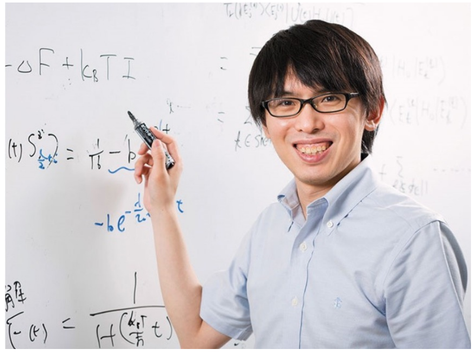
Quantum fluctuation theorem under continuous measurement and feedback

arXiv:2112.09351

Toshihiro Yada (Tokyo University)

Collaborators



Nobuyuki Yoshioka (Tokyo Univ.)   Takahiro Sagawa (Tokyo Univ.)

Quantum Information Entropy in Physics (March 23rd, 2022)

Outline

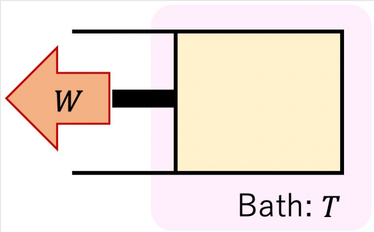
1. Introduction
   - Stochastic thermodynamics
   - Information thermodynamics
2. Summary of our research
   - Motivation
   - Comparison with the previous research
3. Details of our research
   - Setup
   - Derivation of the main result
   - Numerical demonstration

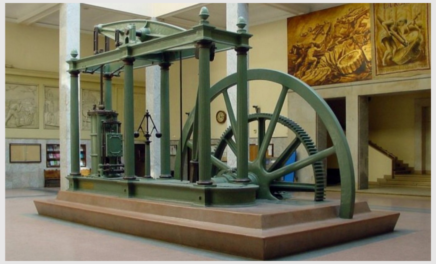
Thermodynamics in small systems

Thermodynamics

Established in 1800s

- Macroscopic systems
- Equilibrium state





extension

The second law (SL) $\sigma \geq 0$

Entropy production: $\sigma \equiv \Delta S - \beta Q$

ΔS: entropy change β: inverse temperature
Q: heat transferred from heat bath

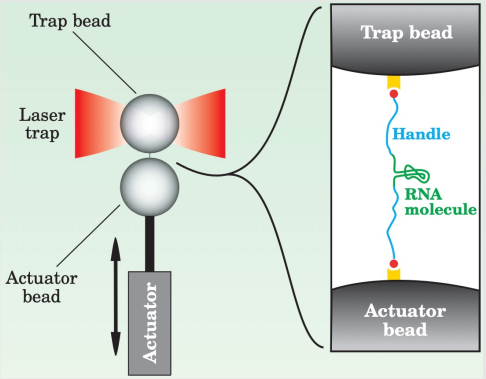
Total entropy increase of system + heat bath

Stochastic thermodynamics

Intensively studied in last 20 years

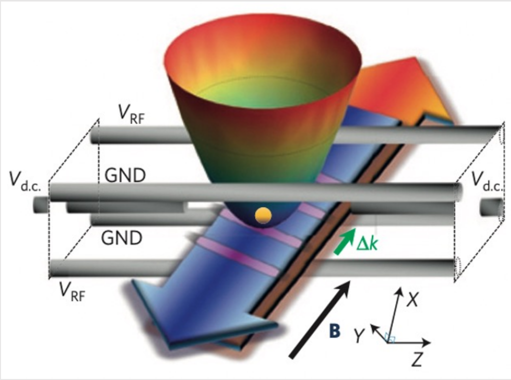
- Small systems
- Nonequilibrium state

Molecule (RNA)



C. Bustamante et al., Physics Today **58**, 7, 43-48 (2005)

Trapped-ion



S. An. et. al., Nature Physics **11**, 193–199 (2015)

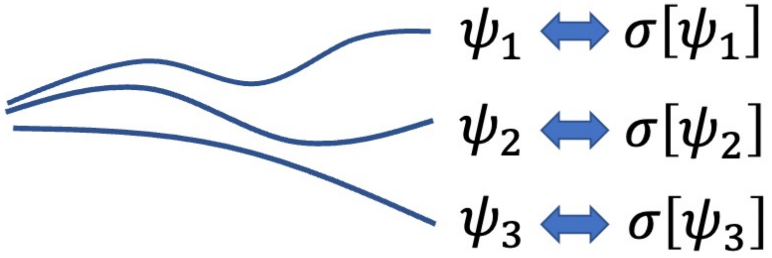
Thermodynamic laws in small systems?

The SL in stochastic thermodynamics

In small systems…

Different trajectories for each trial (fluctuation)

→ Stochastic thermodynamic quantities



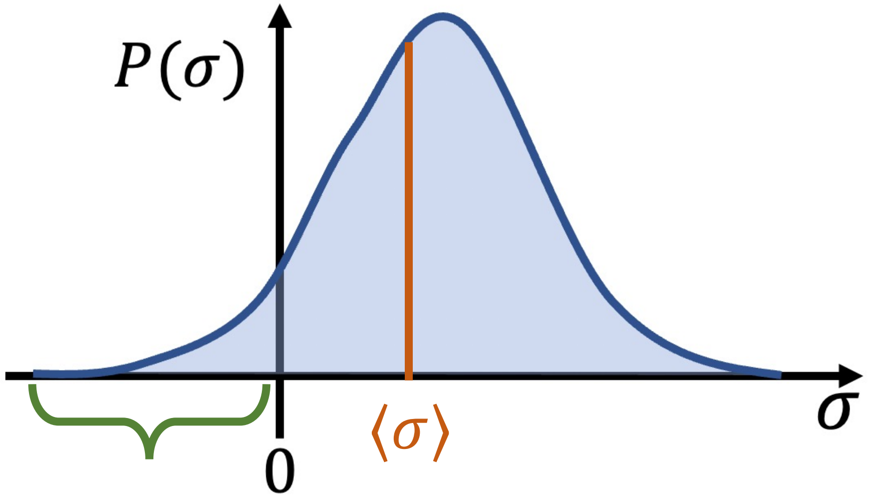
Ensemble level

$\langle\sigma\rangle \geq 0$ ○ the SL

Trajectory level

$\sigma[\psi] < 0$ for ⏟ × the SL

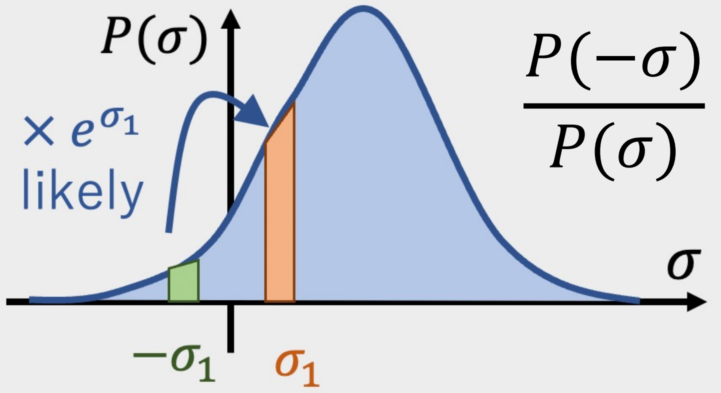
Probability distribution



Characterization at the trajectory level? → The fluctuation theorem

The fluctuation theorem (FT)

Crooks' FT



$$\frac{P(-\sigma)}{P(\sigma)} = e^{-\sigma}$$

integrate

$$\langle e^{-\sigma} \rangle = \int d\sigma\, P(\sigma) e^{-\sigma} = \int d\sigma\, P(-\sigma) = 1$$

Integral FT

$$\langle e^{-\sigma} \rangle = 1$$

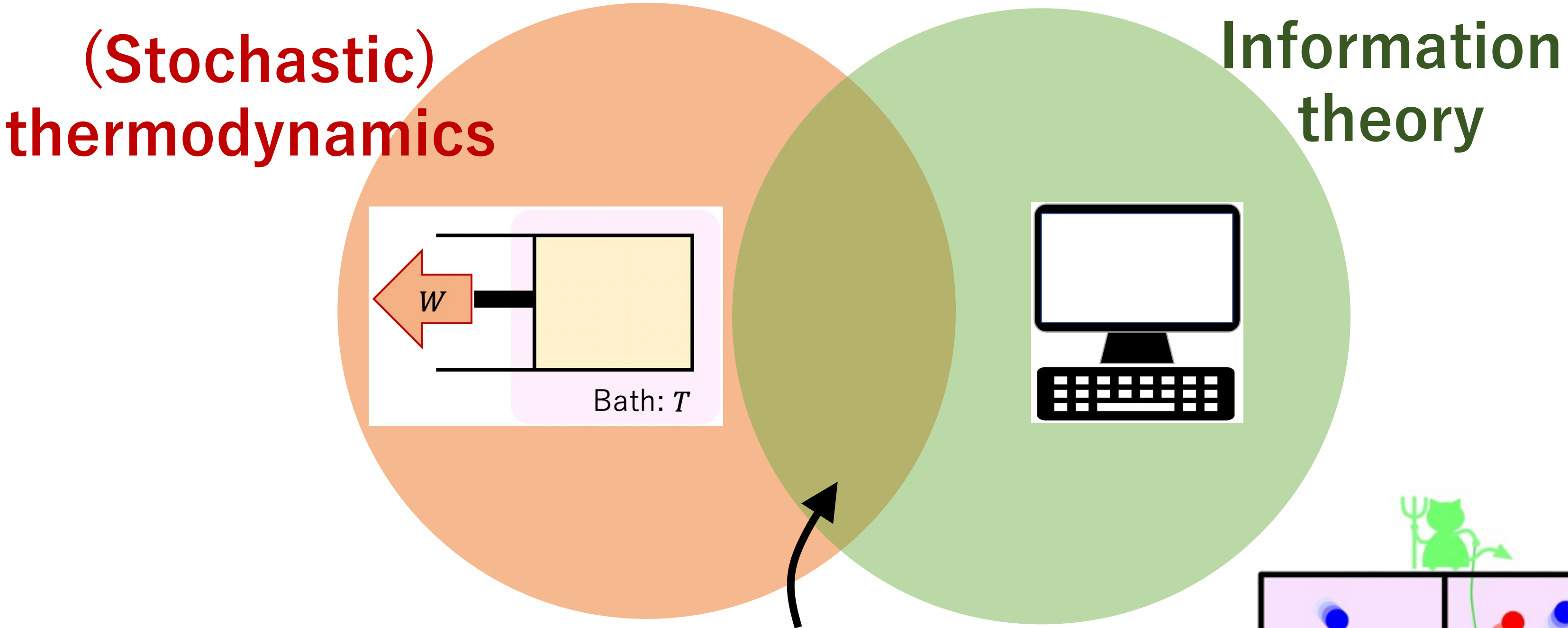
- Implies the SL
  $\because e^{-\langle \sigma \rangle} \leq \langle e^{-\sigma} \rangle = 1 = e^0 \rightarrow \langle \sigma \rangle \geq 0$
- **Generalize this form of FT in our research**

- Universal relation at the trajectory level
- Rich implication
  - The SL
  - Linear response theory
  - Include nonlinear nonequilibrium region

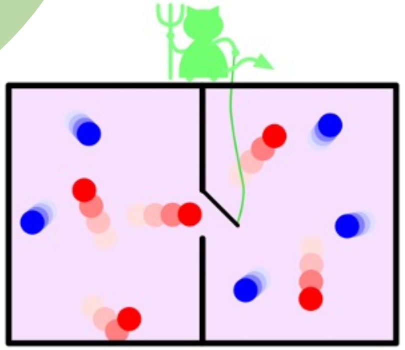
- Gives unified understanding
- Plays the central role

in stochastic thermodynamics

Information thermodynamics



Information thermodynamics
(our research)



Long history
(since 1867)
Maxwell's demon

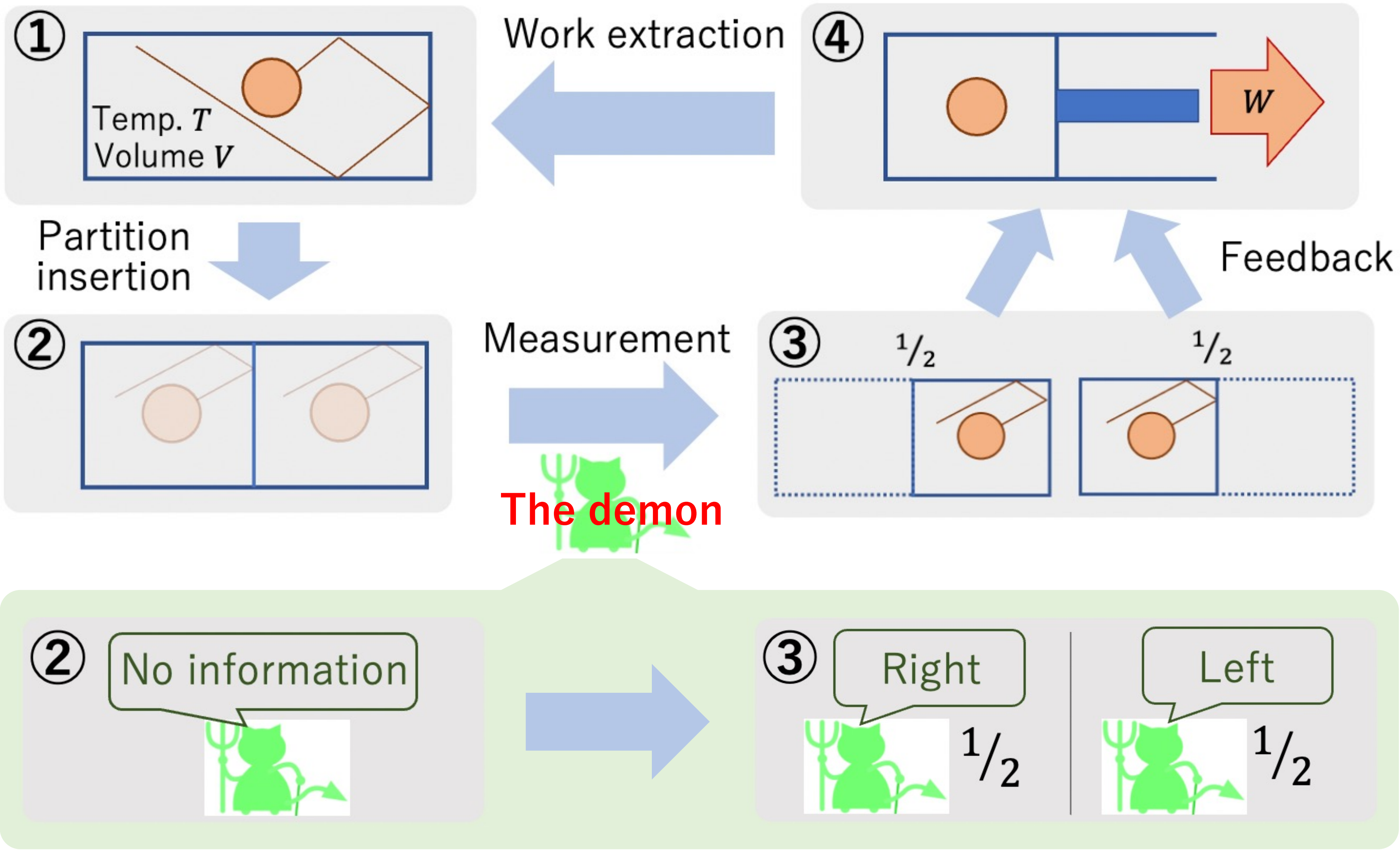
Great progress
in last 15 years

Ordinary thermodynamic laws are violated under information processing

(measurement & feedback)

Szilard engine

Work extraction from a single heat bath
→**Violate the conventional SL**



$$W = \int_{\frac{V}{2}}^{V} dV' \frac{k_B T}{V'} = k_B T \ln 2$$

Feedback

perform different protocols according to measurement outcomes

We need to consider the effect of the measurement.

The generalized thermodynamic laws

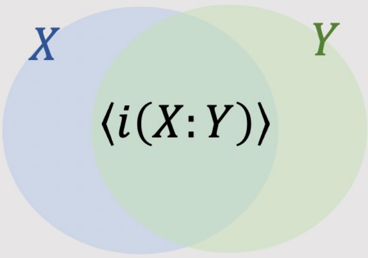
Information obtained by the measurement

<u>Mutual information</u>

$$\langle i(X{:}Y)\rangle \equiv H(X) + H(Y) - H(X,Y)$$

H: Shannon entropy

- Correlation between X and Y

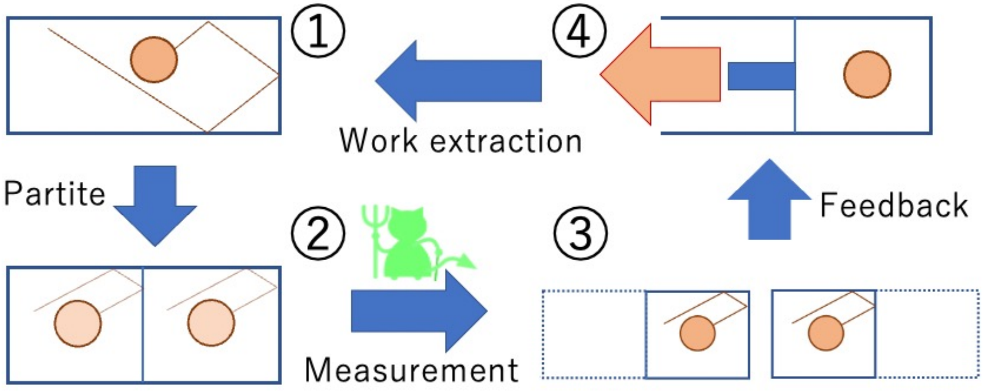


Y \ X	(left)	(right)
Left	$1/2$	0
Right	0	$1/2$

In Szilard engine⋯

$$\langle i(X{:}Y)\rangle = \mathbf{ln2}$$

X: molecule's position
Y: measurement outcome



Violation of thermodynamics

Work extraction: $W = \boldsymbol{k_B T \ln 2}$

Entropy change in ①→③: $\Delta S = \mathbf{ln2}$

The ordinary SL
$\langle W\rangle + \langle \Delta F\rangle \le 0$

The generalized SL
$\langle W\rangle + \langle \Delta F\rangle \le \boldsymbol{k_B T\, \langle i\rangle}$

i : Relevant information

The information gain quantifies the violation of thermodynamic law.

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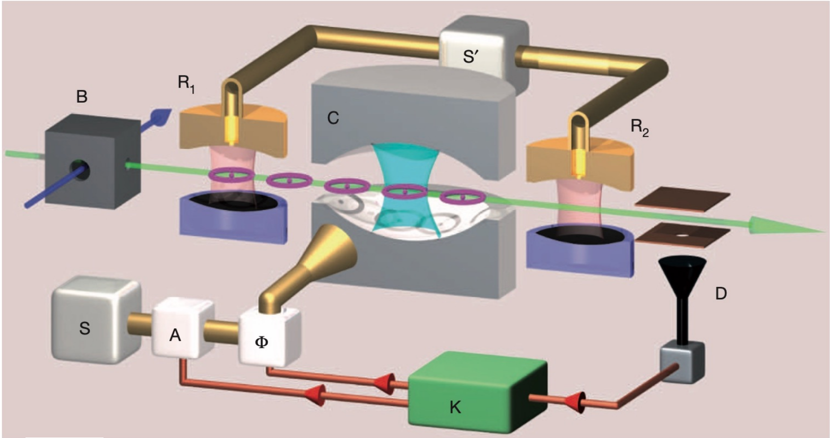
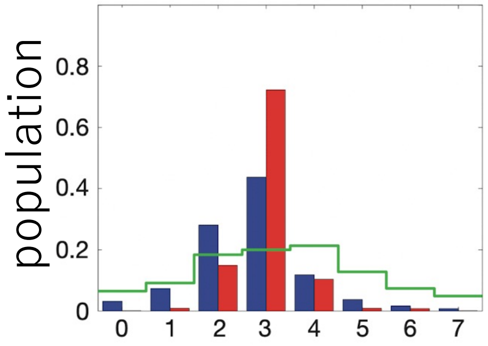
Quantum continuous feedback control

- Target of our research
- **Prospective quantum control method**

Cavity QED

C. Sayrin et. al.,
Nature **477**, 73-77, (2011)





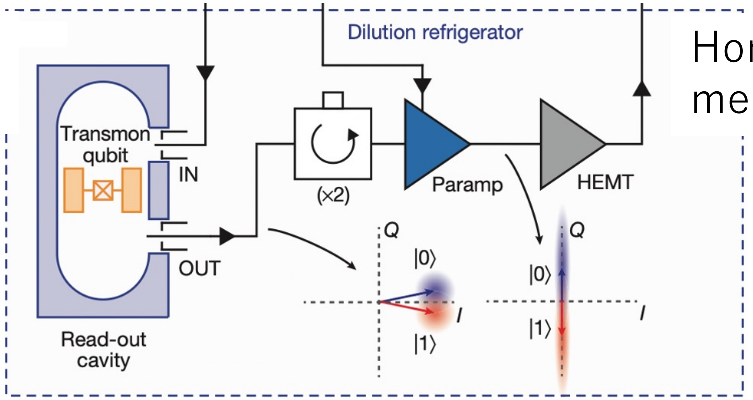
Prepare the photon number state $|n = 3\rangle$

- initial state
- State at fixed time
- Successfully feedback-controlled state

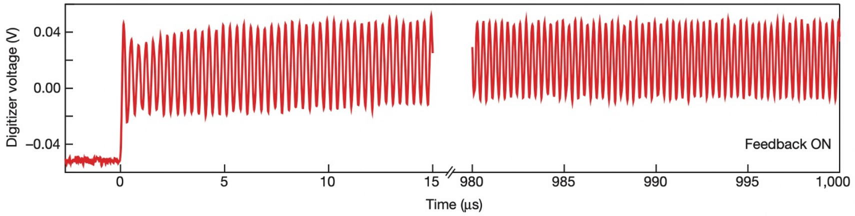
Circuit QED

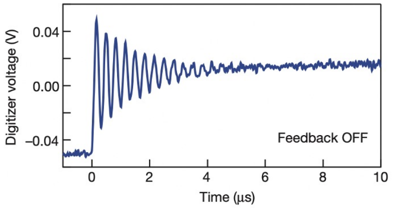
R. Vijay et. al.,
Nature **490**, 77-80 (2012)

Feedback signal



Homodyne measurement





Stabilize Rabi oscillation

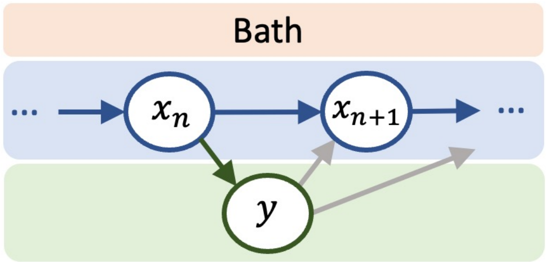
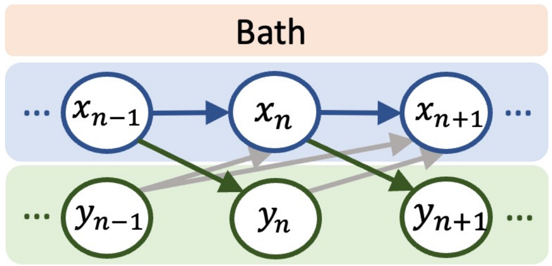
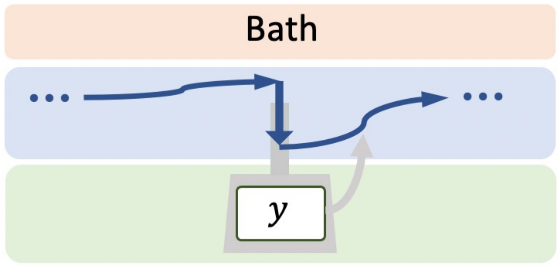
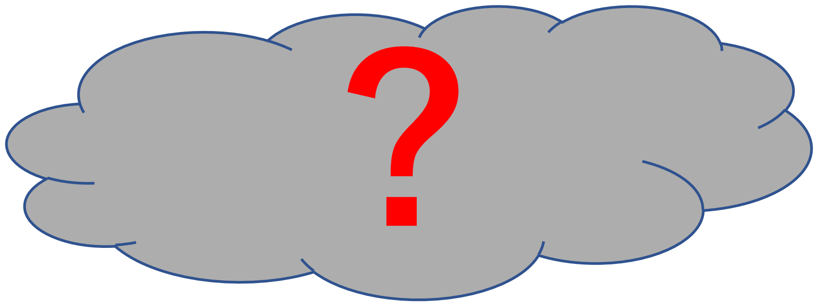
Red line: with feedback
Blue line: without feedback

The previous research and our work

The generalized SL: $\langle\sigma\rangle \geq -\langle\, i\, \rangle$

The generalized FT: $\left\langle e^{-\sigma-i}\right\rangle = 1$

i **: Information obtained by measurement**

	Single measurement and feedback	Continuous measurement and feedback
Classical system	 Sagawa&Ueda PRL 2010 Sagawa&Ueda PRL 2012 **i: mutual information**	 Sagawa&Ueda PRE 2012 Ito&Sagawa PRL 2013 **i: transfer entropy**
Quantum system	 Sagawa&Ueda PRL 2008 Funo et. al. PRE 2013 **i: QC-mutual information**	 **our research**

Summary of our main results (analytically derived)

The generalized SL

$$\langle \sigma \rangle \geq -\langle i_{\mathrm{QC}} \rangle$$

The generalized FT

$$\left\langle e^{-\sigma - i_{\mathrm{QC}}} \right\rangle = 1$$

Reveal the relationship between thermodynamics and information at ensemble level and trajectory level

Measurement & feedback / System	Single	Continuous
Classical	mutual information	transfer entropy
Quantum	QC-mutual information	**QC-transfer entropy**

$\langle i_{\mathrm{QC}} \rangle$: QC-transfer entropy

- **Newly introduced**
- **Total information obtained by continuous measurement**
- **Related with other information measures**

Transfer entropy

Classical systems under
continuous measurement and feedback

$$\langle \boldsymbol{\sigma} \rangle \geq -\langle \boldsymbol{i}_{\mathrm{TE}} \rangle, \ \langle \boldsymbol{e}^{-\boldsymbol{\sigma}-\boldsymbol{i}_{\mathrm{TE}}} \rangle = \mathbf{1}$$

System \ Measurement & feedback	Single	Continuous
Classical	mutual information	transfer entropy
Quantum	QC-mutual information	QC-transfer entropy

Transfer entropy

$$\langle i_{\mathrm{TE}} \rangle \equiv \sum_{n=1}^{N} I(x_n, y_{n+1} | Y_n)$$

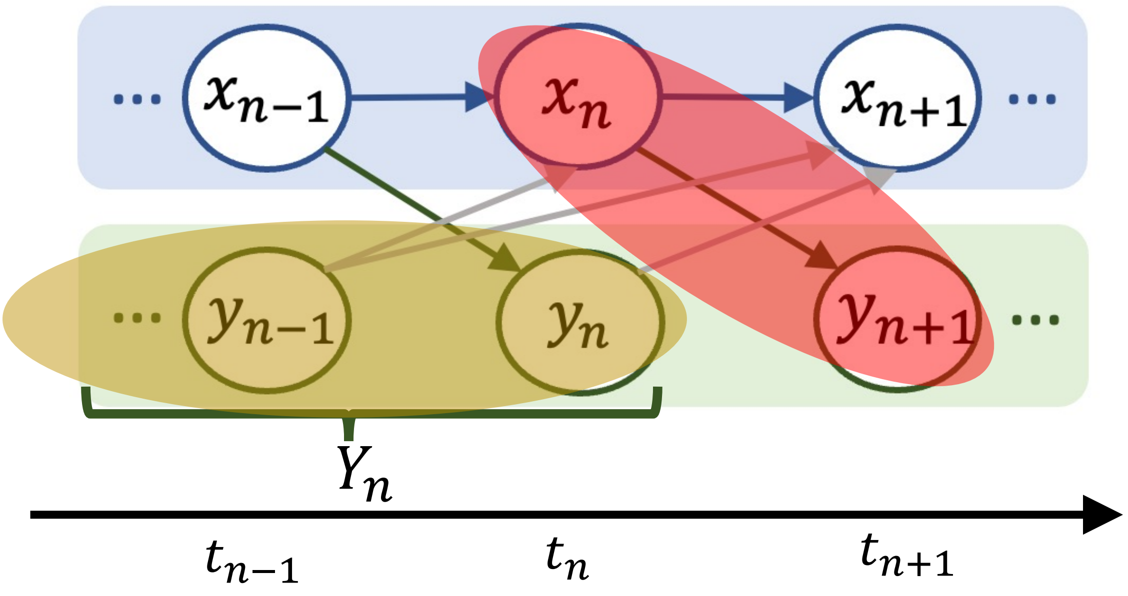
- Conditional mutual information

$$I(x_n, y_{n+1} | Y_n) \equiv H(x_n | Y_n) + H(y_{n+1} | Y_n) - H(x_n, y_{n+1} | Y_n)$$

- Conditional Shannon entropy

$$H(X|Y) \equiv \sum_{X,Y} -p(X,Y) \ln p(X|Y)$$

- **Conditioned on the past measurements**
→Newly obtained information in $[\boldsymbol{t}_{\boldsymbol{n}}, \boldsymbol{t}_{\boldsymbol{n}+1})$



QC-mutual information

Quantum systems under
single measurement and feedback

$$\langle \boldsymbol{\sigma} \rangle \geq -\langle \boldsymbol{I}_{\mathrm{QC}} \rangle, \ \langle e^{-\boldsymbol{\sigma}-\boldsymbol{I}_{\mathrm{QC}}} \rangle = \mathbf{1}$$

$\langle I_{\mathrm{QC}} \rangle \equiv \mathcal{I}_{\mathrm{QC}}$: QC-mutual information

System \ Measurement & feedback	Single	Continuous
Classical	mutual information	transfer entropy
Quantum	QC-mutual information	QC-transfer entropy

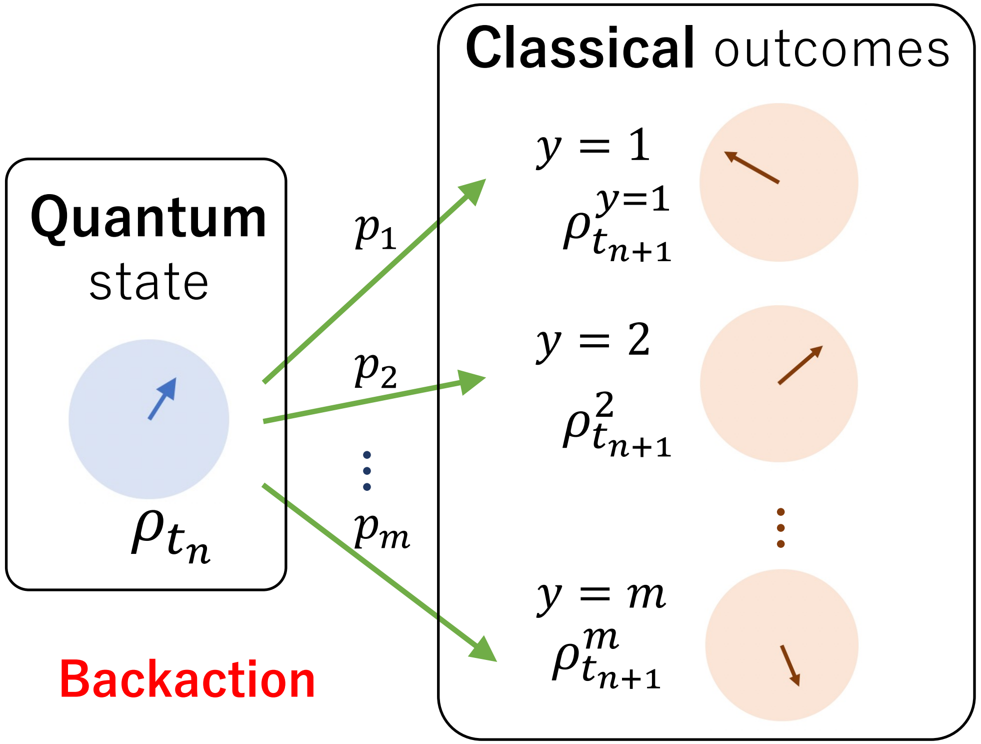
QC-mutual information

(Quantum-classical-mutual information)

$$\mathcal{I}_{\mathrm{QC}}(\rho_{t_n} : y) \equiv S(\rho_{t_n}) - \sum_{y=1}^{m} p_y S(\rho_{t_{n+1}}^{y})$$

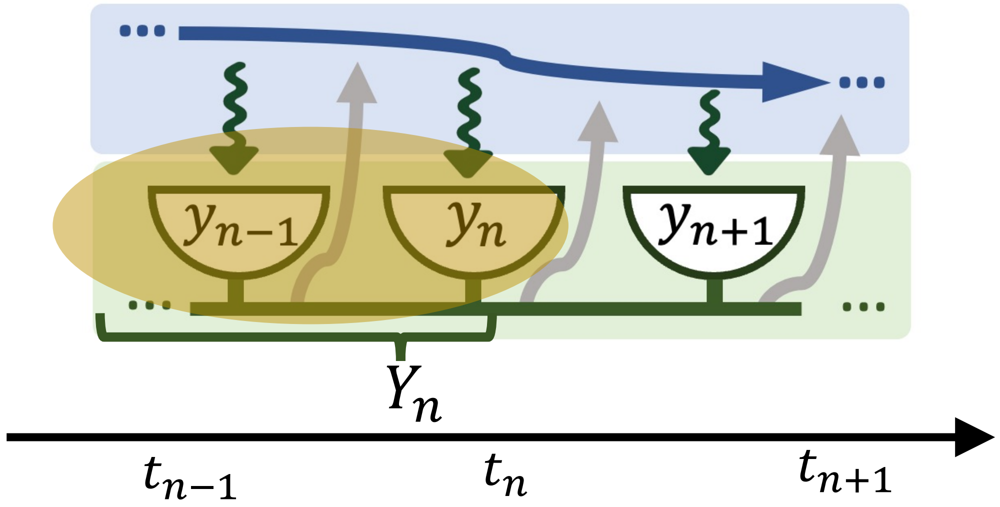
von Neumann entropy: $S(\rho) \equiv -\mathrm{tr}[\rho \ln \rho]$

- Quantum counterpart of Shannon entropy



QC-transfer entropy

$$\langle i_{\mathrm{QC}} \rangle \equiv \sum_{n=0}^{N-1} \sum_{Y_n} P[Y_n] \underline{\mathcal{I}_{\mathrm{QC}}(\rho_{t_n}^{Y_n} : y_{n+1})}$$



- **QC-mutual information** quantifies the information obtained by the measurement in $[t_n, t_{n+1})$
- Conditioned on the past measurement outcomes Y_n

cf. **transfer entropy** $\langle i_{\mathrm{TE}} \rangle \equiv \sum_{n=1}^{N} I(x_n, y_{n+1} | Y_n)$

System \ Measurement & feedback	Single	Continuous
Classical	mutual information	transfer entropy
Quantum	QC-mutual information	**QC-transfer entropy**

Conditional accumulation

Corresponding form

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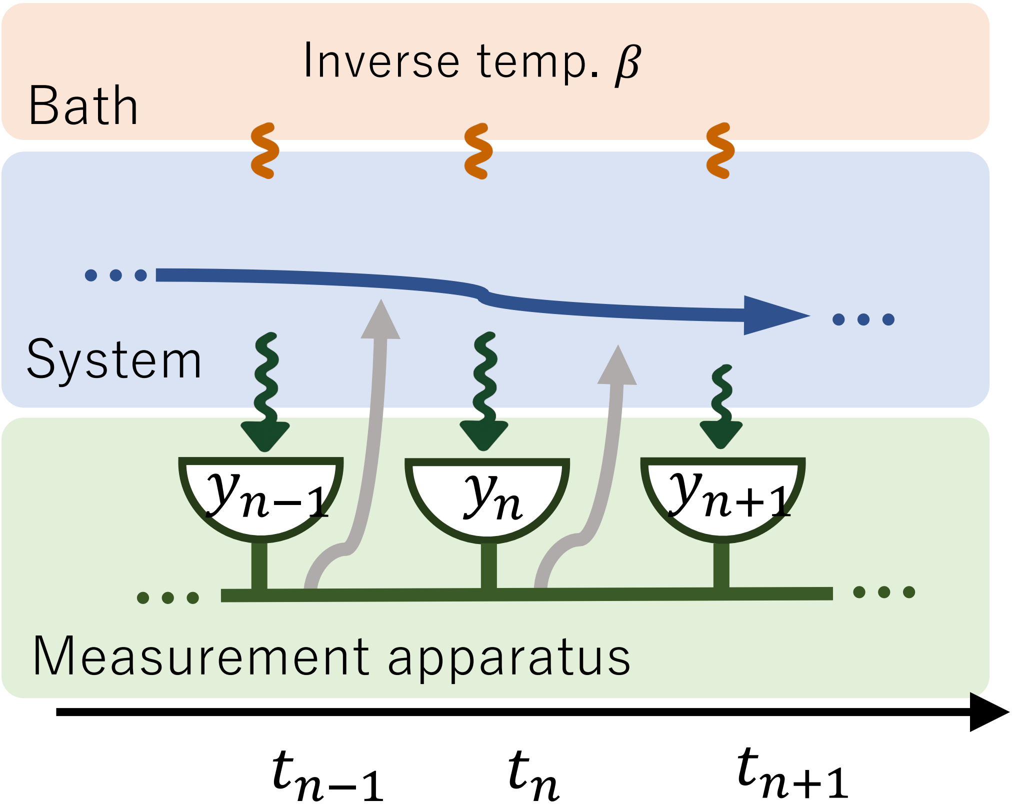
Setup for our research



Time discretization $t_n \equiv n \cdot \Delta t$

Take continuous time limit $\Delta t \to 0, \tau$ const.

<u>Stochastic master equation</u> $(\hbar = 1)$

$$\rho_{t_{n+1}}^{Y_{n+1}} = \rho_{t_n}^{Y_n} + \Delta t \Big\{ -i[H_{t_n} + h_{t_n}, \rho_{t_n}^{Y_n}] + \sum_d \mathcal{D}[L_d]\rho_{t_n}^{Y_n}$$

$$+ \sum_y -\frac{1}{2}\{M_y^\dagger M_y, \rho_{t_n}^{Y_n}\} + \mathrm{Tr}[M_y \rho_{t_n}^{Y_n} M_y^\dagger]\rho_{t_n}^{Y_n} \Big\} + \sum_y \Delta N_y \mathcal{G}[M_y]\rho_{t_n}^{Y_n}$$

$$\left(\mathcal{D}[c]\rho \equiv c\rho c^\dagger - \frac{1}{2}\{c^\dagger c, \rho\},\ \mathcal{G}[c]\rho \equiv \frac{c\rho c^\dagger}{\mathrm{Tr}[c\rho c^\dagger]} - \rho \right)$$

Measurement outcome

y_n: Measurement result at t_n

Y_n: Result until t_n (i.e., $(y_1, y_2, \ldots, y_n)$)

Continuous feedback

Change Hamiltonian $H_{t_n} + h_{t_n}$ according to Y_n

H_{t_n}: System Hamiltonian h_{t_n}: External driving

Details of the derivation of the main result

The goal $\langle e^{-\sigma - i_{\mathrm{QC}}} \rangle = 1$

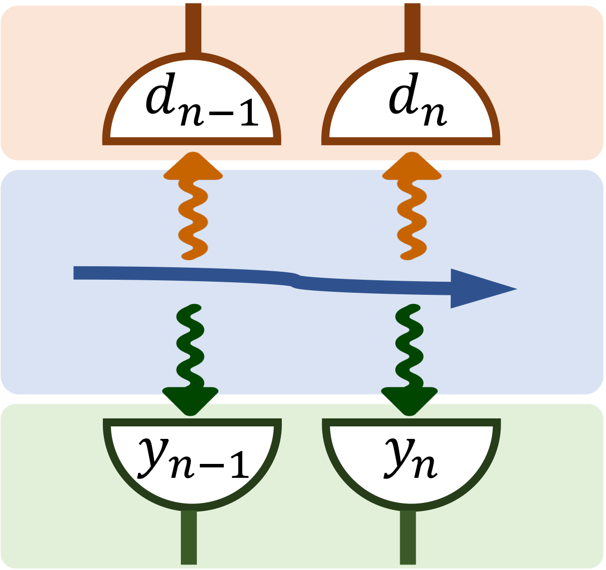
- Entropy production σ
- Information gain i_{QC}

We need to…

- **Properly unravel into trajectories**
- Define stochastic quantities σ, i_{QC}

The standard unraveling

Monitor heat-bath dissipation jump d_n



Trajectory ψ_τ is designated by

- Measurement jump
- Dissipation jump

Stochastic quantities

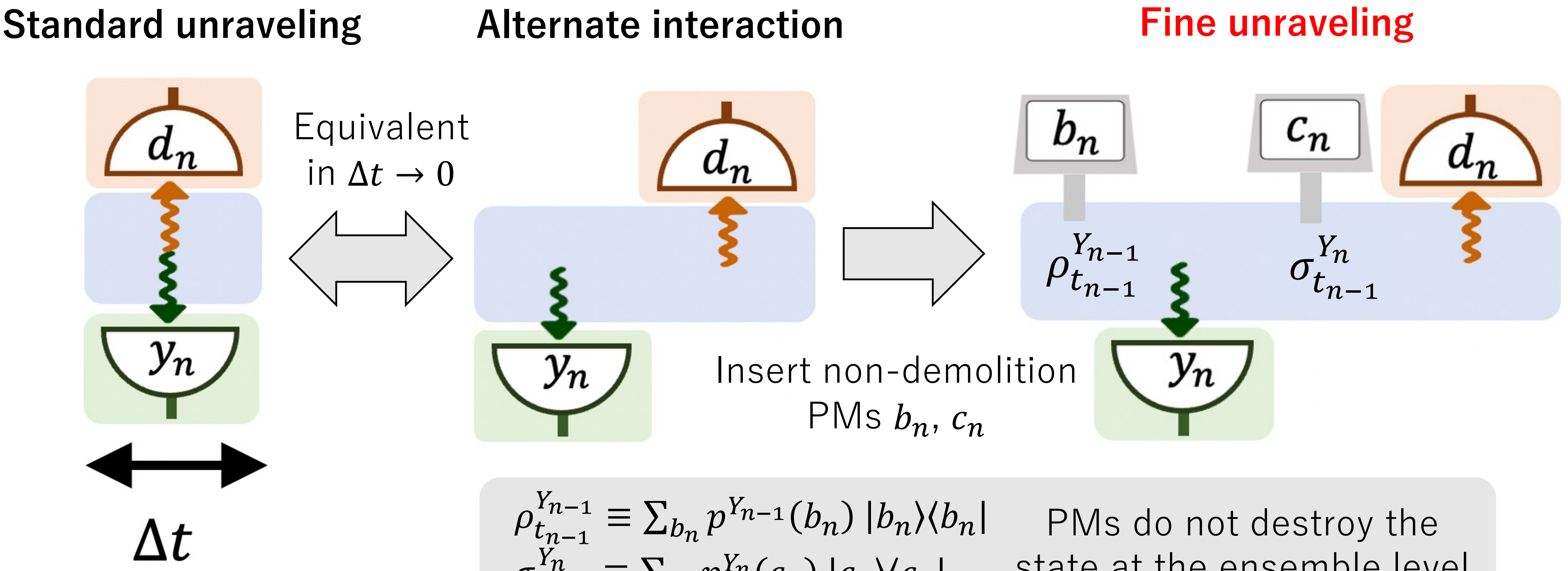
⭕ Entropy production $\sigma[\psi_\tau] \equiv \Delta S[\psi_\tau] - \beta Q[\psi_\tau]$

Heat $Q[\psi_\tau]$ is determined from d_n

❌ Information gain (QC-transfer entropy) $i_{\mathrm{QC}}[\psi_\tau]$

Cannot derive the generalized FT under standard unraveling

Details of the derivation of the main result



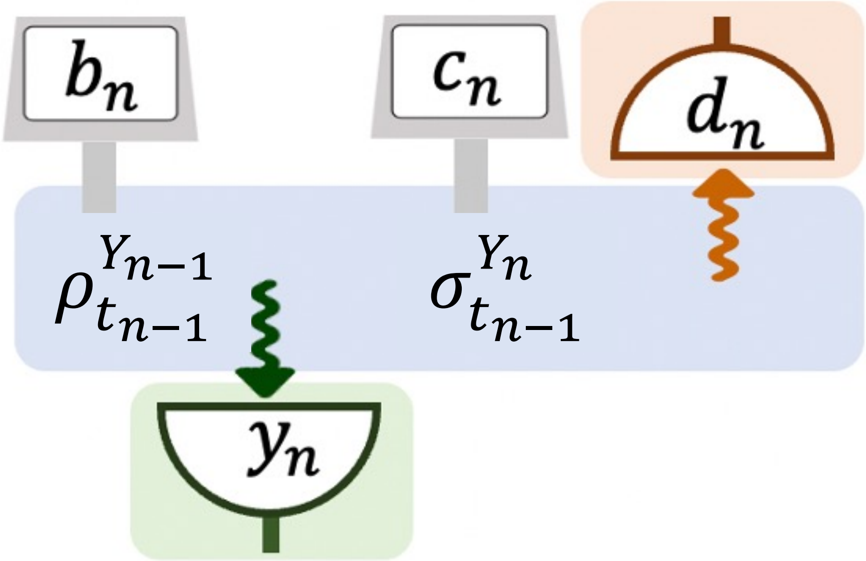
$$\rho_{t_{n-1}}^{Y_{n-1}} \equiv \sum_{b_n} p^{Y_{n-1}}(b_n)\, |b_n\rangle\langle b_n|$$
$$\sigma_{t_{n-1}}^{Y_n} \equiv \sum_{c_n} p^{Y_n}(c_n)\, |c_n\rangle\langle c_n|$$

PMs do not destroy the state at the ensemble level

The fine unraveling reproduces original dynamics, by taking the average over $\{b_n\}_{n=1}^N$, $\{c_n\}_{n=1}^N$ and $\{d_n\}_{n=1}^N$.

Detail of the derivation of the main result

The fine unraveling



The generalized FT

$$\left\langle e^{-\sigma-i_{\mathrm{QC}}}\right\rangle = 1$$

can be derived.

Monitor heat-bath dissipation jump

Insert additional non-demolition PMs

Trajectory (ψ_τ, π_τ) is designated by

- Measurement jump
- dissipation jump
- Inserted PMs $\pi_\tau \equiv \{b_n, c_n\}_{n=1}^N$

Stochastic quantities

⭕ Entropy production $\sigma[\psi_\tau]$

⭕ Information gain (QC-transfer entropy) $i_{\mathrm{QC}}[\psi_\tau]$

Defined in use of b_n, c_n

$$i_{\mathrm{QC}}[\psi_\tau] \equiv \sum_{n=1}^N - \ln p^{Y_{n-1}}(b_n) + \ln p^{Y_n}(c_n)$$

Numerical demonstration

Setting

- Two level system at inverse temperature β
- Calculate the dynamics in $0 \leq t \leq \tau$

- Hamiltonian $H_t = \omega\sigma_z,\ h_t = \epsilon \cos \nu t\ \ \sigma_x$
- Heat-bath interaction $L_\pm = \sqrt{\gamma_\pm}\sigma_\pm$
- Continuous measurement

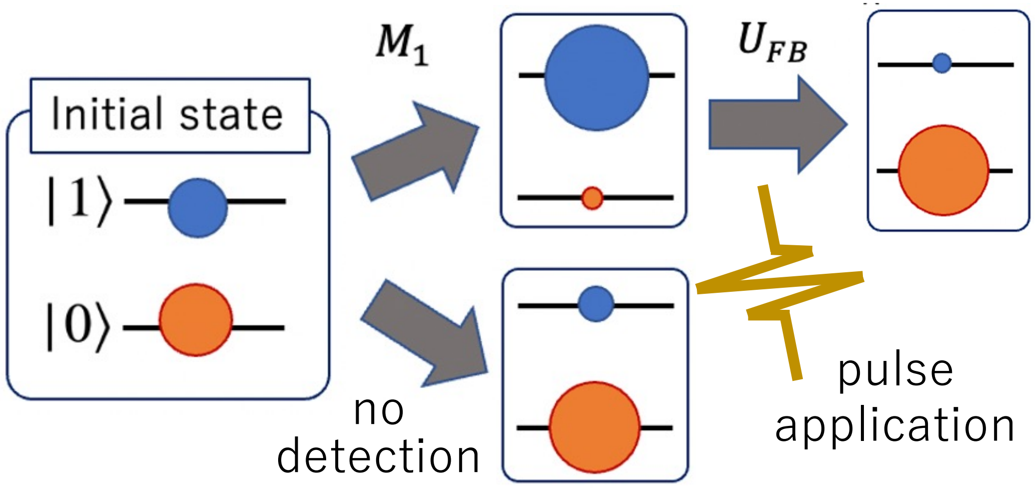
$$M_1 = \sqrt{\gamma_+ + \gamma_-}\begin{pmatrix} 1 & \delta \\ \delta & \delta \end{pmatrix}$$

- Feedback

Apply $U_{FB} = \sigma_x$ right after the detection of M_1
(Sudden pulse)

constants $\omega = 0.3, \epsilon = 0.04, \nu = 0.1\pi, \delta = 0.2,$
$\beta = 1, \gamma_\pm = 0.015\omega\left\{\coth\left(\frac{\beta\omega}{2}\right) \mp 1\right\}$

Feedback protocol



By the measurement and feedback…

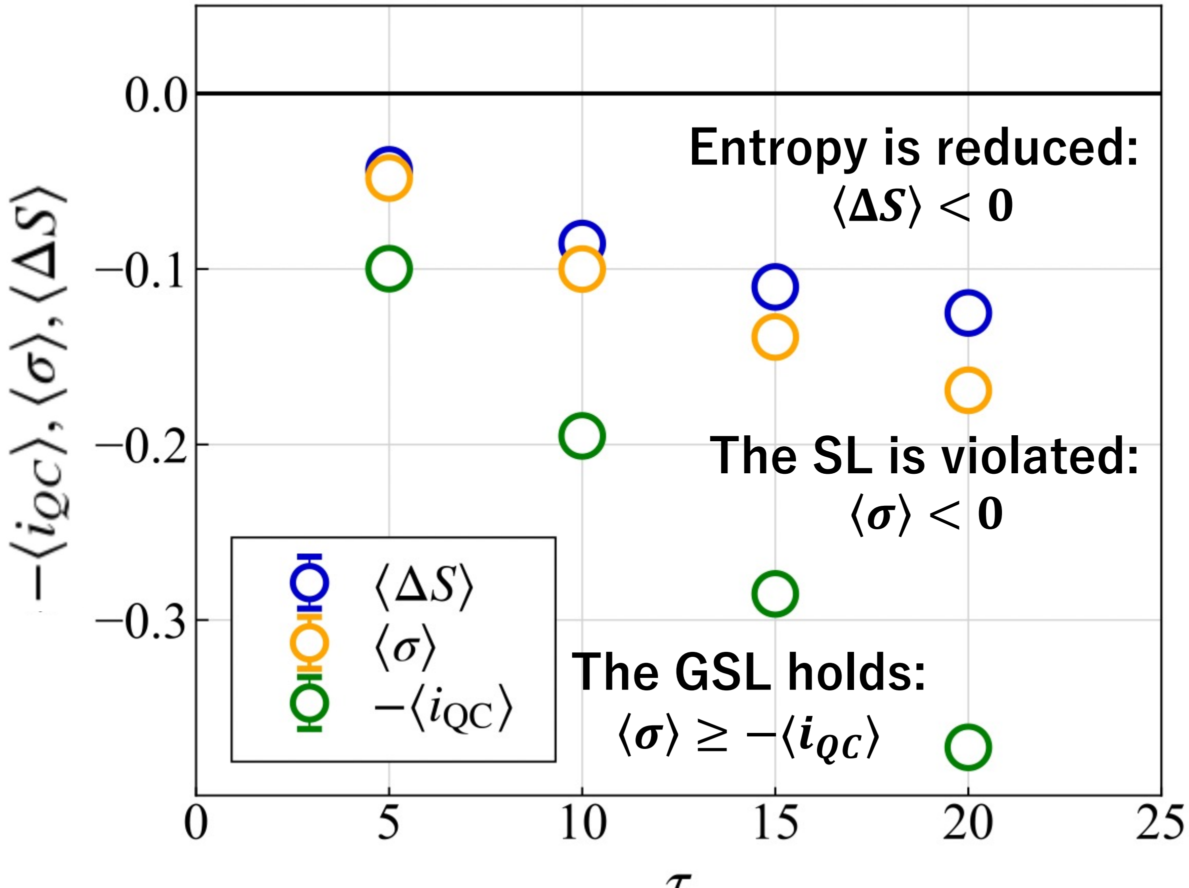
Reduce excited state population → Reduce system entropy

Randomly sampled 1.0×10^5 fine trajectories in this setting (Monte Carlo method)

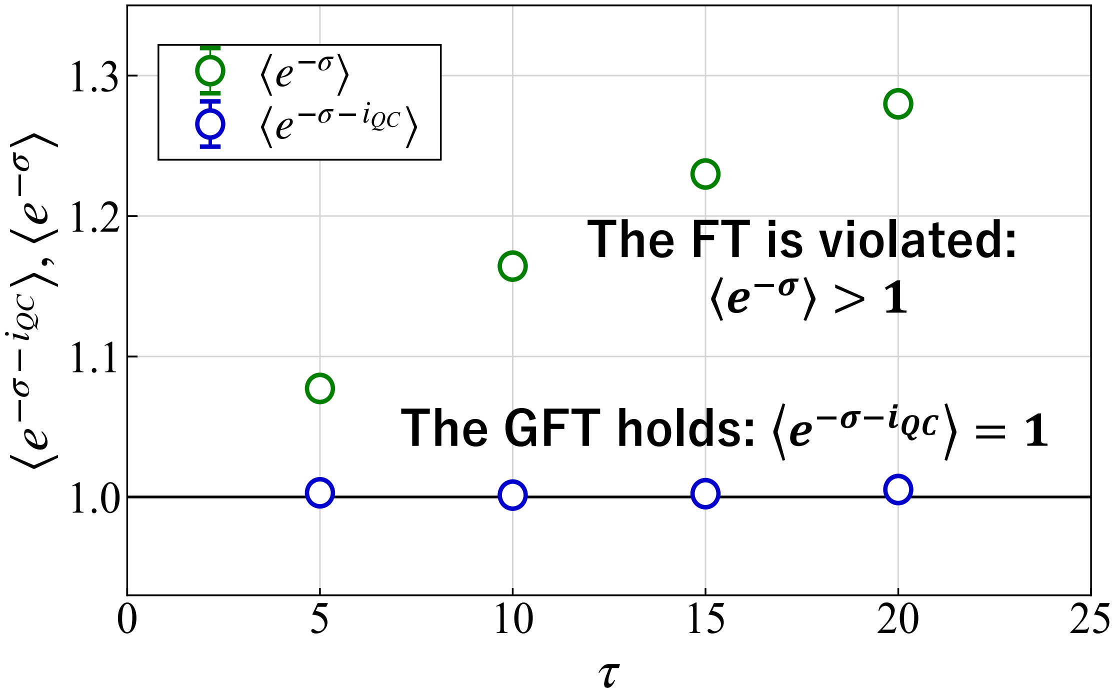
Numerical demonstration

Result

Verification of the generalized SL
(denoted as GSL)



Verification of the generalized FT
(denoted as GFT)



(Error bars are hidden by the plots.)

Summary

arXiv:2112.09351 TY, N. Yoshioka, T. Sagawa

In quantum systems under continuous measurement and feedback, we

- generalize the SL and FT
- introduce the QC-transfer entropy
- perform numerical demonstration

Fundamental relationship between thermodynamics and information

$$\langle \sigma \rangle \geq -\langle i_{\mathrm{QC}} \rangle \qquad \left\langle e^{-\sigma - i_{\mathrm{QC}}} \right\rangle = 1$$

Measurement & feedback / System	Single	Continuous
Classical	**mutual information**	**transfer entropy**
Quantum	**QC-mutual information**	**QC-transfer entropy**

Conditional accumulation