Entanglement entropy in interacting quantum field theories

KEK

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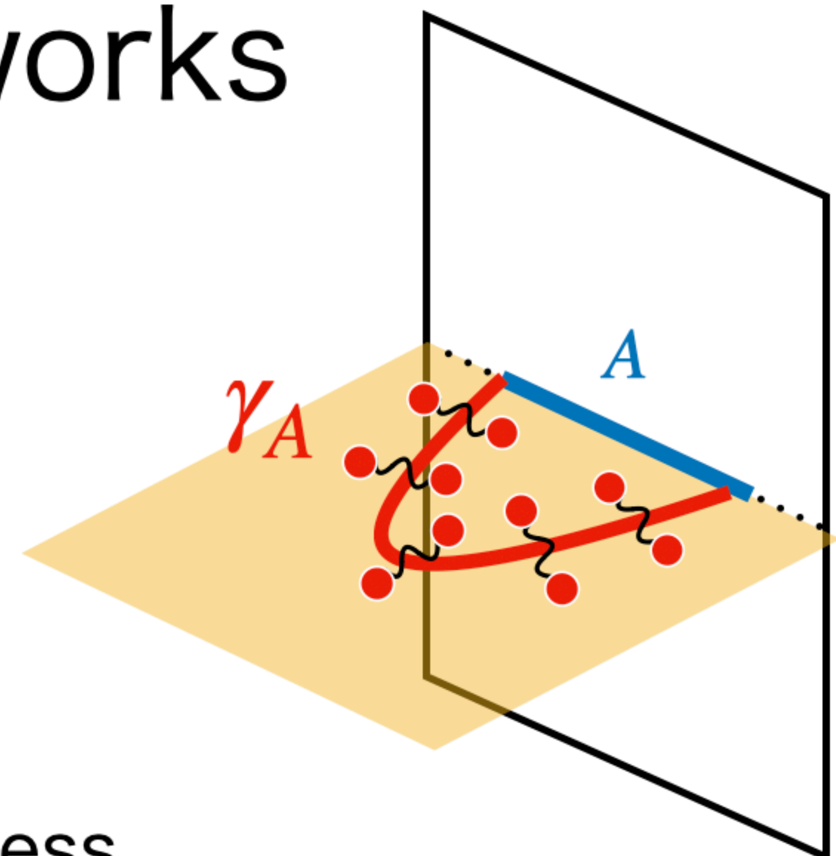
SOKENDAI

Based on S. Iso, **TM**, K. Sakai, 2103.05303 (PRD103, 105010),
S. Iso, **TM**, K. Sakai, 2105.02598 (PRD103, 125019),
S. Iso, **TM**, K. Sakai, 2105.14834 (Symmetry13, 1221)
and some work in progress

Quantum Information Entropy in Physics on Mar. 22, 2022

Tomorrow at Poster session #18

Holographic entanglement distillation in tensor networks



Takato Mori

Based on work in progress
with H. Matsueda (Tohoku U) and H. Manabe (Kyoto U)

Quantum Information Entropy in Physics on Mar. 23, 2022

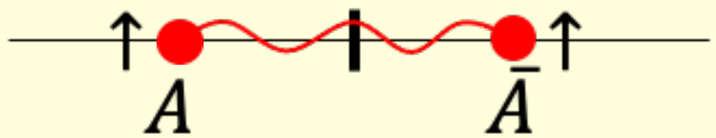
Outline

1. Introduction
2. Replica trick and orbifold method (review)
3. Formulation: Summing over twisted diagrams
4. Propagator/vertex contributions
   - Diagrammatic analysis
   - Computation of twisted links and its interpretation
   - Resummation over perturbative contributions
5. Wilsonian RG and EE
6. Exact calculation: Asymptotic Equipartition Property

   (speculative)

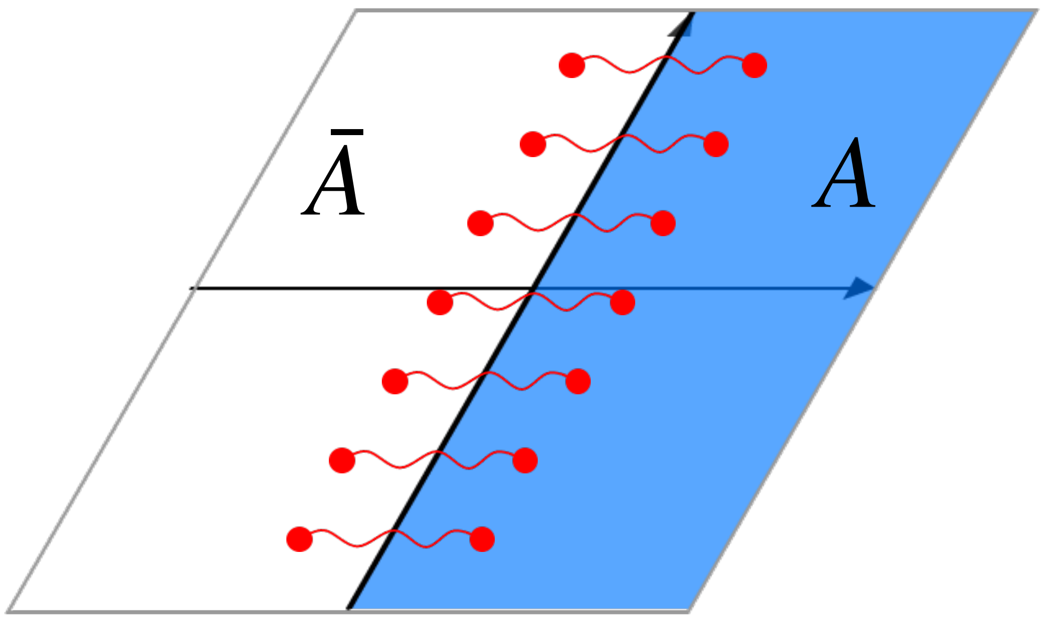
Entanglement entropy (EE)

$$S_{\mathrm{EE}} = -\,\mathrm{tr}_A\left(\rho_A \log \rho_A\right), \quad \rho_A = \mathrm{tr}_{\bar{A}}\rho$$



✓ characterize quantum corr. between subsytems; but NOT an observable

➡ Relation to (flat-space) observables?



Suppose we know correlation functions in flat space, how is EE expressed in terms of them?

Entanglement entropy (EE)

$$S_{\mathrm{EE}} = -\,\mathrm{tr}_A\left(\rho_A \log \rho_A\right), \quad \rho_A = \mathrm{tr}_{\bar{A}}\rho$$

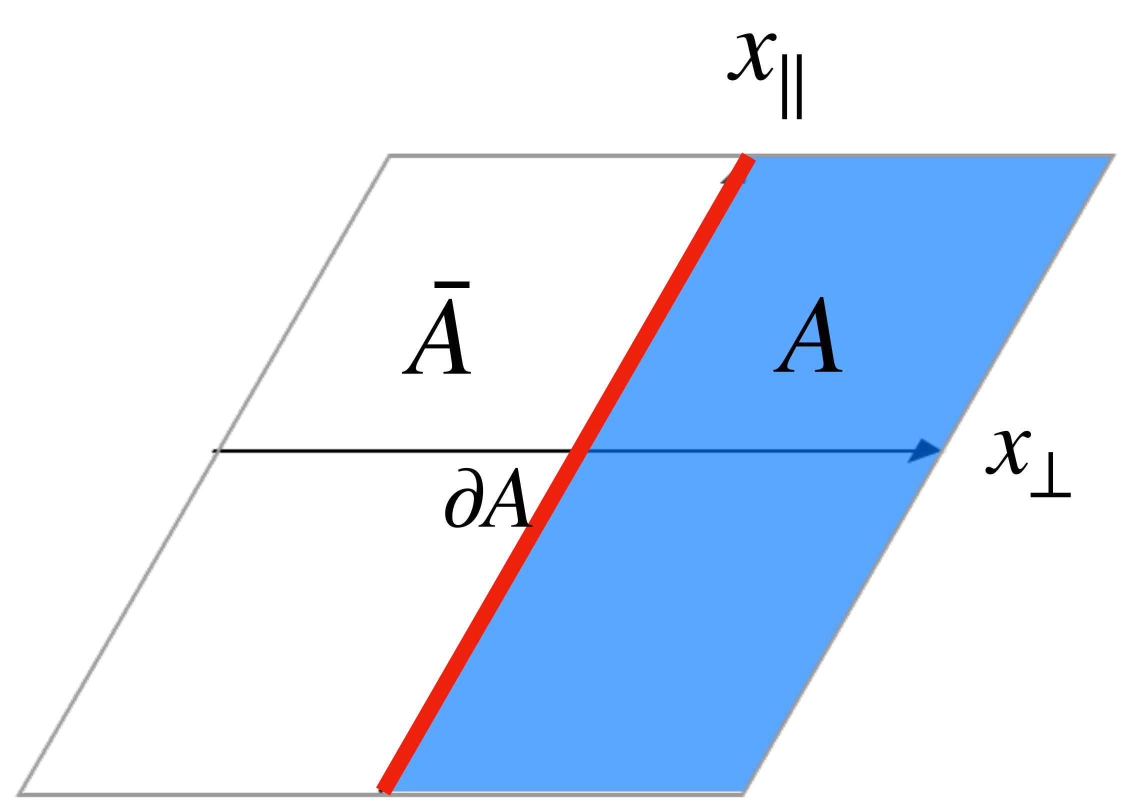
Setup:

vacuum (ground state) EE of massive interacting QFTs with half-space subregion

not Gaussian, not CFT,
no gravity dual

➡Utilize **QFT techniques**

perturbation theory,
Feynman diagrams



$x^0 = 0$ (constant time slice)

Summary of our results

- Dominant part of EE in low energy come from propagators & vertices

  They can be resummed to all orders (e.g. ϕ^4 theory):

1. Propagator contributions

$$S_{\text{prop}} = \frac{\text{vol}(\partial A)}{12} \int^{\Lambda} \frac{d^{d-1}k_{\parallel}}{(2\pi)^{d-1}} \log\left[G_{\phi\phi}(\mathbf{0}, k_{\parallel})\right]$$ in terms of 2pt fn of fund. op.

2. Interaction vertex contributions

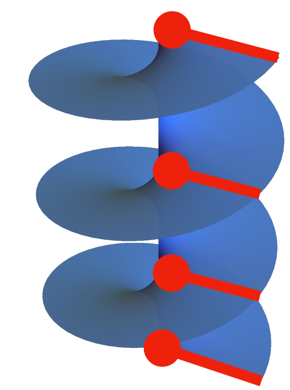
$$S_{\text{vert}} = \frac{\text{vol}(\partial A)}{12} \int^{\Lambda} \frac{d^{d-1}k_{\parallel}}{(2\pi)^{d-1}} \left[\log\left(1 + \left(-\frac{3\lambda_4}{2}\right) G_{\phi^2\phi^2}(\mathbf{0}, k_{\parallel})\right)\right]$$ 2pt fn of comp. op.

- Generalized to any locally interacting theories

Orbifold method

[Nishioka-Takayanagi, 2007]
[He-Numasawa, et al., 2014]

Replica trick: $S_{\mathrm{EE}} = -\lim_{n\to 1} \partial_n \left(\mathrm{tr}_A \rho_A^n\right) = \lim_{n\to 1} \partial_n \left(F_n/n\right)$



$\times\, \mathbb{R}^{d-1}$

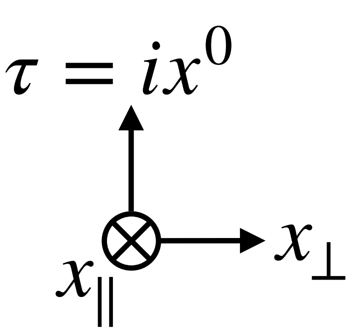
analytical cont. $M = 1/n \in \mathbb{Z}$

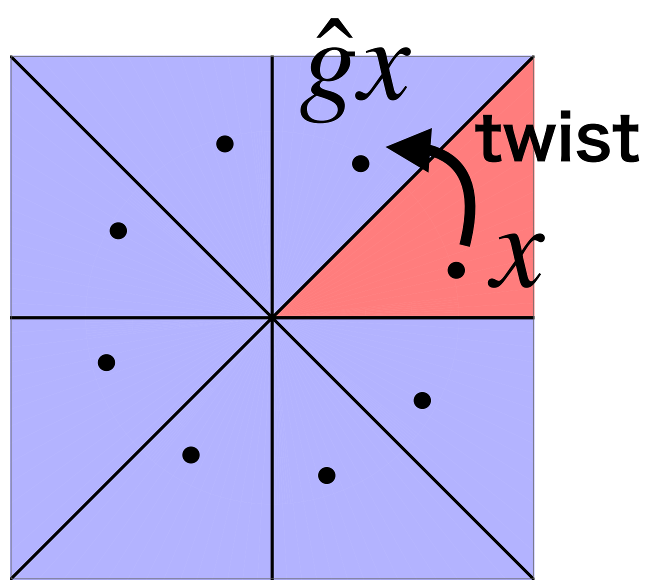
$$\boxed{S_{\mathrm{EE}} = -\frac{\partial}{\partial M}\left(MF^{(M)}\right)\Big|_{M\to 1}}$$

$F^{(M)}$: Free energy on $\mathbb{Z}_M$ orbifold $\mathbb{R}^2/\mathbb{Z}_M \times \mathbb{R}^{d-1}$

=connected bubbles

$\boldsymbol{x} = (\tau, x_\perp)$ $\quad x_\parallel$





$(M = 8)$

Identify $\mathbb{R}^2$ by projection $\hat{P} \equiv \frac{1}{M}\sum_{j=0}^{M-1} \hat{g}^j$

$\Leftrightarrow$ **Sum over twists** $\{\hat{g}^j\}$

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(speculative)

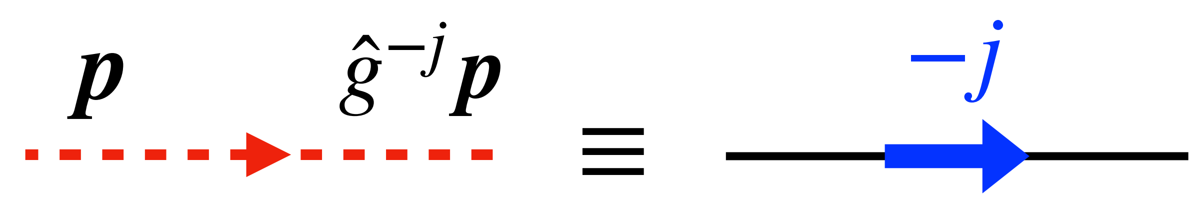
QFTs on $\mathbb{Z}_M$ orbifold

- Action on $\mathbb{Z}_M$ orbifold (e.g. scalar ϕ^4 theory)

$$\int \frac{d^2x}{M} d^{d-1}x_\parallel \left[\frac{1}{2}\phi\hat{P}\left(-\Box + m^2\right)\hat{P}\phi + \frac{\lambda}{4}(\hat{P}\phi)^4\right]$$

- Propagator

$$G_0^{(M)}(x, y) = M\langle x\,|\,(\hat{P}G_0\hat{P})\,|\,y\rangle$$



$$= \sum_{j=0}^{M-1} G_0(\hat{g}^j x, y) = \sum_{j=0}^{M-1} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{i(\hat{g}^{-j}\boldsymbol{p}\cdot\boldsymbol{x} - \boldsymbol{p}\cdot\boldsymbol{y}) + ip_\parallel\cdot(x_\parallel - y_\parallel)}}{p^2 + m^2}$$

- Interaction vertex

$$-\frac{6}{M}\lambda$$

Calculate EE based on the flat-space Feynman rule + summation over twists

Aside: Position space interpretation of twisted link

Consider a link $U(x,y) = U(x-y)$.

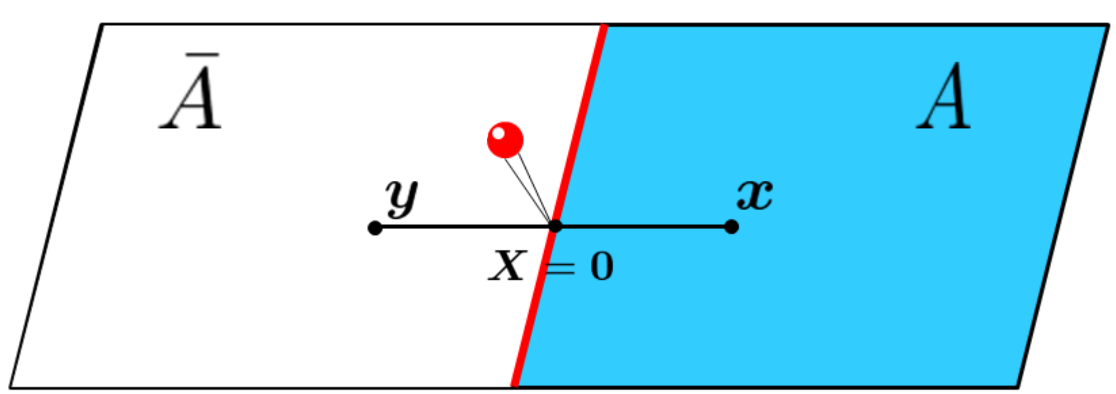
$$U(\hat{g}^n x - y) = e^{\cot\theta_n \boldsymbol{r}\times\partial_{\boldsymbol{X}}/2} \frac{1}{4\sin^2\theta_n} \int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} e^{ik_\parallel\cdot r_\parallel} \frac{1}{\left(-\partial_{\boldsymbol{X}}^2/4\sin^2\theta_n\right) + {k_\parallel}^2 + m^2} \delta^2(\boldsymbol{X}),$$

where $r = x - y$、$\boldsymbol{X} = (\boldsymbol{x}+\boldsymbol{y})/2$、$\theta_n = n\pi/M$

translational invariance $\partial_X = 0$

$$\sum_{n=1}^{M-1} \frac{1}{\sin^2\theta_n} = \frac{M^2-1}{3}$$

$$\Rightarrow \boxed{U^{(M)}(x,y) = \sum_{j=0}^{M-1} U(\hat{g}^j x - y) = U(x-y) + \frac{M^2-1}{12M}\delta^2(\boldsymbol{X}) U_\parallel(r_\parallel)}$$



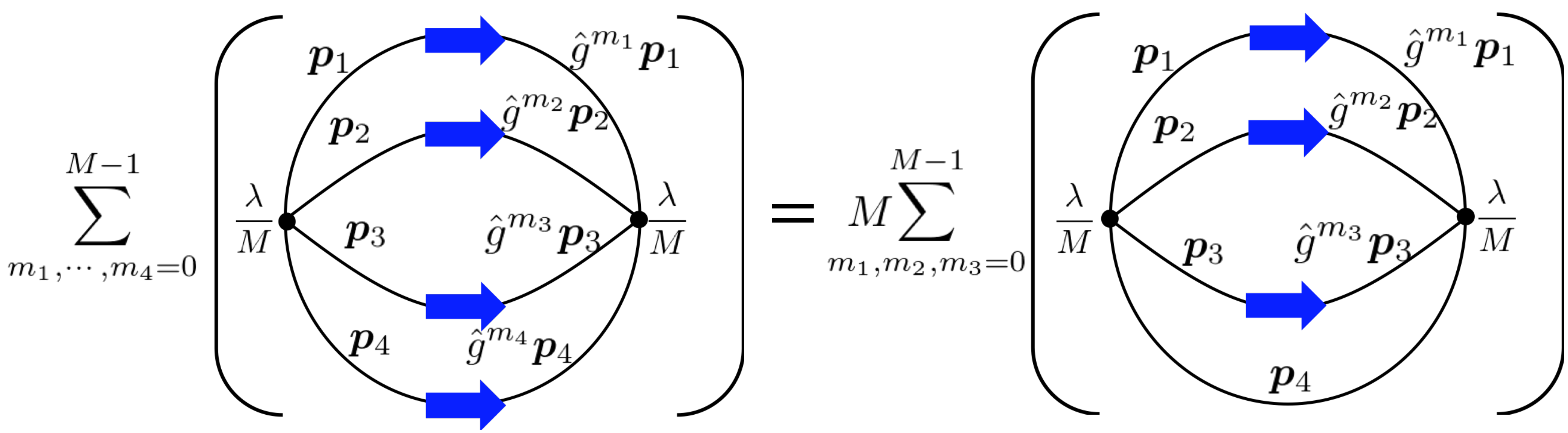
$$\left(U_\parallel(r_\parallel) \equiv \int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} \tilde{U}(\boldsymbol{0}, k_\parallel) e^{ik_\parallel\cdot r_\parallel} \right)$$

Redundancy in twisted diagrams

- CAUTION: Summation over twists must be treated carefully

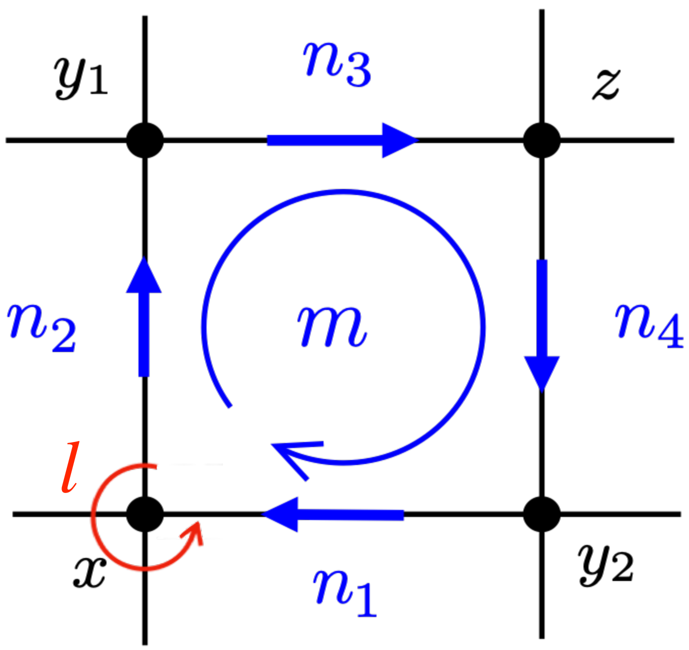
☹Some twists can be eliminated by redundancies!

$$\delta^2(\hat{g}^{m_1}\boldsymbol{p}_1 + \hat{g}^{m_2}\boldsymbol{p}_2 + \cdots) = \delta^2\left(\hat{g}^{n}\{\hat{g}^{m_1}\boldsymbol{p}_1 + \hat{g}^{m_2}\boldsymbol{p}_2 + \cdots\}\right)$$



Twisted loop momentum (flux) is fundamental

Feynman diagrams: composed of plaquettes



(lattice-like)

$\mathbb{Z}_M$ gauge theory on Feynman diagrams

Invariant: Sum of twists around plaquette (flux)

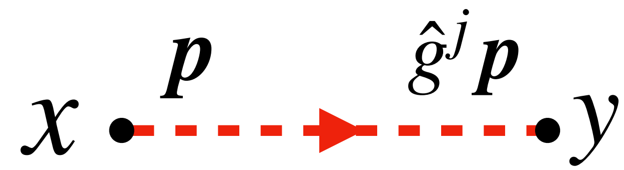
$$m = n_1 + n_2 + n_3 + n_4 = (n_1 - l) + (n_2 + l) + n_3 + n_4 \mod M$$

flux ~ twist of loop momentum

Sum over **indep.** twist config. = Sum over all possible **flux** config.

Propagator/vertex contributions
—Diagrammatic analysis

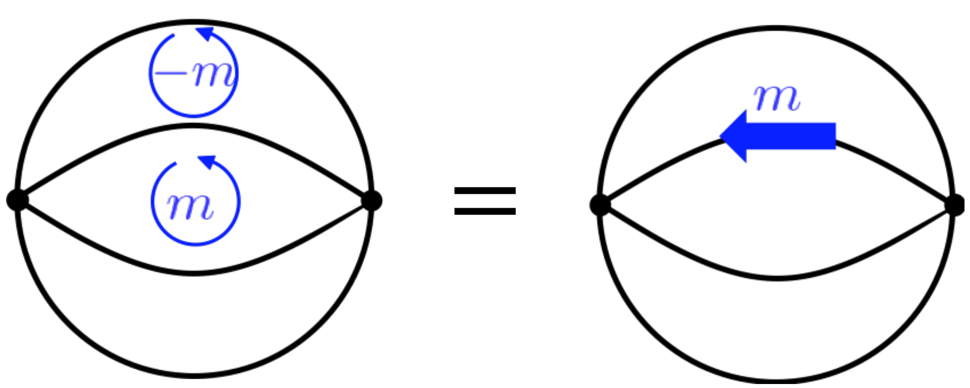
Dominant contrib. in low energy→diagrams with a single twisted link

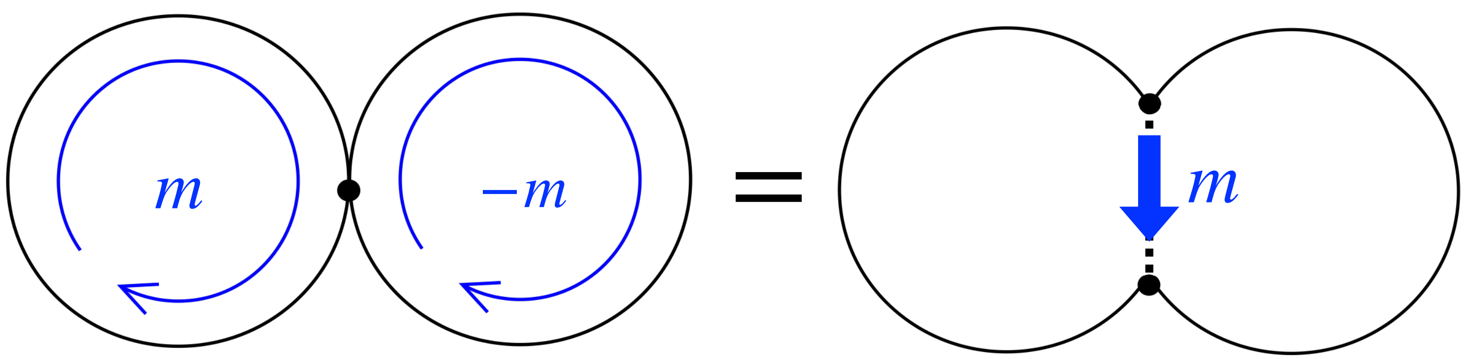


$$G(\boldsymbol{x}-\boldsymbol{y}) \to G(\hat{g}^j \boldsymbol{x}-\boldsymbol{y})$$

$$\int d\boldsymbol{x}\phi^4(\boldsymbol{x}) = \int d\boldsymbol{x}d\boldsymbol{y}\phi^2(\boldsymbol{x})\phi^2(\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{y})$$

$$\to \int d\boldsymbol{x}d\boldsymbol{y}\phi^2(\boldsymbol{x})\phi^2(\boldsymbol{y})\delta(\hat{g}^j \boldsymbol{x}-\boldsymbol{y})$$





link = propagator

link = opened vertex

Propagator/vertex contributions —Resummation

We can resum propagator/vertex contribution to all orders using Dyson equation and by carefully dealing with redundancies due to twists

$$S_A = S_{\rm prop} + S_{\rm vertex}$$

$$\boxed{\begin{aligned} S_{\rm prop} &= -\frac{V_{d-1}}{12}\int^{\Lambda}\frac{d^{d-1}k_{\parallel}}{(2\pi)^{d-1}}\left[\log\left(G^{-1}/\Lambda^2\right)\right](\mathbf{0},k_{\parallel}) \\ S_{\rm vertex} &= \frac{V_{d-1}}{12}\int^{\Lambda}\frac{d^{d-1}k_{\parallel}}{(2\pi)^{d-1}}\left[\log\left(1+\left(-\frac{3\lambda_4}{2}\right)G_{\phi^2\phi^2}\right)\right](\mathbf{0},k_{\parallel}) \end{aligned}}$$

Propagator/vertex contributions to all orders

||

Contributions from renormalized 2pt func. of fundamental op. ϕ/composite op. $[\phi^2]$

We successfully found *single* twist contributions to all orders!

Contributions from other terms (*multiple* twists)?

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(speculative)

Wilsonian RG

- Along Wilsonian RG, quant. fluctuations are integrated out

$$\int \mathcal{D}\phi_{k\leq\Lambda} e^{-S_\Lambda[\phi_{k\leq\Lambda}]} \overset{\substack{\text{integrate}\\ \text{out}}}{=} \int \mathcal{D}\phi_{k\leq\Lambda'<\Lambda} e^{-S'_{\text{eff}}[\phi_{k\leq\Lambda'<\Lambda}]} \overset{\text{rescale}}{=} \int \mathcal{D}\phi_{k\leq\Lambda} e^{-S_{\text{eff}}[\phi_{k\leq\Lambda}]}$$

- Higher-order interaction terms in S_{eff} take loop corrections into the account as it flows to IR

$$\mathscr{L}_{\text{eff}} = \frac{1}{2}\left(\partial\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \lambda'\frac{\phi^6}{\Lambda^2}\cdots \qquad (d+1=4)$$

- Multiple twists necessarily involve multiple loops

➡ $\underbrace{\text{Single twists}}_{\text{prop.\&vertex contributions from } S_{\text{eff}}}$ contribute to EE the most in the IR limit

Exact computation of entanglement entropy

- The difficulty of the exact calculation lies the analytical continuation of the following discrete summation:

$$B(L) = \frac{1}{M} \sum_{j_1,\cdots,j_L=0}^{M-1} \int \prod_{l=1}^{L} \left[\frac{d^{d+1} p_l}{(2\pi)^{d+1}} \right] I\left(\{p_l, \hat{g}^{j_l} p_l\}\right) \delta^{d+1}\left(\sum_{l=1}^{L} (1-\hat{g}^{j_l}) p_l \right)$$

For each twisted link,

$$U(\hat{g}^n x - y) = e^{\cot\theta_n \boldsymbol{r}\times\partial_X/2} \frac{1}{4\sin^2\theta_n} \int \frac{d^{d-1}k_\parallel}{(2\pi)^{d-1}} e^{ik_\parallel \cdot r_\parallel} \frac{1}{\left(-\partial_{\boldsymbol{X}}^2/4\sin^2\theta_n\right) + {k_\parallel}^2 + m^2} \delta^2(\boldsymbol{X}),$$

Summing over n is not possible for finite integer M unless $\partial_X = 0$ (which only holds for a single twist).

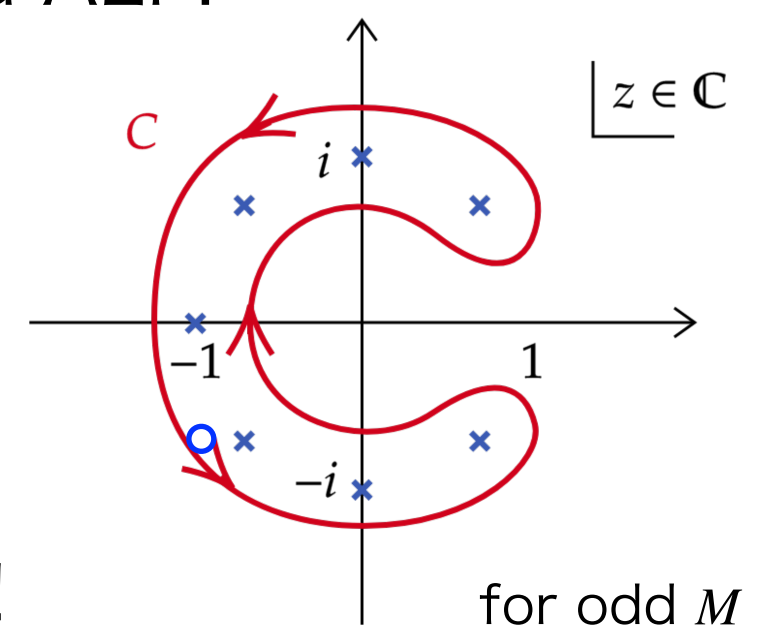
Exact computation of entanglement entropy

- In $M \to \infty$ (Renyi-0 entropy a.k.a. min-entropy)
  the discrete sum is replaced by an integral (but too naive)

$$\sum_{n=1}^{M-1} U(\hat{g}^n x - y) \to \left(\int_{\varepsilon_M}^{\pi(1-\varepsilon_M)} + \int_{\pi(1+\varepsilon_M)}^{2\pi(1-\varepsilon_M)} \right) \frac{d\theta}{2\pi} e^{\cot(\theta/2)\boldsymbol{r}\times\partial_X/2} \frac{1}{-\partial_X^2 + 4(k_{\parallel}{}^2 + m^2)\sin^2(\theta/2)} \delta^2(\boldsymbol{X})$$

- But this is divergent in the strict min-entropy limit.
  If this is resolved, then we can calculate EE via AEP:

$$S(\rho) = \lim_{\epsilon\to 0}\lim_{N\to\infty} \frac{1}{N} S_{\epsilon}^{\min}(\rho^{\otimes N})$$



N~(# of independent fields) (e.g. $O(N)$ model);

ϵ~the amount of discarded eigenvalues$\cdots$

Here we have already regularized theory by Λ!

Entropy in a cut-off QFT might be a m/Λ-smoothed entropy!

Summary

[2103.05303, 2105.02598, 2105.14834]

- Focus: Vacuum EE of a half space in interacting QFTs
- Method: $\mathbb{Z}_M$ gauge theory on Feynman diagrams

  (summing over twists of “loop momenta”)
- There exists contributions from interactions in addition to propagators

  ➡given in terms of renormalized 2pt func. of fund./comp. operators

  correlation func.

  ➡relation to **observables**!

(What we didn’t discuss)

- The area law of EE and Rényi entropy
- Can be generalized to any locally interacting theories
- Prop. & vertex contrib can be unified in a matrix form

Future work

Work in progress:

- Behavior of EE along RG flow:
  Proposal for a new c-function based on half-space EE;
  c-theorem apart from fixed points
- Relation and generalization with real-time formalism
- Smoothed min-entropy in QFT

Open problems/Outlook:

- Other than half space? Excited states?
- Any cutoff-independent quantity based on half-space EE? $\rightarrow$ comparison with numerical computation