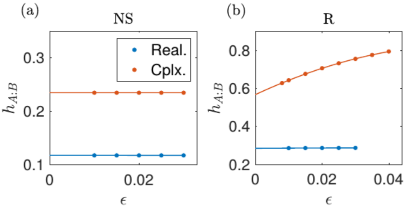
Main result – Reflected entropy

- We computed the Markov gap $h_{A:B} = R_{A:B} - I_{A:B}$ using the bulk-boundary correspondence:

$$h_{A:B} = \frac{c}{3} \ln 2, \quad c\text{: central charge.}$$

- Combined with numerics in lattice models, $h_{A:B} \geq (c/3) \ln 2$ in general.

- Agrees with the recent claim [Zou, Siva, Soejima, Mong, Zaletel (2110.11965)]

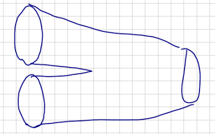


$((c/3) \ln 2 = 0.116, 0.231$ for $c = 1$ and $c = 1/2)$.

- Modular commutator: $i\langle[K_{AB}, K_{BC}]\rangle = \pi c/3$ Kim-Shi-Kato-Albert (21)

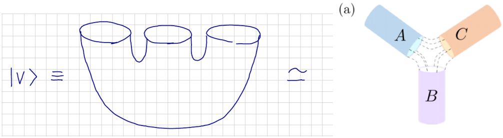
Multipartition topological phases using string field theory

- String field theory $=$ many-body (second quantized) string theory
- Interaction vertices in string field theory



[Witten (86), Gross-Jevicki (87), LeClair-Peskin-Preitschopf (89), ...]

- Vertex state $|V\rangle \simeq$ Topological ground state $|\Psi\rangle$ near the entangling boundary by utilized bulk-boundary correspondence

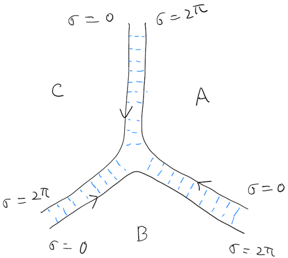


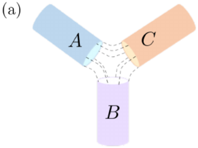
- Generalized [Qi-Katsura-Ludwig (12)] for tripartition geometry

Tripartition by "vertex state"

- Need to find the "ground" states near the entangling boundaries; three copies of edge theories interact

$$[T^I(\sigma) - T^{I+1}(2\pi - \sigma)]|V\rangle = 0 \quad I = A, B, C$$

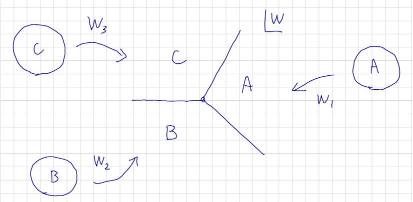




Vertex state

- Borrow ideas from string field theory ("vertex states"). Vertex state $|V\rangle \in \mathcal{H}^3_{edge}$ satisfies

$$\langle V| \left(O_\alpha|0\rangle_1 \otimes O_\beta|0\rangle_2 \cdots O_\gamma|0\rangle_N\right) = \left\langle\, \omega_1[O_\alpha]\,\omega_2[O_\beta]\,\omega_3[O_\gamma]\,\right\rangle_{\mathbb{C}}$$



- We constructed $|V\rangle$ explicitly for chiral p-wave superconductor and Chern insulators. From $e^{-\epsilon H_{edge}}|V\rangle$, we can construct the ground state and reduced density matrix near the entangling boundaries

Entanglement negativity

- *Entanglement negativity* and *logarithmic negativity*, using *partial transpose*

  [Peres (96), Horodecki-Horodecki-Horodecki (96), Vidal-Werner (02), Plenio (05) ...]

$$\mathcal{N}(\rho) := \sum_{\lambda_i < 0} |\lambda_i| = \frac{1}{2}\left(||\rho^{T_B}||_1 - 1\right),$$

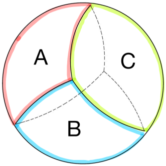
$$\mathcal{E}(\rho) := \log(2\mathcal{N}(\rho) + 1) = \log||\rho^{T_B}||_1.$$

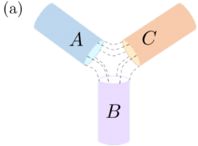
- Good entanglement measure since LOCC monotone. Entanglement entropy is LOCC monotone only only for pure states.

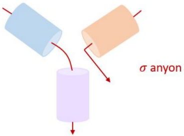
- For mixed states, negativity can extract quantum correlations only.

Tripartition setup

[Liu-Sohal-Kudler-Flam-SR (21)]



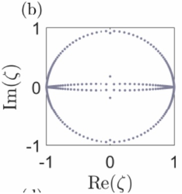


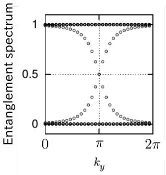


- Studied chiral p-wave superconductor (the Ising TQFT) with $c = 1/2$ and the integer quantum Hal state with $c = 1$.

Negativity spectrum

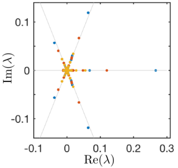
- For fermionic systems, the eigenvalues of $\rho_{A\cup B}^{T_B}$ are complex. Nontrivial distribution of the eigenvalues of $\rho_{A\cup B}^{T_B}$





c.f. Entanglement spectrum [SR-Hatsugai (06)]

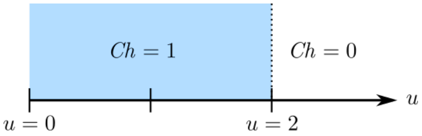
- C.f. (1+1)D fermionic CFTs (6-fold structure) [Shapourian-Ruggiero-SR-Calabrese (19)] (1+1)D topological superconductor (8-fold structure) [Inamura-Kobayashi-SR (19)]



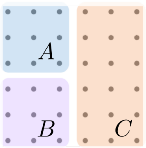
Lattice Chern insulator calculations

- Lattice fermion model $f_{\mathbf{r}} = (f_{\uparrow\mathbf{r}}, f_{\downarrow\mathbf{r}})$:

$$H = \frac{-i}{2} \sum_{\mathbf{r}} \sum_{\mu=x,y} \left[f_{\mathbf{r}}^{\dagger} \tau_{\mu} f_{\mathbf{r}+\mathbf{a}_{\mu}} - f_{\mathbf{r}+\mathbf{a}_{\mu}}^{\dagger} \tau_{\mu} f_{\mathbf{r}} \right] + \frac{1}{2} \sum_{\mathbf{r}} \sum_{\mu=x,y} \left[f_{\mathbf{r}}^{\dagger} \tau_{z} f_{\mathbf{r}+\mathbf{a}_{\mu}} + f_{\mathbf{r}+\mathbf{a}_{\mu}}^{\dagger} \tau_{z} f_{\mathbf{r}} \right] + u \sum_{\mathbf{r}} f_{\mathbf{r}}^{\dagger} \tau_{z} f_{\mathbf{r}},$$

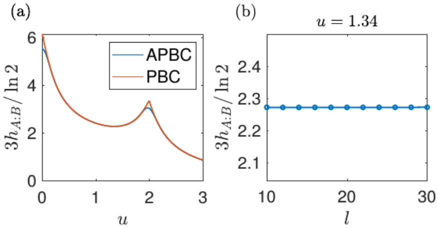


(b)



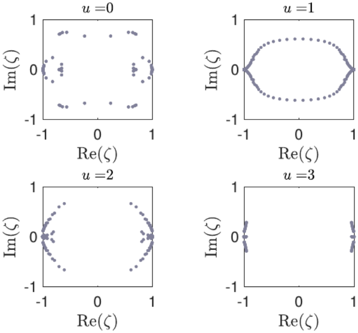
Reflected entropy

- $h_{A:B}$ is minimal in the topological phase around $u = 1.34$
- $h_{A:B} \sim (c/3) \ln 2 \times 2.3$
- Note that there are four trijunctions, as opposed to two in the edge theory calculations. May result in a factor of 2.

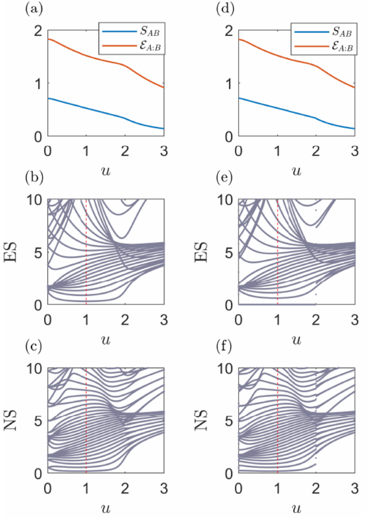


- See also: [Zou, Siva, Soejima, Mong, Zaletel (2110.11965)]

Negativity spectrum

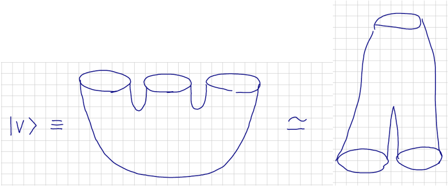


Entropy and Negativity



- "Trivial" for $|u| \to \infty$
- "Circular distribution" deep in the Ch=1 phase

Wave function overlap

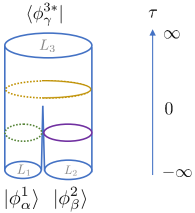


- Vertex state $\leftrightarrow$ Wave function overlap $A_{\alpha\beta\gamma}$

$$A_{\alpha\beta\gamma} = \langle V| \Big(|\phi^1_\alpha\rangle |\phi^2_\beta\rangle |\phi^3_\gamma\rangle\rangle \Big) \quad = \quad \langle \phi^*_\gamma | \phi_\alpha \phi_\beta \rangle$$

- Wave function overlap $A_{\alpha\beta\gamma}$ can be used to extract OPE coefficients (and more) of (1+1)d lattice quantum systems at conformal critical point [Zou-Vidal arXiv:2108.09366] [Liu-Zou-SR arXiv:2203.xxxxx]

Wavefunction overlap and OPE



- Wave function overlap can be mapped on to a 3pt function on a plane:

$$\frac{A_{\alpha\beta\gamma}}{A_{\mathbb{1}\mathbb{1}\mathbb{1}}} = \langle \phi_\alpha^1(-\infty)\phi_\beta^2(-\infty)\phi_\gamma^{3*}(+\infty)\rangle_{\text{Pants}}$$

$$= \text{Jacobian} \cdot \langle \phi_\alpha^1(w_1)\phi_\beta^2(w_2)\phi_\gamma^{3*}(w_3)\rangle_{\mathbb{C}}$$

$$\propto C_{\alpha\beta\gamma}$$

Finite size corrections

- Finite size corrections:

$$\frac{A_{\alpha\beta\gamma}}{A_{\mathbb{1}\mathbb{1}\mathbb{1}}} = \tilde{A}^{(0)}_{\alpha\beta\gamma} + \sum_{p_{\alpha\beta\gamma}>0} \tilde{A}^{(p)}_{\alpha\beta\gamma} L^{-p_{\alpha\beta\gamma}}$$

$p_{\alpha\beta\gamma}$: operator content of the $\mathbb{Z}_2$ orbifold theory

- Explicitly, when $L_3 = L_1 + L_2$ and $L_1 = L_2 = L$,

$$A_{\mathbb{1}\mathbb{1}\mathbb{1}} \propto L^{-c/8}$$

and

$$A_{\alpha\beta\gamma} = \sum_{\delta,\hat{\chi}=0,1} a_{(\delta,\hat{\chi})}\, 2^{-2\Delta_\alpha - 2\Delta_\beta + 2\Delta_\gamma} C_{\alpha\beta\gamma}\, L^{-\Delta_{(\delta,\hat{\chi})}}$$

Application: Haagerup model

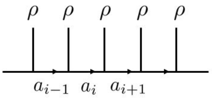
Interacting anyon chain for the Haagerup fusion category
[Huang-Lin-Ohmori-Tachikawa-Tezuka (21); Vanhove-Lootens-Damme-Wolf-Osborne-Haegeman-Verstraete (21)] :

$$H = -\sum_i P_\rho^{(i)},$$

$$P_c^{(i)} |a_{i-1} a_i a_{i+1}\rangle = \sum_{a'_i} [F^{a_{i-1}\rho\rho}_{a_{i+1}}]_{a_i c} [F^{a_{i-1}\rho\rho}_{a_{i+1}}]^*_{a'_i c} |a_{i-1} a'_i a_{i+1}\rangle$$

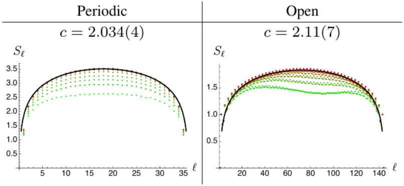
Hilbert space:

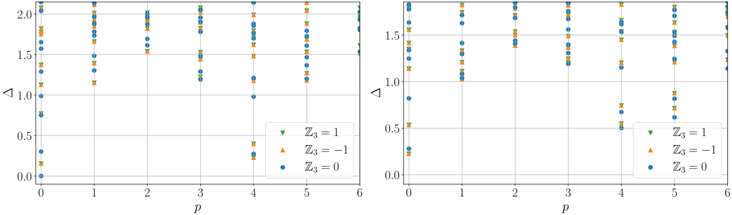
$$|a_1 a_2 \cdots a_N\rangle =$$

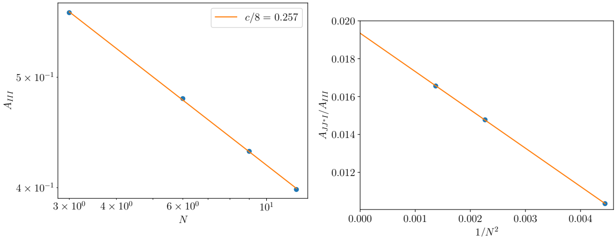


[Feiguin et al (06)]

Realizes a CFT with $c \simeq 2$? 2d CFT which has the Haagerup fusion category as its symmetry. Which CFT?







$|J\rangle$: lowest eigenstate with spin 1 at size $N = 3n + 1$. $|J^*\rangle$: lowest eigenstate with spin 1 at size $N = 3n + 2$; are they chiral currents? Expect:

$$\frac{A_{JJ^*\mathbb{1}}}{A_{\mathbb{1}\mathbb{1}\mathbb{1}}} = 2^{-2\Delta_J - 2\Delta_{J^*} - 2\Delta_{\mathbb{1}}} C_{JJ^*\mathbb{1}} = \frac{1}{16}$$

Obtained $\frac{A_{JJ^*\mathbb{1}}}{A_{\mathbb{1}\mathbb{1}\mathbb{1}}} \simeq 0.02$ numerically.

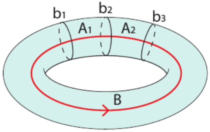
Summary/Outlook

- New tripartition setup and new calculations of entanglement quantities
- They may capture topological data beyond topological entanglement entropy, e.g., Abelian v.s. Non-abelian, total central charge.
- Finite-T topological transition can be detected by negativity [Hart-Castelnovo(18);Lu-Hsieh-Grover(19)]
- May have an implication on numerics (tensor-networks)
- May have an implication in string field theory?
- Other entanglement quantities, such as odd entropy, entanglement of purification, etc?
- Experiments: Many-body interference or randomized measurements [Islam et al (15)] [Kaufman et al (16)] [Lukin et al (18)] [Brydges (19)]

Negativity for topological liquid

[Lee-Vidal (13), Castelnovo (13), Wen-Matsuura-SR (16), Wen-Chang-SR (16) Lim-Asasi-Teo-Mulligan (21)]

- Generic state on a torus: $|\psi\rangle = \sum_a \psi_a |\mathfrak{h}_a\rangle\rangle$



- Mutual information and negativity:

$$I_{A_1:A_2} = \frac{\pi c}{12}\frac{l_2}{\epsilon} - 2\ln\mathcal{D} + 2\sum_a |\psi_a|^2 \ln d_a - \sum_a |\psi_a|^2 \ln |\psi_a|^2$$

$$\mathcal{E}_{A_1:A_2} = \frac{\pi c}{16}\frac{l_2}{\epsilon} - \ln\mathcal{D} + \ln\Big(\sum_a |\psi_a|^2 d_a\Big)$$

$\mathcal{E}$ is dependent on ψ_a only for non-Abelian topological order (for Abelian topological order, $d_a = 1$ for all a).