ANALYTIC BOOTSTRAP IN 2D BOUNDARY CFT

Caltech

Yuya Kusuki

Based on [arXiv:2112.10984]

Conformal Field Theory

Invariant under

- translations
- rotations
- dilatations
- special conformal transformations

Helpful:
Correlation function is almost fixed by symmetry

CFT appears in, for example,

- critical points in condensed matter theory.
- dual of quantum gravity

What can we do by CFT? One example is

- answer why critical exponents are rational.

2D Ising model

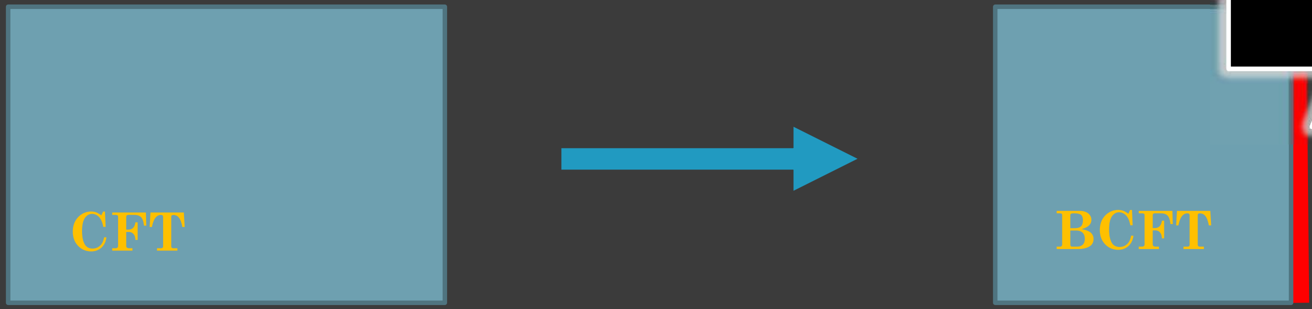
$\eta = \frac{1}{4}$

$\nu = 1$

Boundary CFT

$$T_{xt} = 0\Big|_{bdy}$$

→no momentum flow across bdy



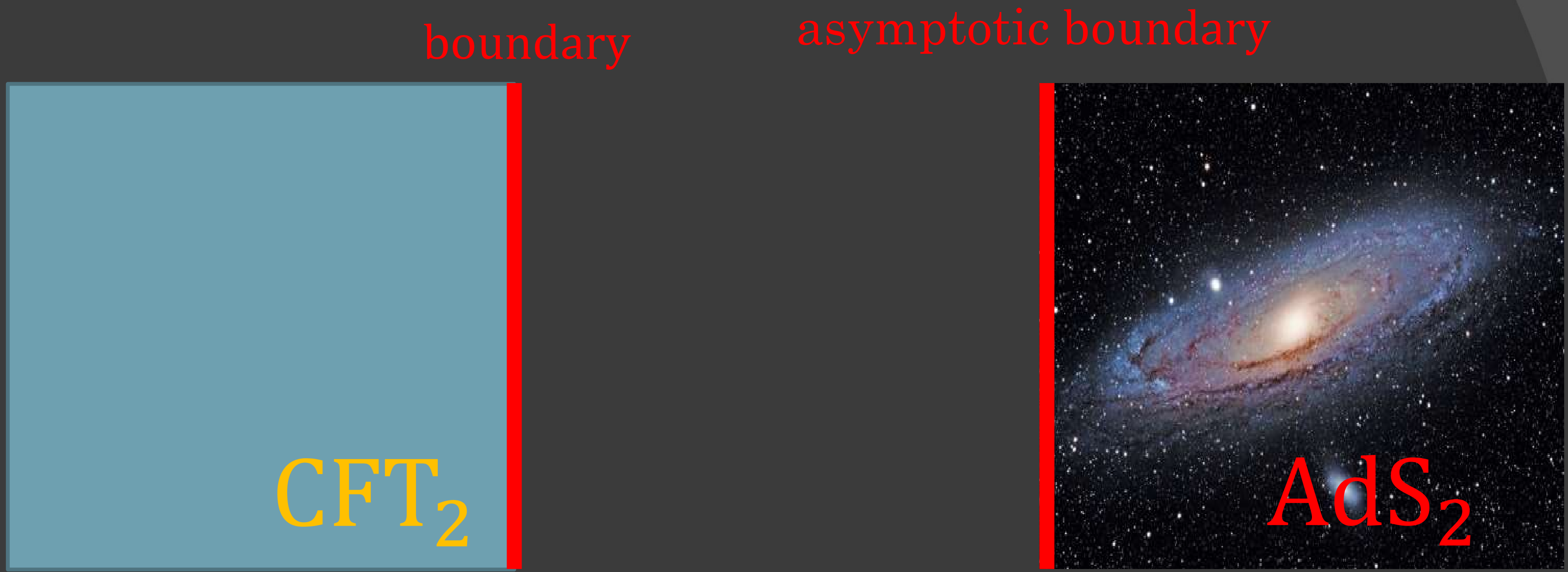
Boundary breaks conformal symmetry…

Conformal Symmetry of BCFT is part of conformal symmetry preserving the bdy. position.

$SO(2,d)$ → $SO(2,d-1)$

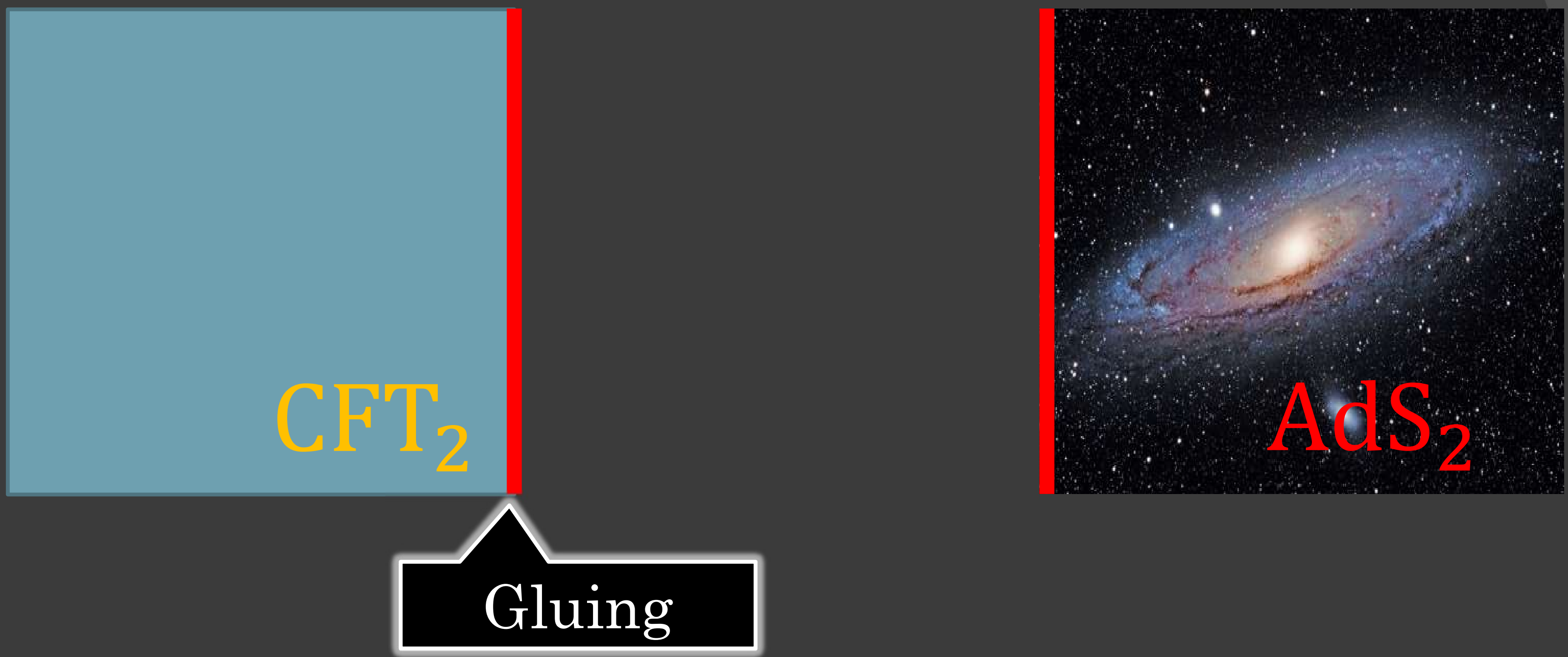
Why do BCFTs attract attention recently?

Big Goal

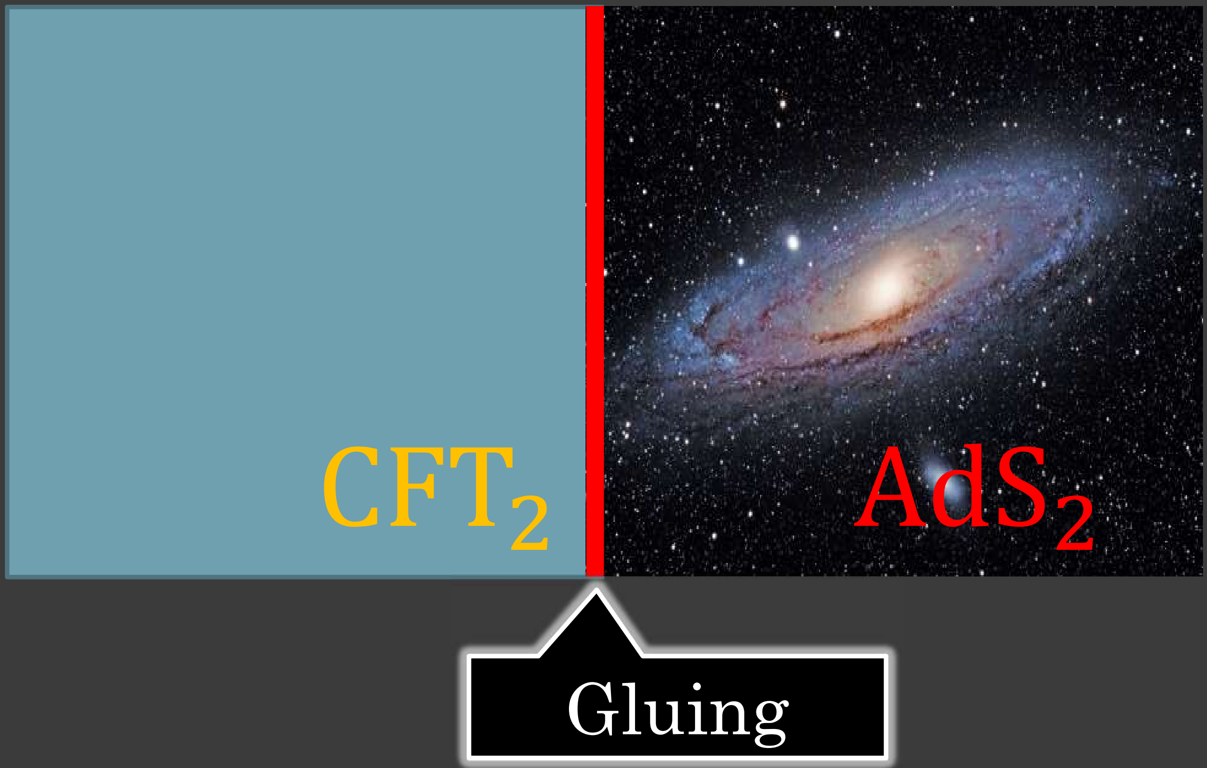


CFT coupled to gravity in AdS_2

Big Goal



Big Goal



Big Goal

Comment:
Entropy of radiation from BH can be detected by EE between CFT_2 & AdS_2

Why interesting?
EE can be defined on flat space, not on curved space

Big Goal

Point: More tractable to understand Quantum Gravity

Problem: NOT so explored, NO knowledge

Purpose: New method to explore

Small Goal

BCFT is now very interesting for the big goal.

Q. What is known in (irrational) BCFT?

A. Many unexplored parts!

On this background, we will provide one technique to explore BCFT.

Small Goal

BCFT is now very interesting for the big goal.

Q. What is known in (irrational) BCFT?

A. Many unexplored parts!

In particular, very limited information even in asymptotic regime

This motivates us to develop ***conformal bootstrap*** in BCFTs

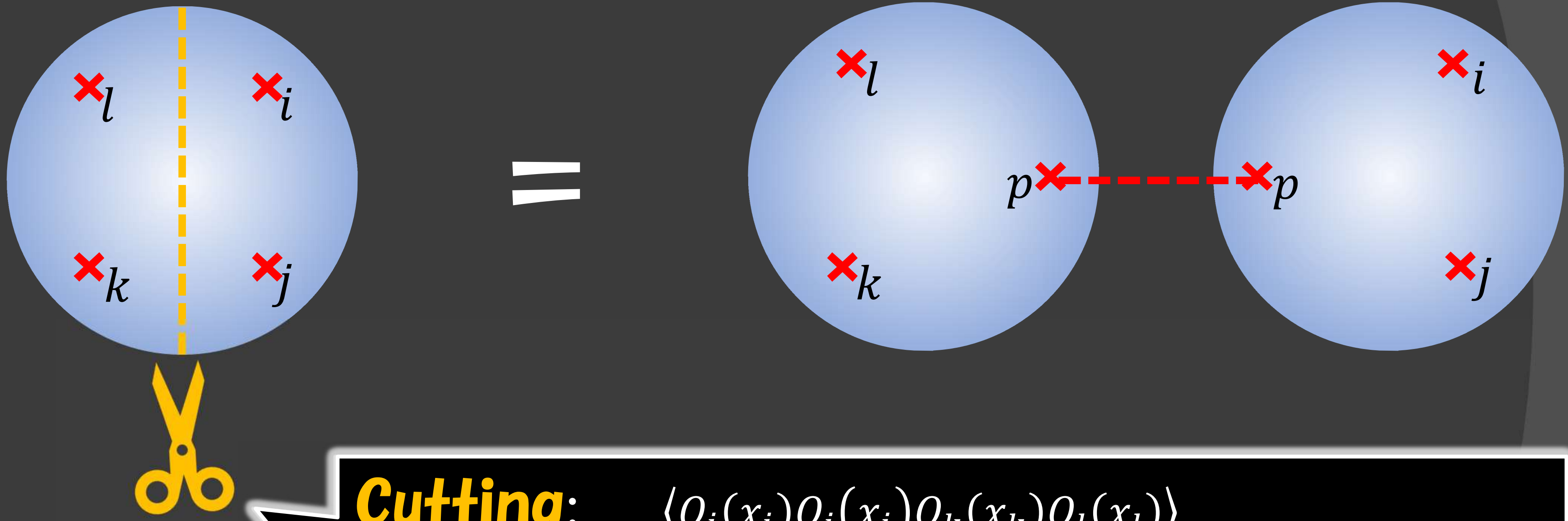
Useful to identify unknown information (DoF & OPE) in CFT

Review

How to evaluate c

or equivalently, using bulk-bulk-bulk OPE

$$\phi_i(0)\phi_j(x) \sim \sum_p C_{ijp} \left| x^{h_p - h_i - h_j} \right|^2 \phi_p + \cdots$$



Cutting:

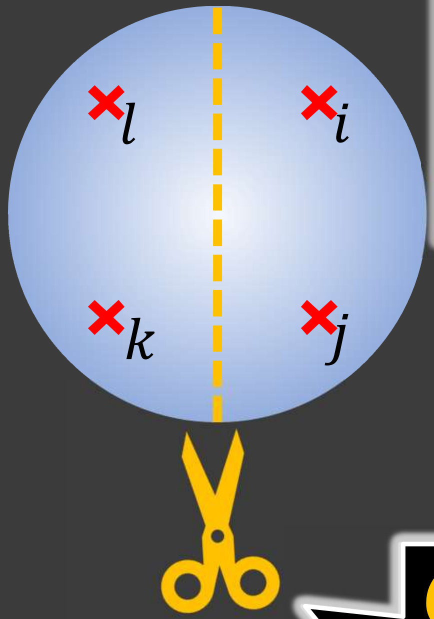
$$\langle O_i(x_i) O_j(x_j) O_k(x_k) O_l(x_l) \rangle$$

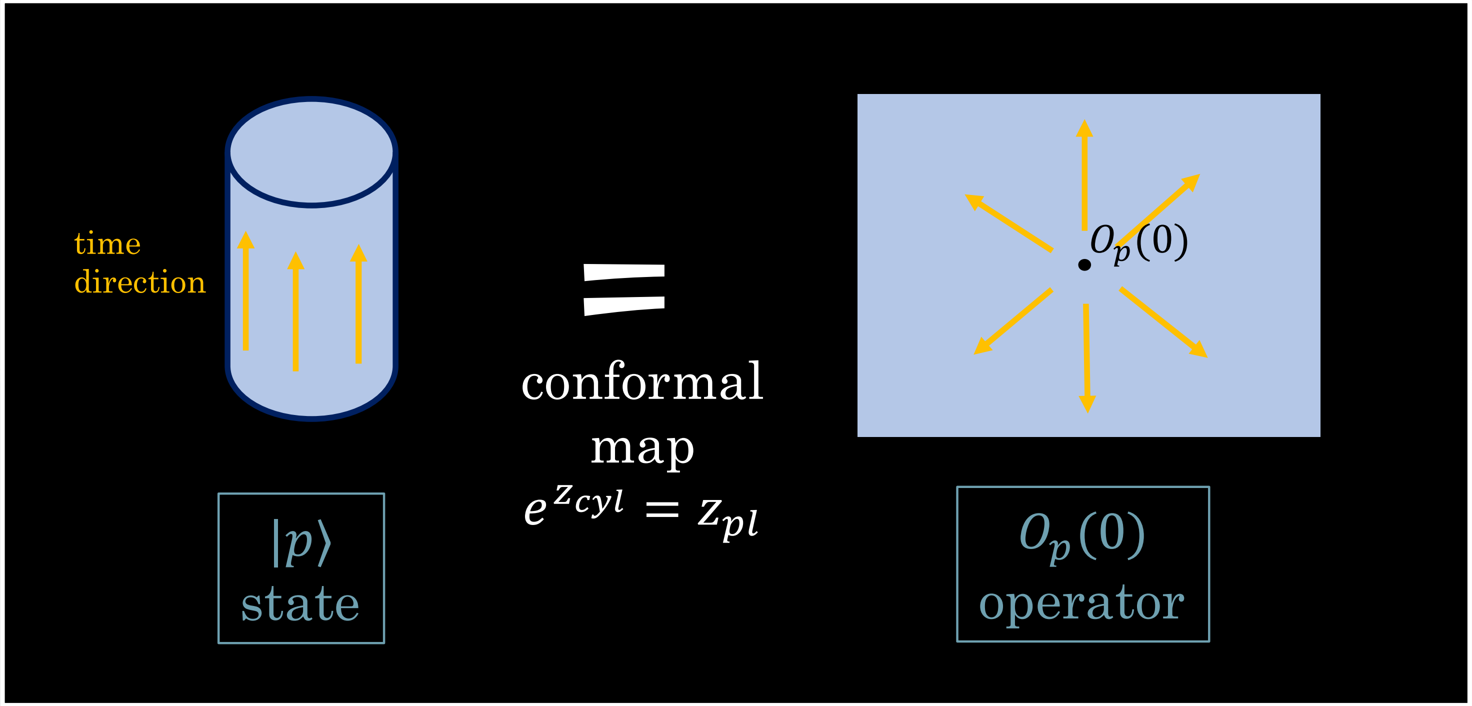
$$= \sum_p \langle O_i(x_i) O_j(x_j) | p \rangle \langle p | O_k(x_k) O_l(x_l) \rangle$$

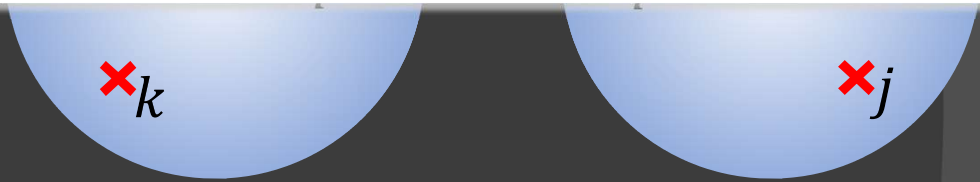
$$= \sum_p \langle O_i(x_i) O_j(x_j) O_p(0) \rangle \langle O_p(\infty) O_k(x_k) O_l(x_l) \rangle$$

Review

How to evaluate c







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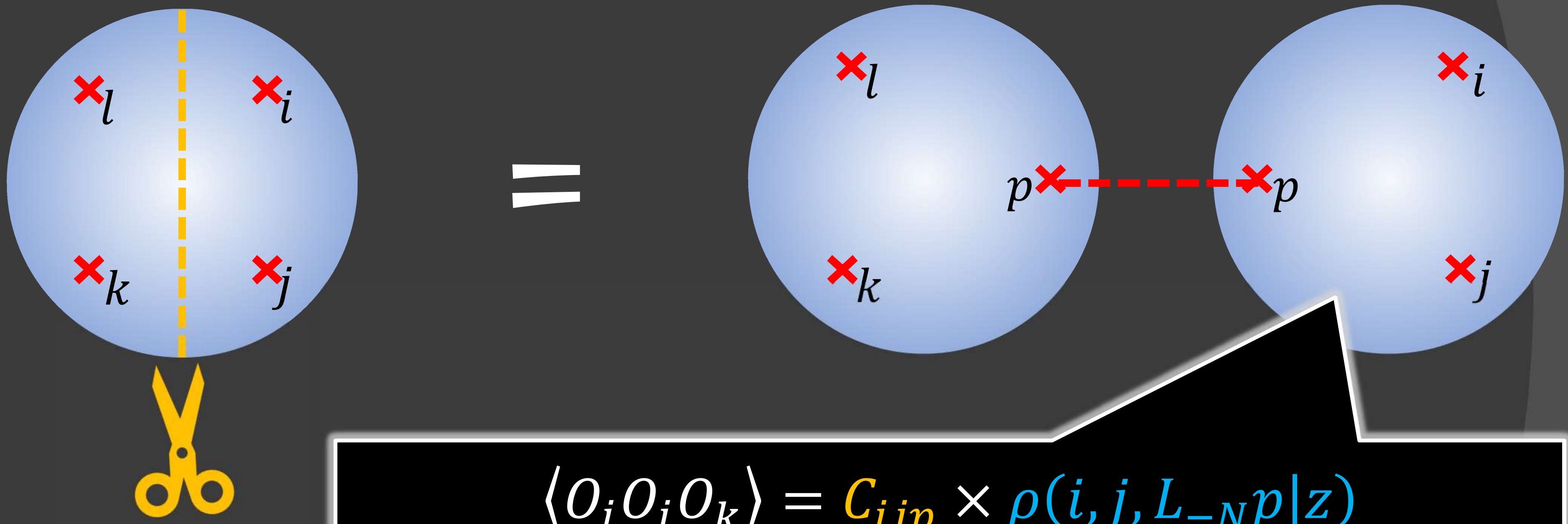
Review of CFT

How to evaluate correlation function in CFT



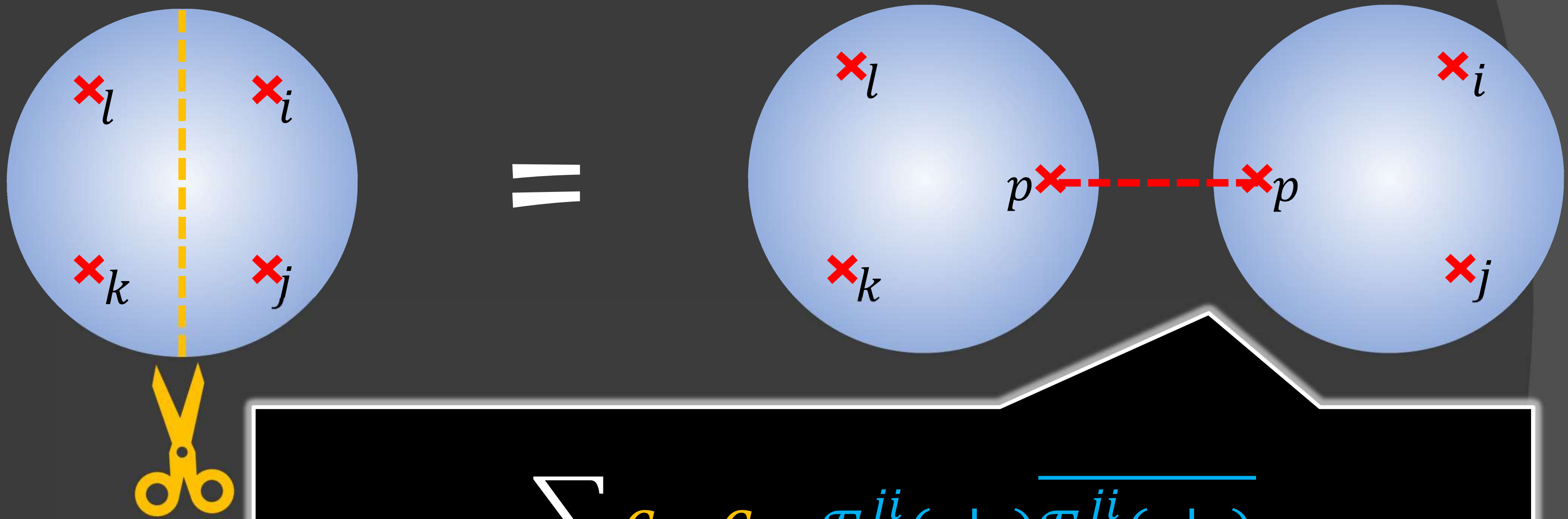
$$\langle O_i O_j O_k \rangle = C_{ijp} \times \rho(i, j, L_{-N}p|z)$$

theory-dependent theory-independent

ρ is completely fixed by conformal sym.
Does not depend on theory

Review of CFT

How to evaluate correlation function in CFT



$$\sum_p C_{ijp} C_{klp} \mathcal{F}^{ji}_{kl}(p|z) \overline{\mathcal{F}^{ji}_{kl}(p|z)}$$

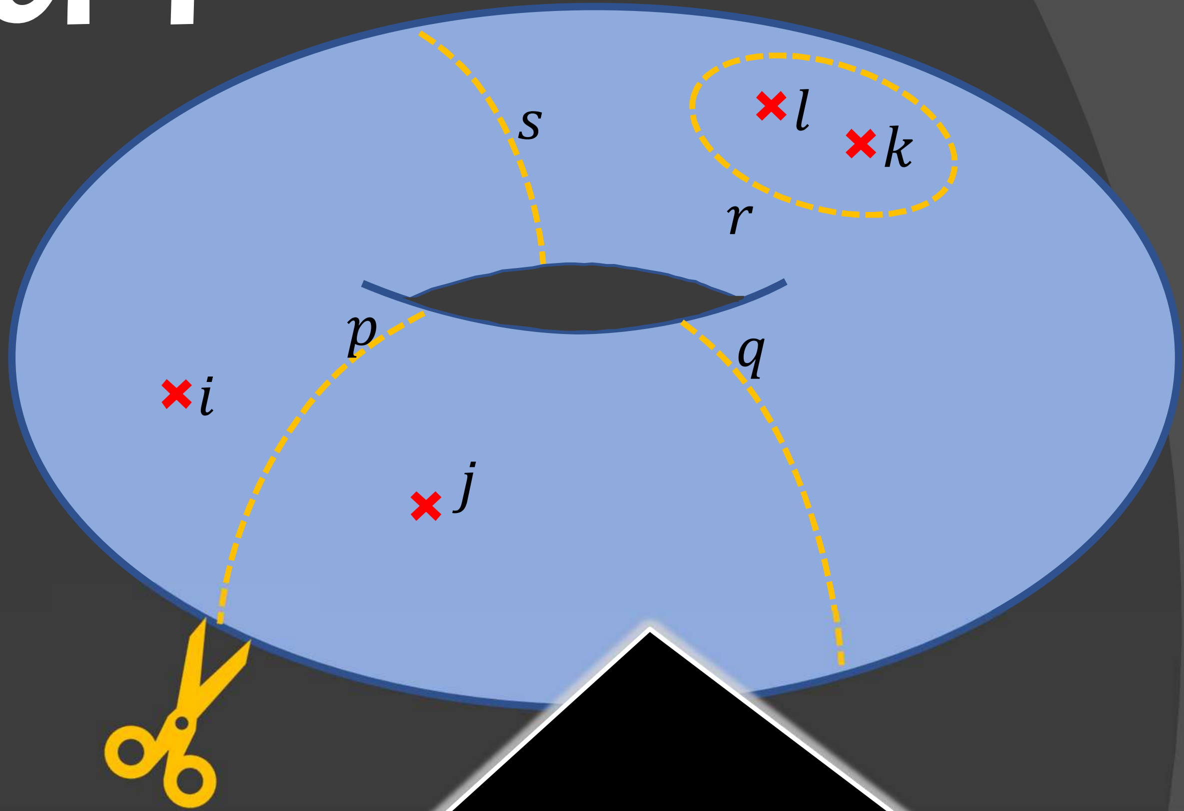
theory-dependent theory-independent

$\mathcal{F}^{ji}_{kl}$ is completely fixed by conformal sym.

since $\mathcal{F}^{ji}_{kl}(p|z) = \sum_N \rho(i, j, L_{-N}p) \times \rho(k, l, L_{-N}p)$

Review of CFT

More general

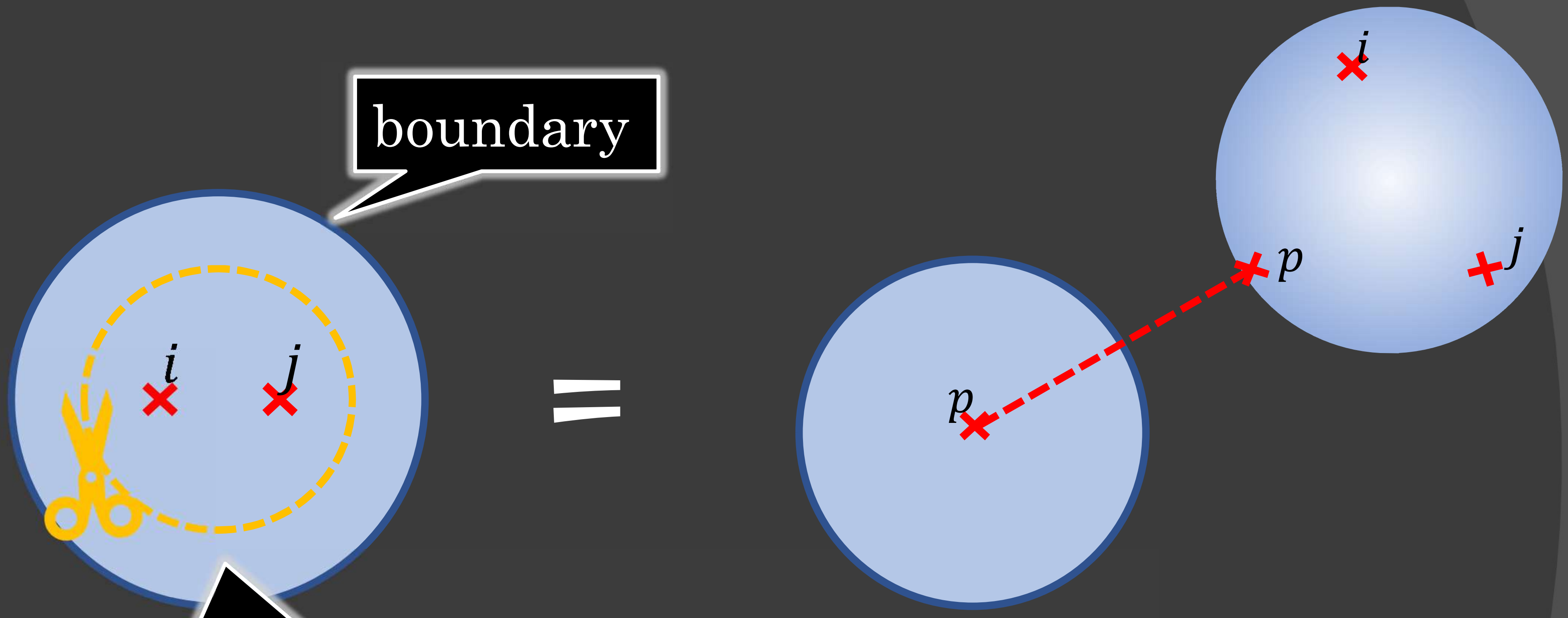


$$\sum_{p,q,r,s} C_{sip}C_{pjq}C_{qrs}C_{rlk}\mathcal{F}^{ji}_{kl}(p,q,r,s|\mathbb{z})\overline{\mathcal{F}^{ji}_{kl}(p,q,r,s|\mathbb{z})}$$

theory-dependent   theory-independent

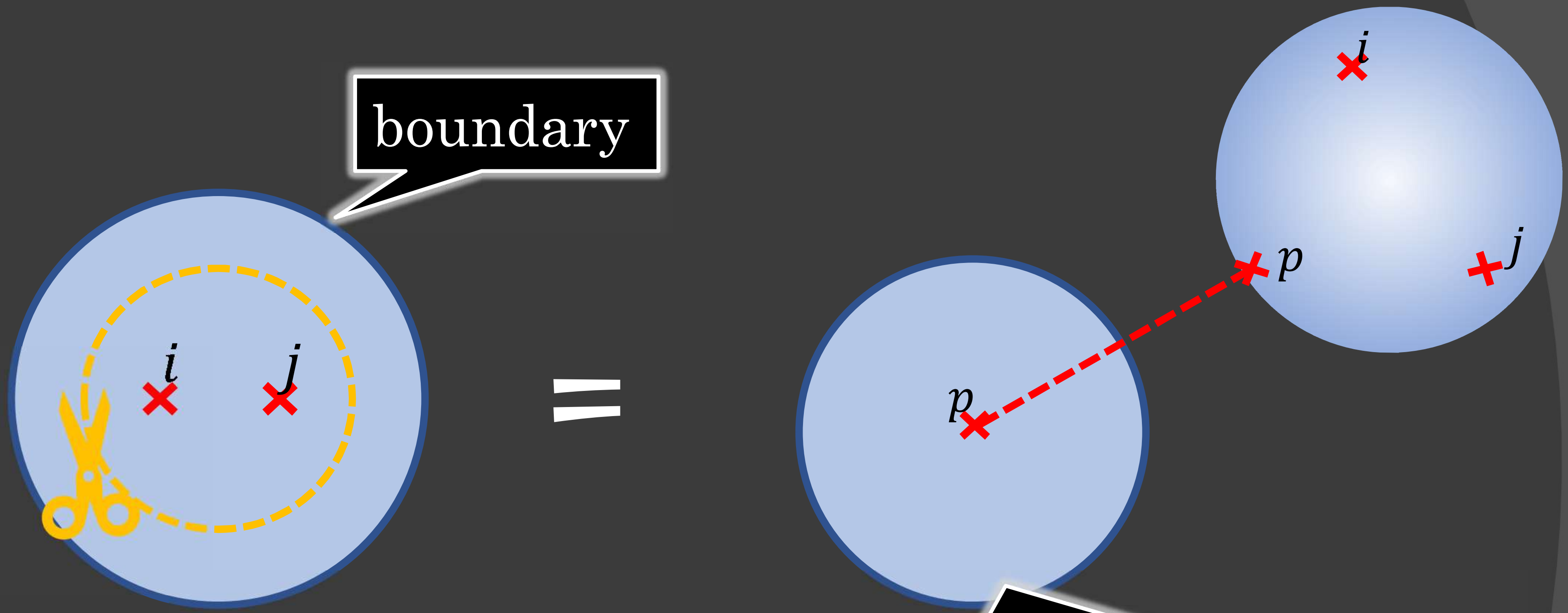
Any correlator can be evaluated by **cutting** with ρ & C_{ijk}

Review of BCFT [Lewellen]



Cutting:
Inserting (bulk operator) complete set

Review of BCFT [Lewellen]

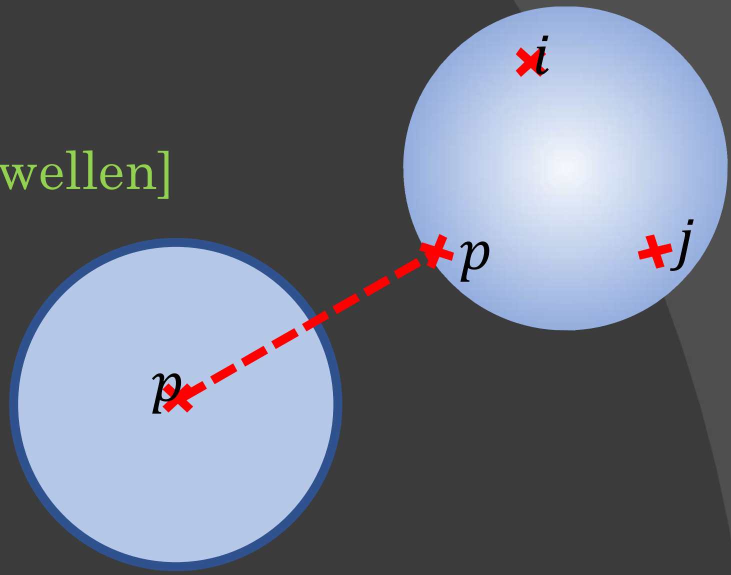


$$\sum_p C_{p0} C_{ijp} \mathcal{F}^{ji}_{\bar{j}\bar{i}}(p|z)$$

$\mathcal{F}^{ji}_{\bar{j}\bar{i}}$ is fixed by conformal sym. & mirror method

Review of BCFT [Lewellen]

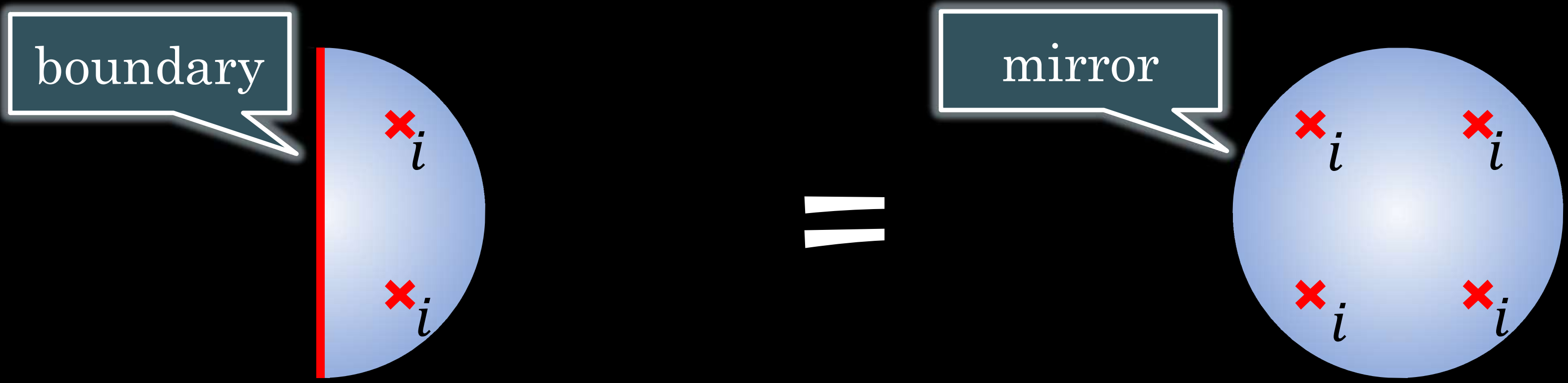
$$\sum_p C_{p0} C_{ijp} \mathcal{F}^{ji}_{\bar{j}\bar{i}}(p|z)$$



Note:

$\mathcal{F}^{ji}_{\bar{j}\bar{i}}(p|z)$ = Virasoro block.

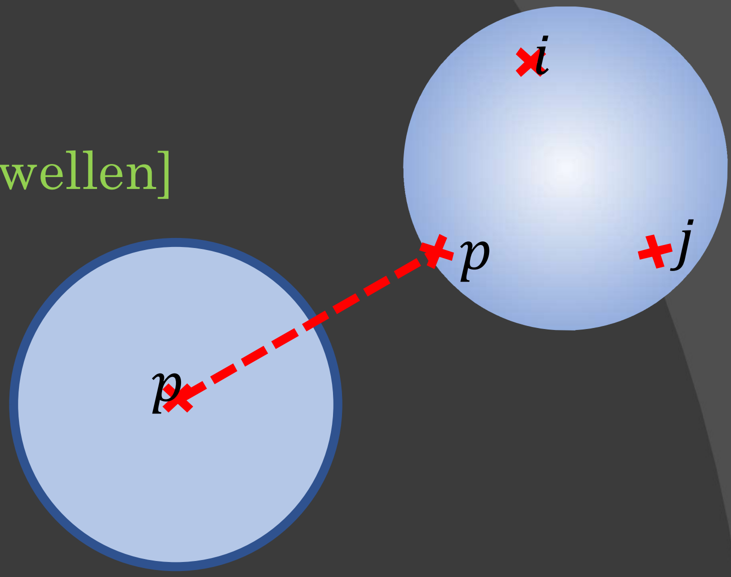
Because Ward id (with bdy) is equivalent to Ward id (without bdy) by mirror method



kinematic part = conformal block

Review of BCFT [Lewellen]

$$\sum_p C_{p0} C_{ijp} \mathcal{F}_{\bar{j}\bar{i}}^{ji}(p|z)$$



Note:

$\mathcal{F}_{\bar{j}\bar{i}}^{ji}(p|z)$ = Virasoro block.

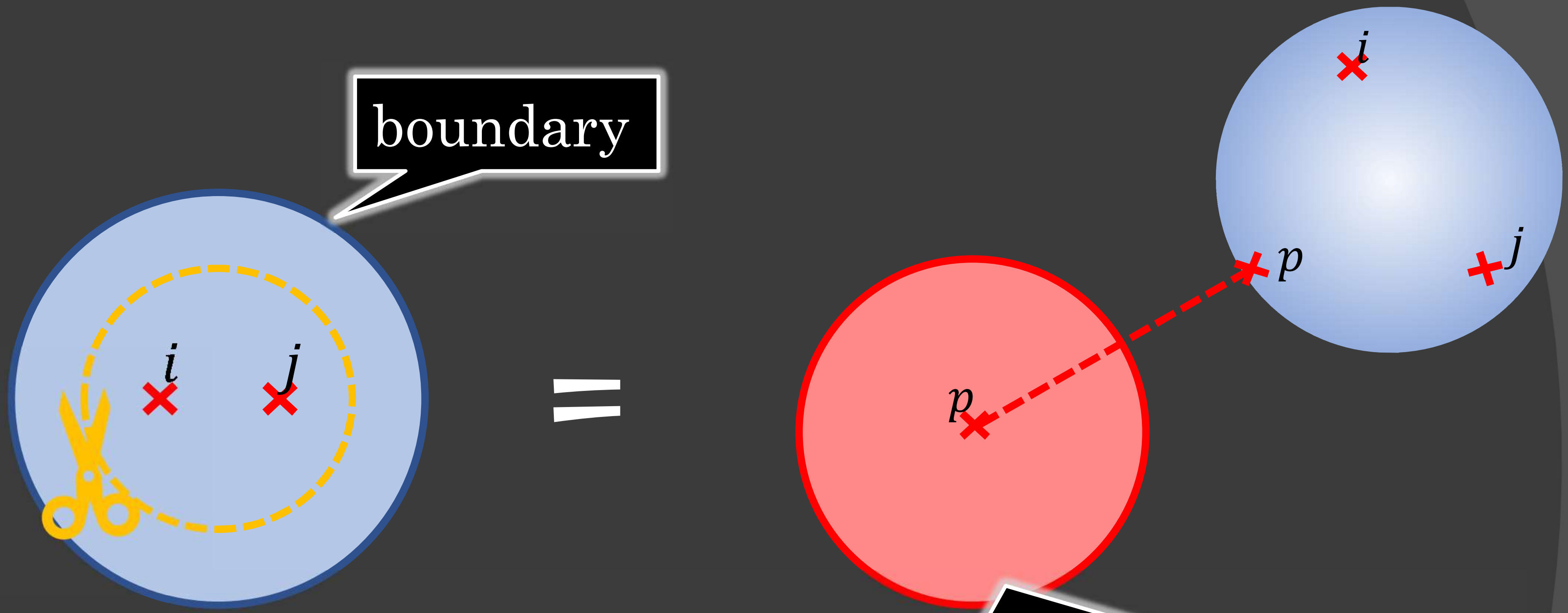
Because Ward id (with bdy) is equivalent to Ward id (without bdy) by mirror method

$$\sum_{p,\bar{p},N,\bar{N}} \langle \phi_i | \phi_j | L_{-N} \phi_p \rangle \langle \phi_{\bar{i}} | \phi_{\bar{j}} | L_{-\bar{N}} \phi_{\bar{p}} \rangle \langle L_{-N} L_{-\bar{N}} \phi_{p,\bar{p}} \rangle_{disk}$$

$$= \sum_{p,\bar{p},N,\bar{N}} \langle \phi_i | \phi_j | L_{-N} \phi_p \rangle \langle \phi_{\bar{i}} | \phi_{\bar{j}} | L_{-\bar{N}} \phi_{\bar{p}} \rangle \langle L_{-N} \phi_p \, | L_{-\bar{N}} \phi_{\bar{p}} \rangle$$

$$= \sum_{p,N} \langle \phi_i | \phi_j | L_{-N} \phi_p \rangle \langle \phi_{\bar{i}} | \phi_{\bar{j}} | L_{-N} \phi_p \rangle$$

Review of BCFT [Lewellen]



$$\sum_p {\color{red}C_{p0}} C_{ijp} \mathcal{F}^{ji}_{\bar{j}\bar{i}}(p|z)$$

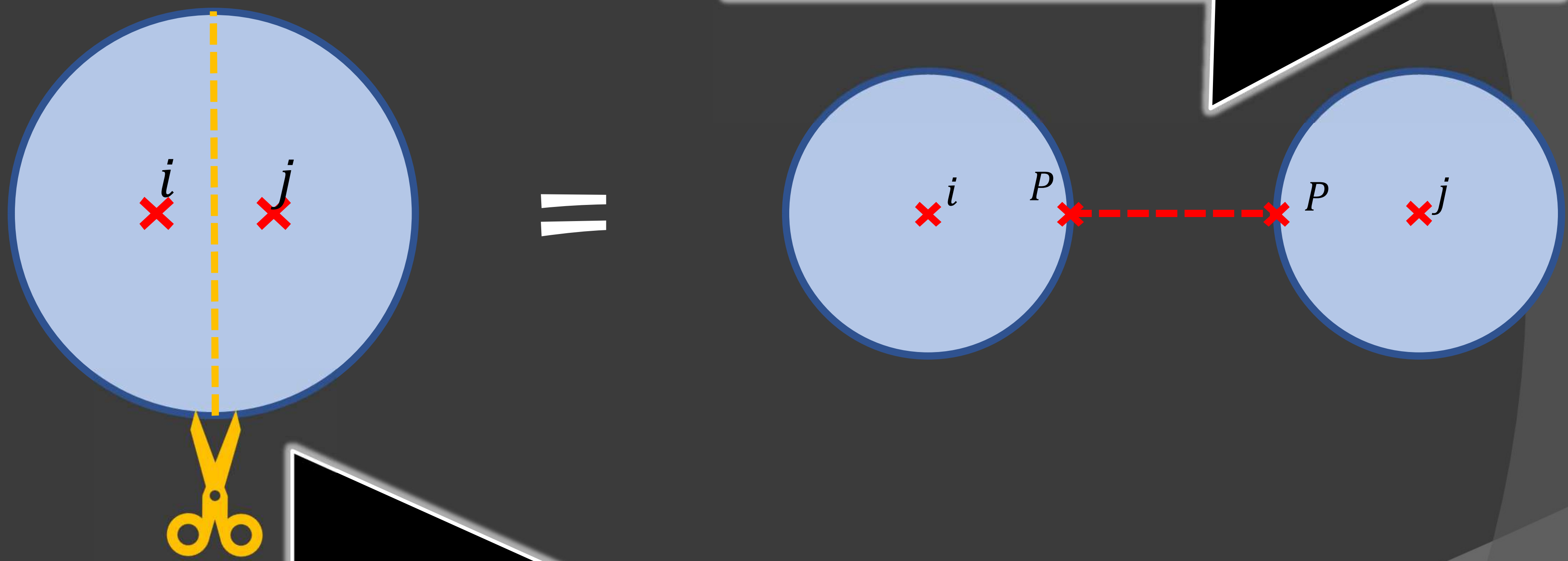
New ingredient: bulk-boundary OPE coef.

Review

or equivalently, using bulk-boundary OPE

$$\phi_i(z) \sim \sum_P C_{iP}(2\Im z)^{h_P-h_i-\bar{h}_i}\phi_P(\Re z) + \cdots$$

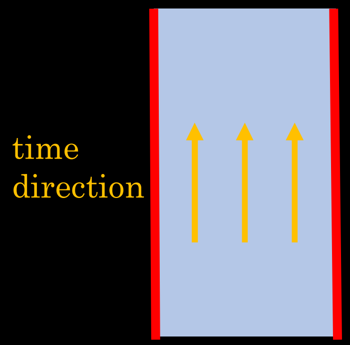
bulk-boundary OPE coef.

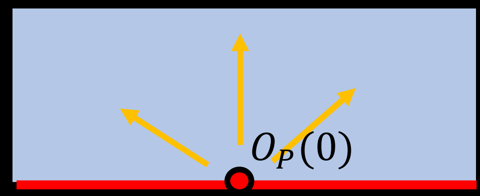


Cutting:

Inserting (**boundary** operator) complete set

Review

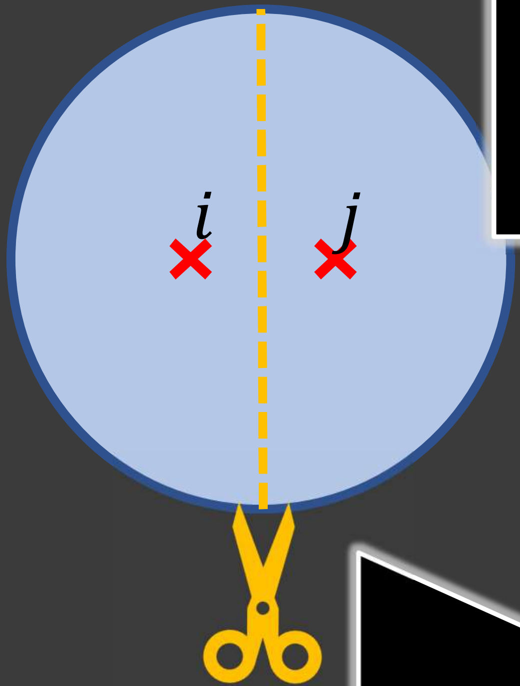


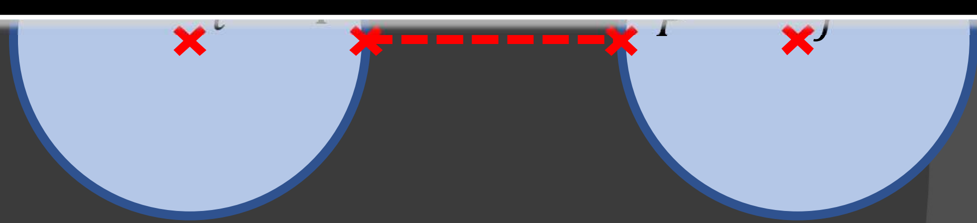


$$e^{z_{cyl}} = z_{pl}$$

$|P\rangle$
state
(in open string sector)

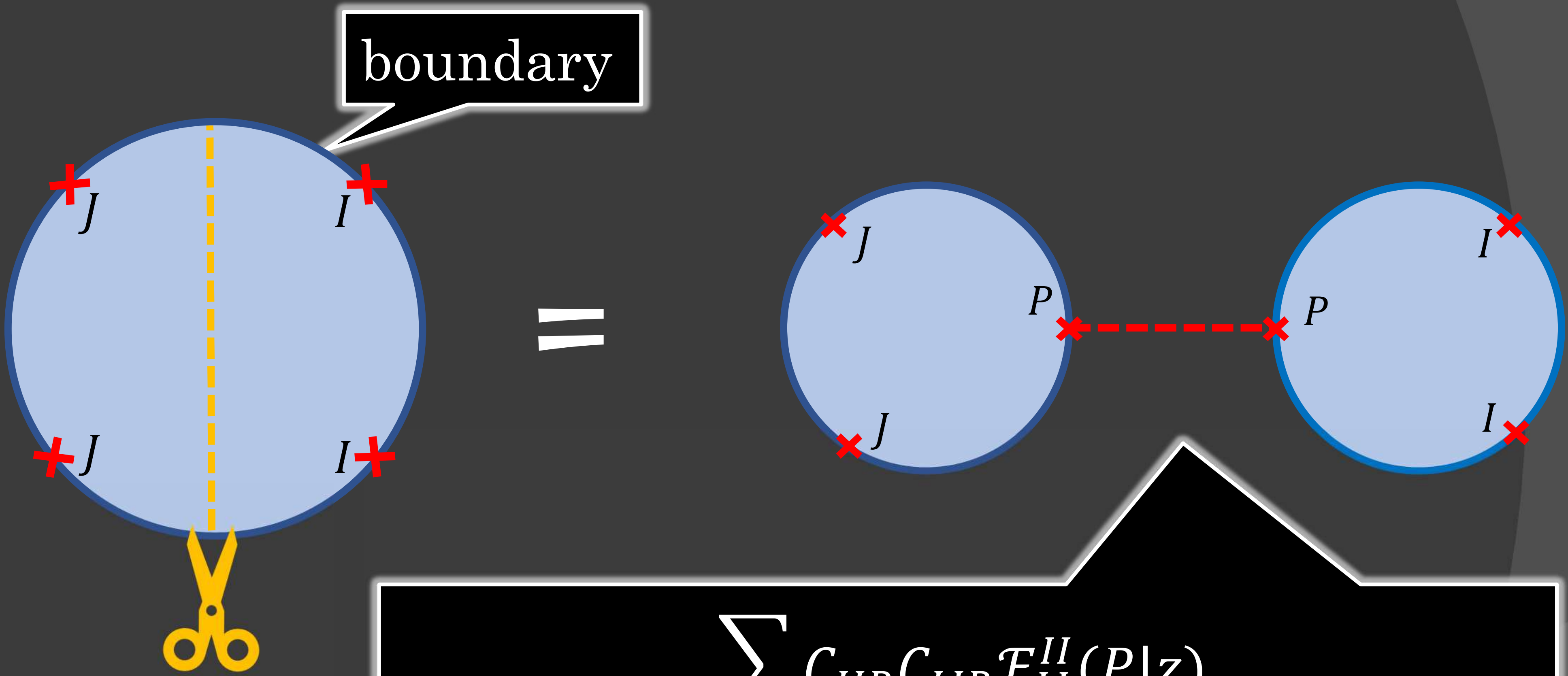
$O_P(0)$
boundary
operator
(lives only on boundary)





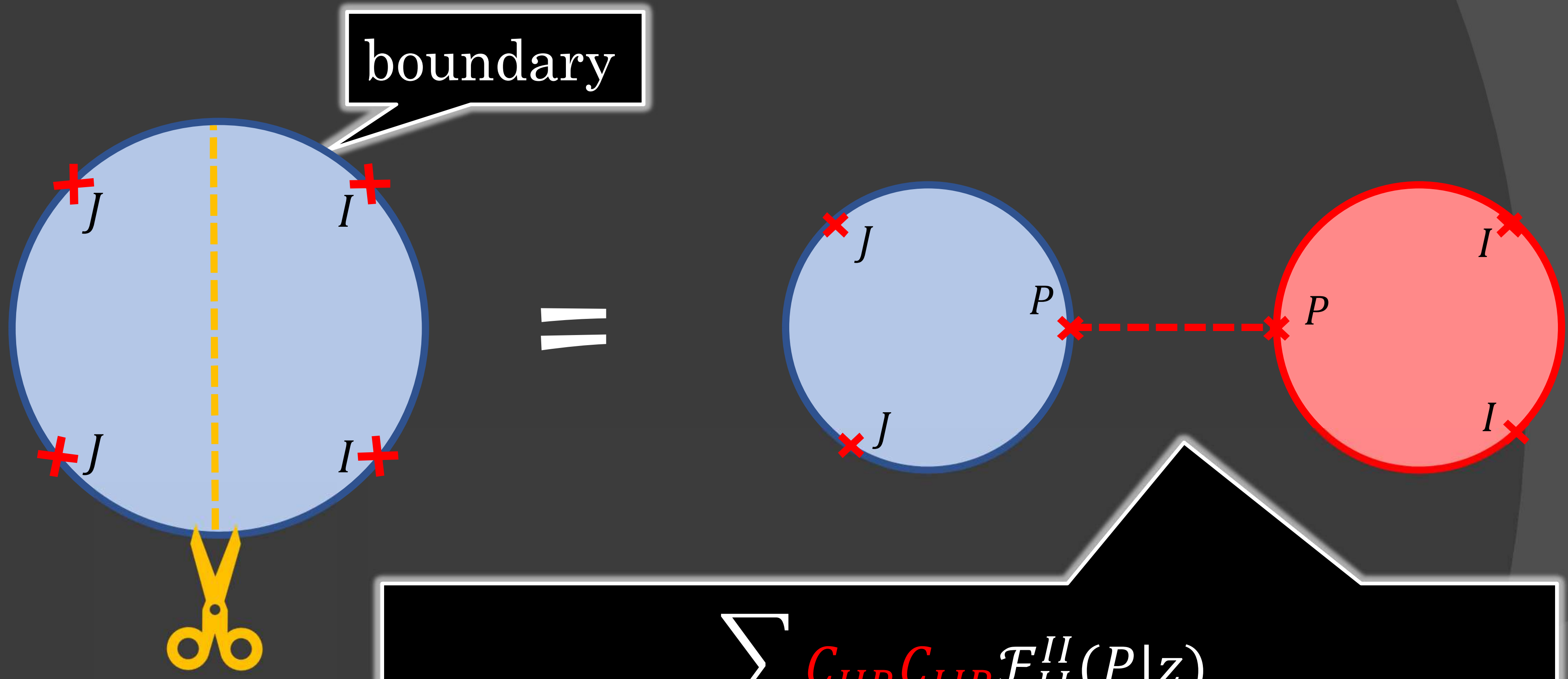
Cutting:
Inserting (**boundary** operator) complete set

Review of BCFT [Lewellen]



$$\sum_P C_{IIP} C_{JJP} \mathcal{F}^{II}_{JJ}(P|z)$$

Review of BCFT [Lewellen]

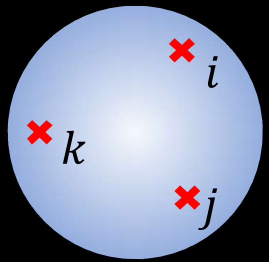


$$\sum_P C_{IIP} C_{JJP} \mathcal{F}^{II}_{JJ}(P|z)$$

New ingredient: bdy-bdy-bdy OPE coef.

As the first step, it would be interesting to give the asymptotic formula, which may have the potential to understand the braneworld holography.

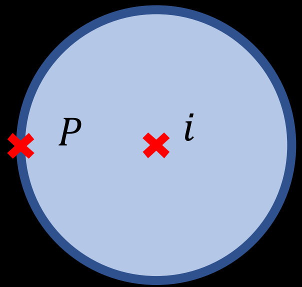
Ingredients in BCFT



[Collier, Maloney, Maxfield, Tsiares]

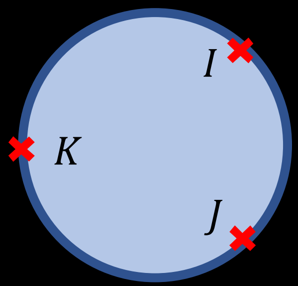
$\equiv C_{ijk}$

Bulk-bulk-bulk OPE coefficient



$\equiv C_{iP}$

Bulk-boundary OPE coefficient



$\equiv C_{IJK}$

Bdy-bdy-bdy OPE coefficient

[Cardy]

$\rho(h,h)$

Bulk primary spectrum

$\rho^{bdy}(h)$

Bdy primary spectrum

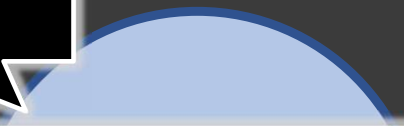
[Collier, Mazac, Wang]

g

Boundary entropy

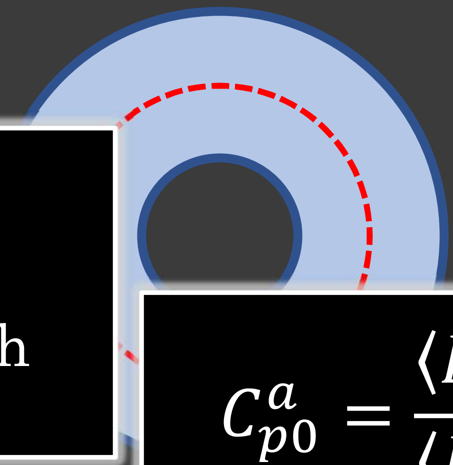
Bootstrap in BCFT

Annulus
(Not Torus)



$$\rho(\alpha) = \sum_i D_i \delta(\alpha - \alpha_i)$$

where D_i is degeneracy of primaries with momentum α_i



$$C^a_{p0} = \frac{\langle B^a | p \rangle}{\langle B^a | 0 \rangle}$$

$$\int \mathrm{d}\alpha_P \; \rho^{bdy}(\alpha_P)\chi_P(\tau) \qquad = \qquad g^2 \int \mathrm{d}\alpha_p \; \rho(\alpha_p)\overline{(C_{p0})^2}\chi_p\left(-\frac{1}{\tau}\right)$$

where

$$c = 1 + 6Q^2, \qquad Q = b + \frac{1}{b}, \qquad h_i = \alpha_i(Q - \alpha_i).$$

Bootstrap in BCFT

Same method with no boundary
[Kusuki]
[Collier, Gobeil, Maxfield, Perlmutter]
[Collier, Maloney, Maxfield, Tsiares]
etc.

$$\int \mathrm{d}\alpha_P\ \rho^{bdy}(\alpha_P)\chi_P(\tau) = g^2 \int \mathrm{d}\alpha_p\ \rho(\alpha_p)\overline{\left(C_{p0}\right)^2}\chi_p\left(-\frac{1}{\tau}\right)$$

Step 1. vacuum approximation in $\tau \to i0$

$$\int \mathrm{d}\alpha_P\ \rho^{bdy}(\alpha_P$$

Virasoro character is universal.
Therefore, kernel is also universal

Step 2. modular transformation

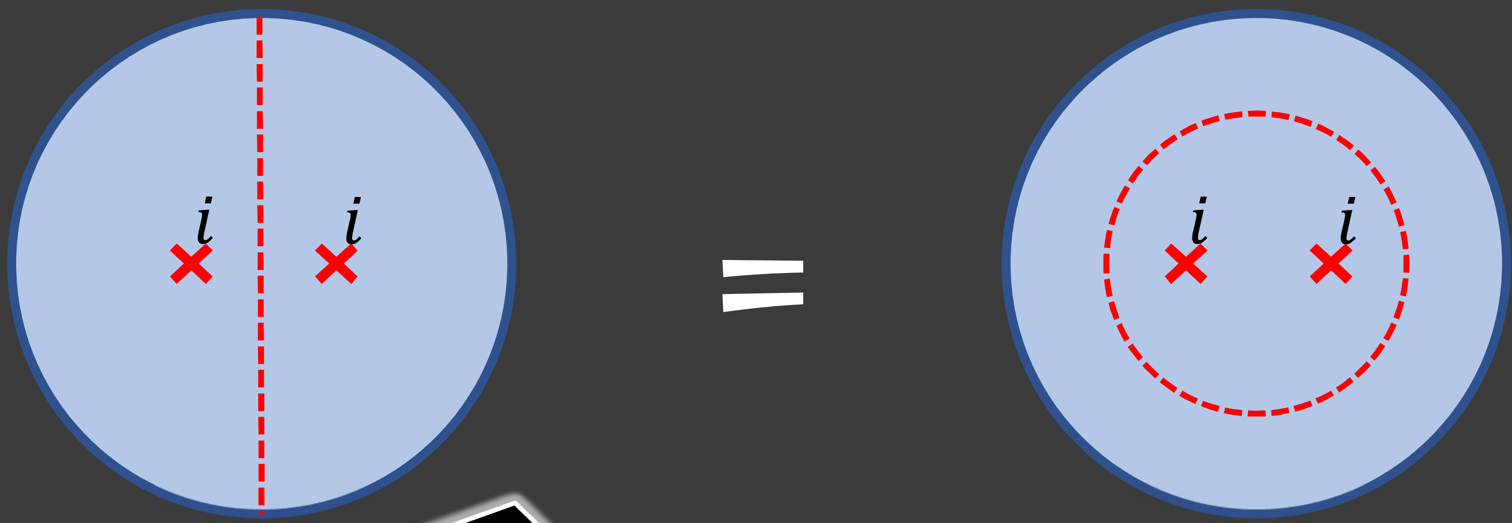
$$\chi_0\left(-\frac{1}{\tau}\right) = \int \mathrm{d}\alpha_P\, S_{0P}\chi_P(\tau)$$

BCFT ver. of Cardy formula

Step3. coefficient comparison

$$\rho^{bdy}(\alpha_P) \simeq g^2 S_{0P} \qquad (h_P \to \infty)$$

Bootstrap in BCFT



$$\int \mathrm{d}\alpha_P \;\rho^{bdy}(\alpha_P)\overline{(C_{iP})^2}\mathcal{F}^{ii}_{ii}(P|z) = \int \mathrm{d}\alpha_p \;\rho(\alpha_p)\overline{C_{iip}C_{p0}}\mathcal{F}^{ii}_{ii}(p|1-z)$$

where

$$c = 1 + 6Q^2, \qquad Q = b + \frac{1}{b}, \qquad h_i = \alpha_i(Q - \alpha_i).$$

Bootstrap in BCFT

$$\int d\alpha_P\ \rho^{bdy}(\alpha_P)\overline{(C_{iP})^2}\mathcal{F}^{ii}_{ii}(P|z) = \int d\alpha_p\ \rho(\alpha_p)\overline{C_{iip}C_{p0}}\mathcal{F}^{ii}_{ii}(p|1-z)$$

Step 1. vacuum approximation in z → 1

$$\int d\alpha_P\ \rho^{bdy}(\alpha_P)\overline{(C_{iP})^2}\mathcal{F}^{ii}_{ii}(P|z) \simeq \mathcal{F}^{ii}_{ii}(0|1-z)$$

Step 2. fusion transformation

closed form is given by Ponsot & Teschner

$$\mathcal{F}^{ii}_{ii}(0|1-z) = \int d\alpha_P\, F_{0P}\begin{bmatrix} i & i \\ i & i \end{bmatrix}\mathcal{F}^{ii}_{ii}(P|z)$$

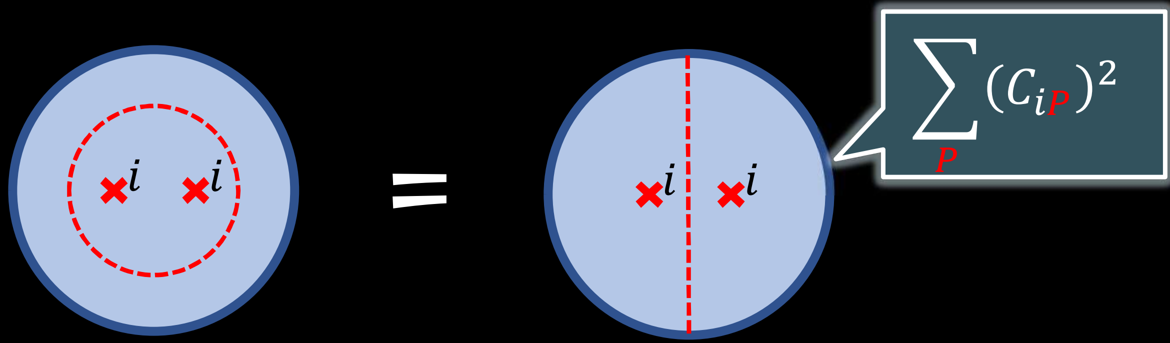
Step3. coefficient comparison

$$\overline{(C_{iP})^2} \simeq g^{-2}{S_{0P}}^{-1}F_{0\alpha_P}\begin{bmatrix} \alpha_i & \alpha_i \\ \alpha_i & \alpha_i \end{bmatrix} \qquad (h_P \to \infty)$$

Bootstrap in BCFT

Bulk-boundary OPE coefficient

- Light-Heavy



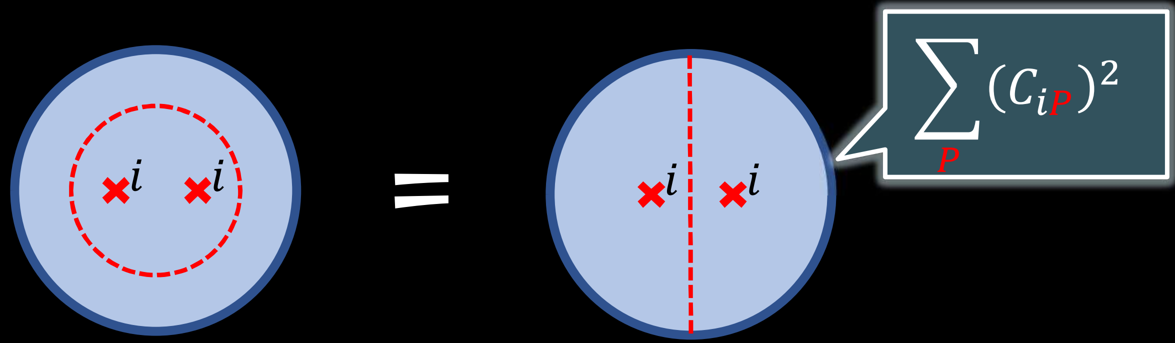
Note: How to find the bootstrap equation?

- We can extract information about a heavy state P from a sum over P (by out method or inverse Laplace transformation)
- Bootstrap equation should have sums over states corresponding to heavy

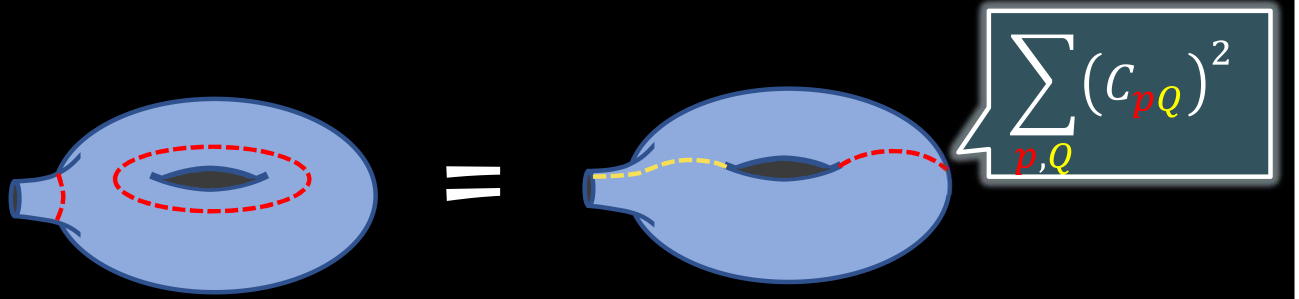
Bootstrap in BCFT

Bulk-boundary OPE coefficient

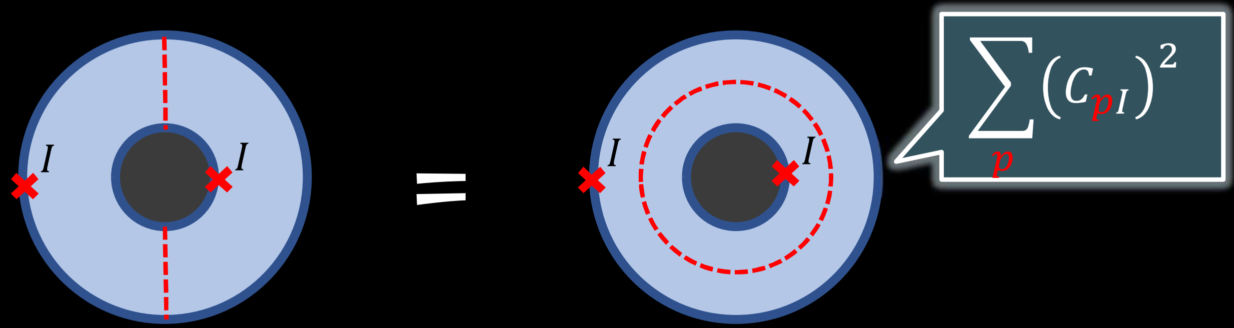
- Light-Heavy



- Heavy-Heavy
  (or large spin)



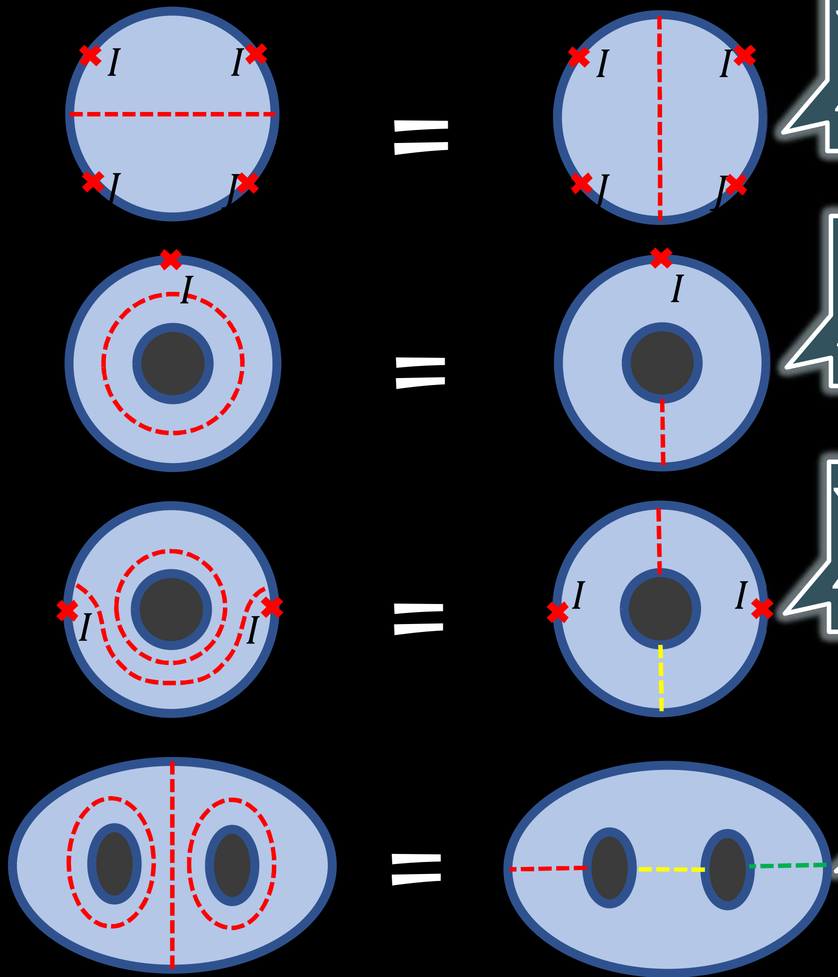
- Heavy-Light
  (or large spin)



Bootstrap in BCFT

Bdy-bdy-bdy OPE coefficient

- H-L-L
- H-H-L
  $(H_1 = H_2)$
- H-H-L
  $(H_1 \neq H_2)$
- H-H-H



$$\sum_{P} (C_{IJP})^2$$

$$\sum_{P} (C_{PPI})^2$$

$$\sum_{P,Q} (C_{PQI})^2$$

$$\sum_{P,Q,R} (C_{PQR})^2$$

Comments

Assume HKS sparse condition,

$$\rho^{bdy}(h_P), \rho^{bulk}(h_P) \leq \mathrm{e}^{2\pi h_P}, \qquad h_P \leq \frac{c}{24}$$

- ρ^{bdy} follows **Cardy formula**

$$\rho^{bdy}(h_P) \simeq \mathrm{e}^{2\pi\sqrt{\frac{c}{6}\left(h_P - \frac{c}{24}\right)}}, \qquad h_P \geq \frac{c}{12}$$

- H-H-L OPE coef. follows **ETH**

$$C_{PQI} \simeq \mathrm{e}^{-\frac{1}{2}S\left(\frac{E_P+E_Q}{2}\right)}, \qquad h_P \to \infty \text{ with } |h_P - h_Q| \text{ fixed}$$

$$\text{where } S(E) = 2\pi\sqrt{\frac{c}{6}E} \text{ and } E_I = h_I - \frac{c}{24}$$

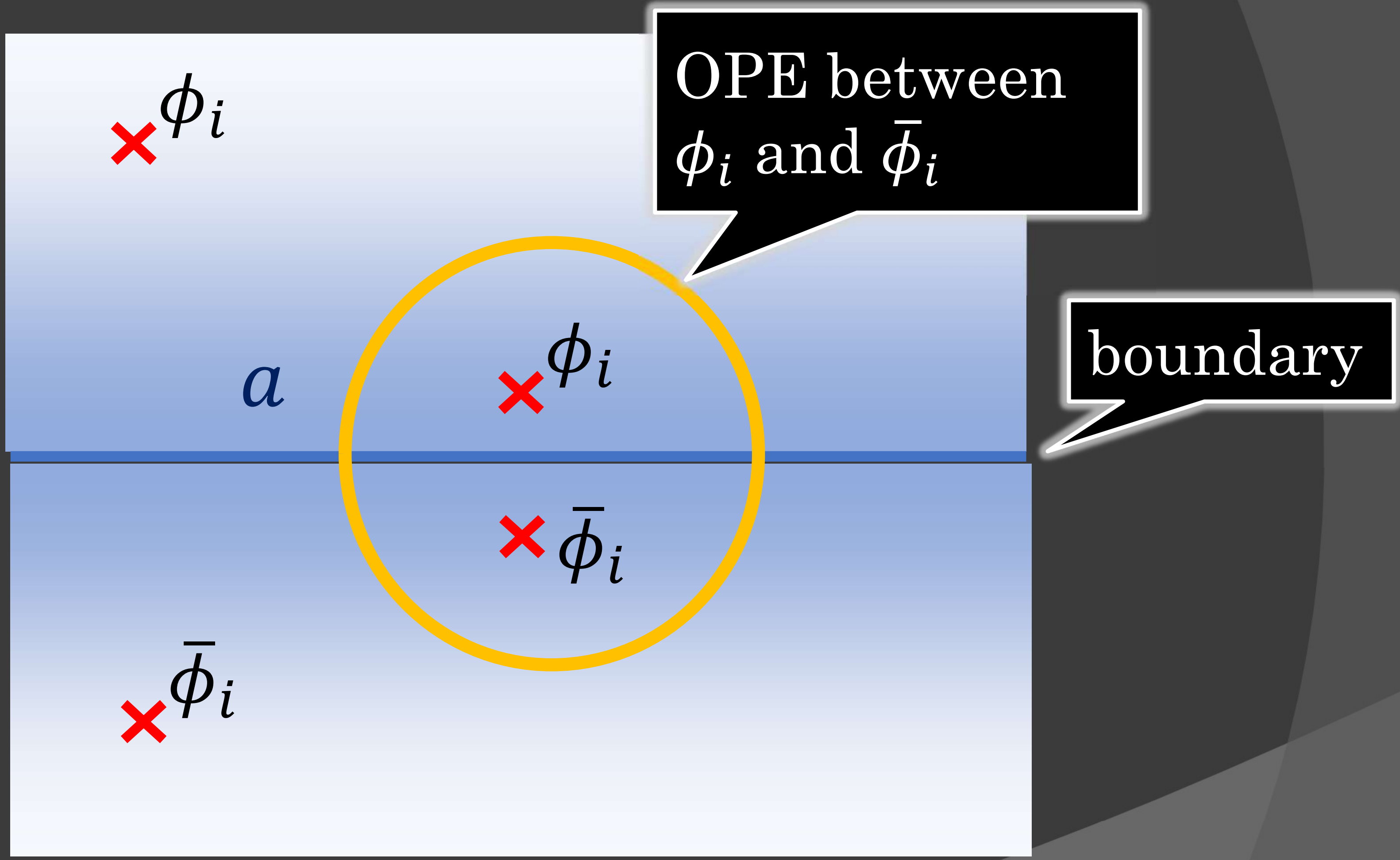
Discussion

As the next step, we hope to understand a relation between braneworld (island model, …) & BCFT (moving mirror, …) from boundary bootstrap results!

- More input on bootstrap
  - localization on brane ($g \gg 1$) — [Karch, Randall], [Cooper, Rozali, Swingle], etc.
  - causality (large gap) — [Reeves, Rozali, Simidzija, Sully, Waddell, Wakeham], etc.
  - brane self-intersection — [Cooper, Rozali, Swingle], [Geng, Lust, Mishra, Wakeham], etc.
- Application of new techniques developed in CFT

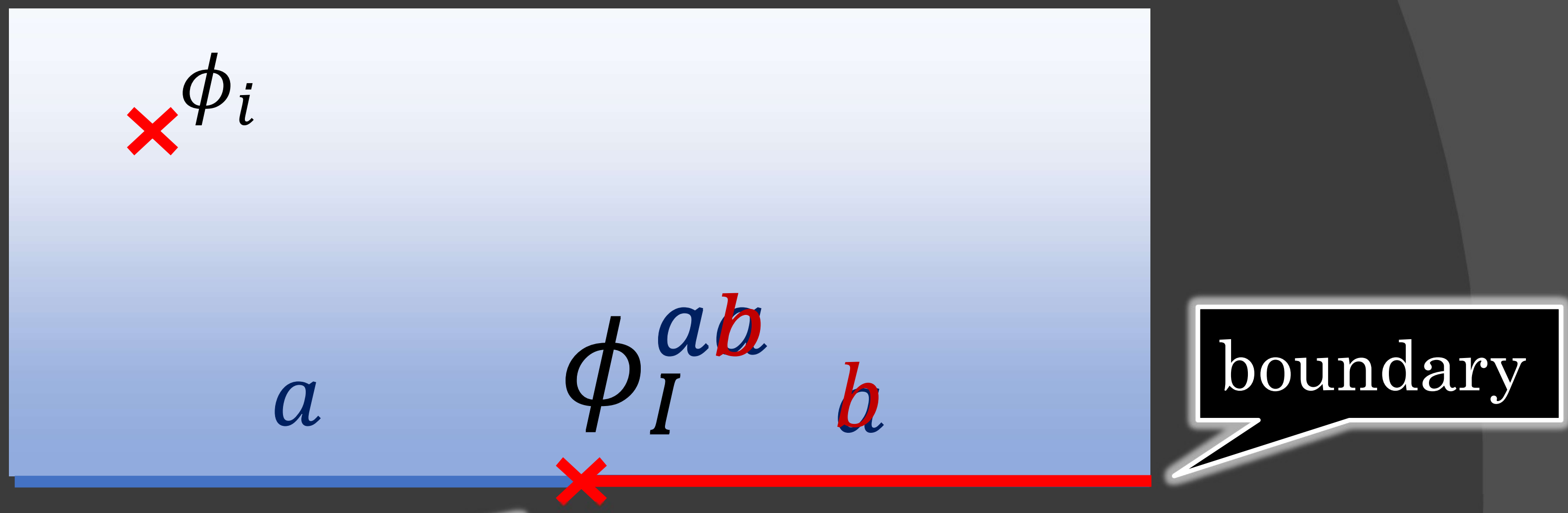
Appendix

Review of BCFT



Review of BCFT

i: *bulk*
I: *boundary*

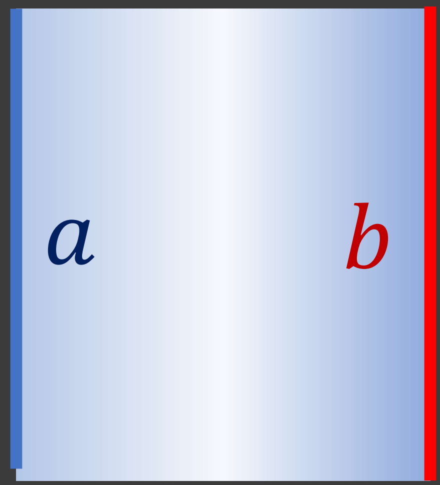


New ingredient (boundary primary)

Primary operator living on boundary,
which can change boundary condition.
Same transformation law under conformal mapping.

Review of BCFT

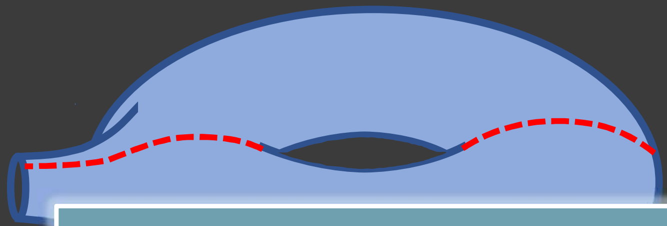
state – operator like mapping



Conformal weight of ϕ_I^{ab}

= Energy corresponding to the state on the strip

Lightcone Bootstrap in BCFT



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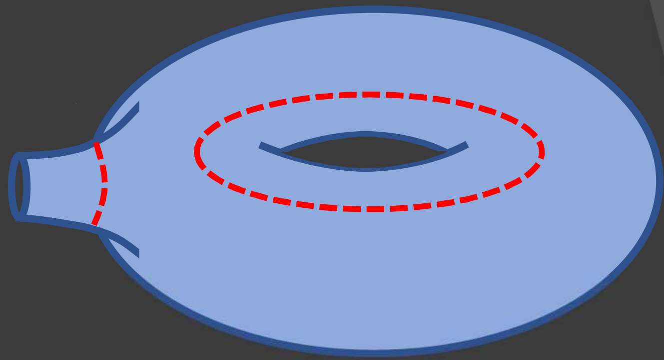
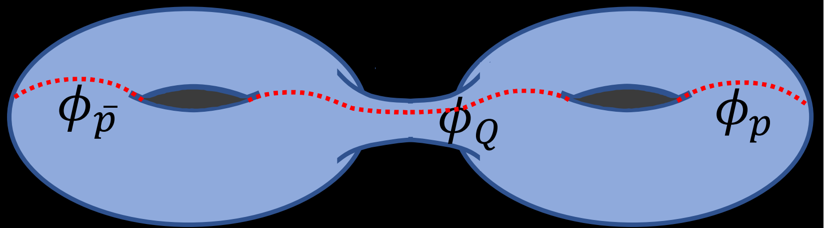
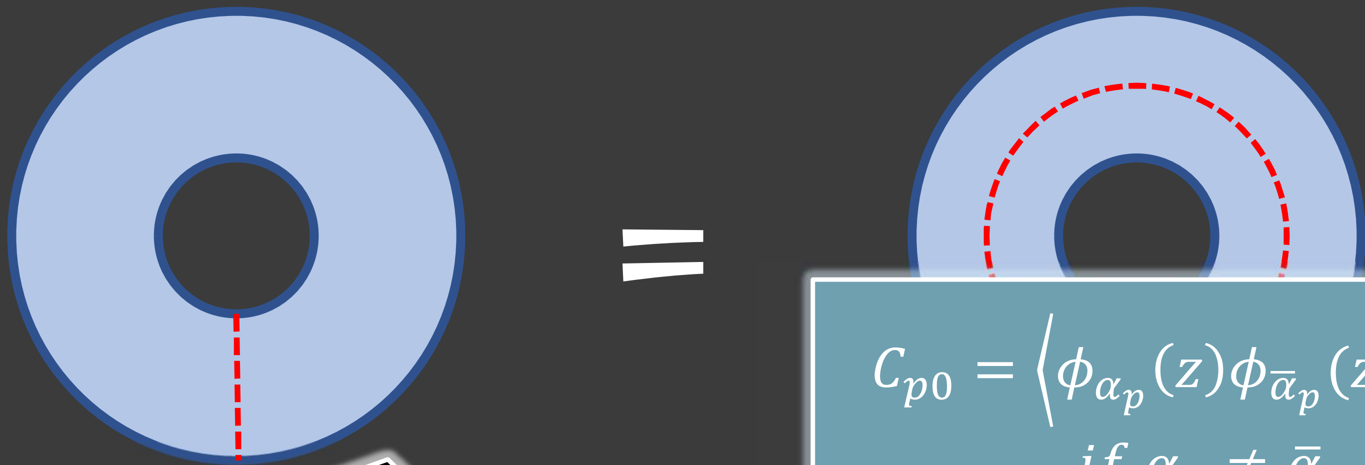


For intermediate state p,
$\alpha_p \neq \bar{\alpha}_p$ is possible.

$$\int \mathrm{d}\alpha_p \int \mathrm{d}\bar{\alpha}_p \int \mathrm{d}\alpha_Q \, \rho(\alpha_p, \bar{\alpha}_p) \rho^{bdy}(\alpha_Q) \overline{(C_{pQ})^2}$$

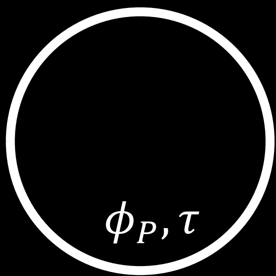


Bootstrap in BCFT

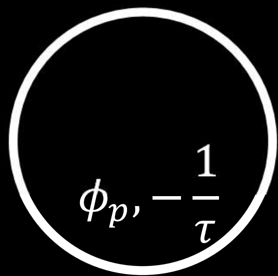


$$C_{p0} = \left\langle \phi_{\alpha_p}(z)\phi_{\bar{\alpha}_p}(z^*) \right\rangle = 0$$
$$if\ \alpha_p \neq \bar{\alpha}_p$$

$$\int \mathrm{d}\alpha_P\ \rho^{bdy}(\alpha_P)$$



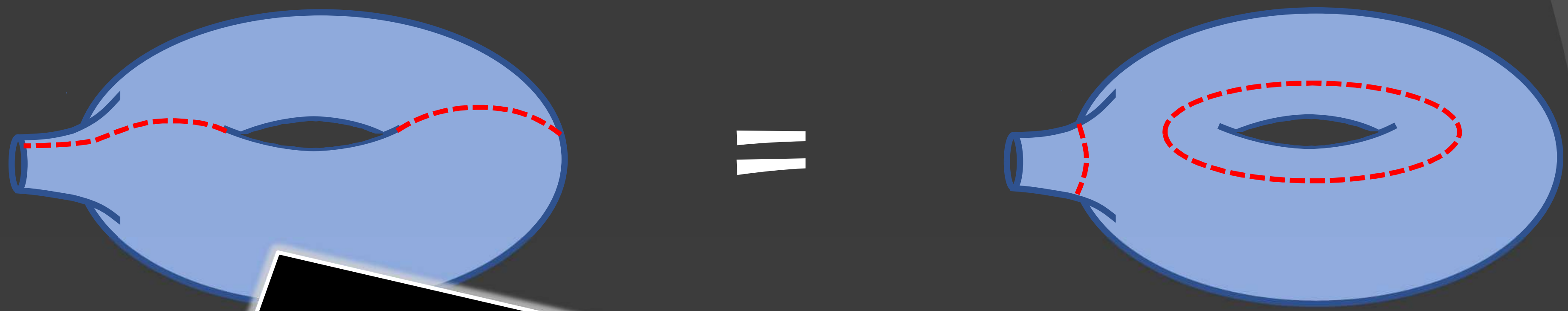
$$= g^2 \int \mathrm{d}\alpha_p\ \rho(\alpha_p) \overline{\left(C_{p0}\right)^2}$$



where

$$c = 1 + 6Q^2, \qquad Q = b + \frac{1}{b}, \qquad h_i = \alpha_i(Q - \alpha_i).$$

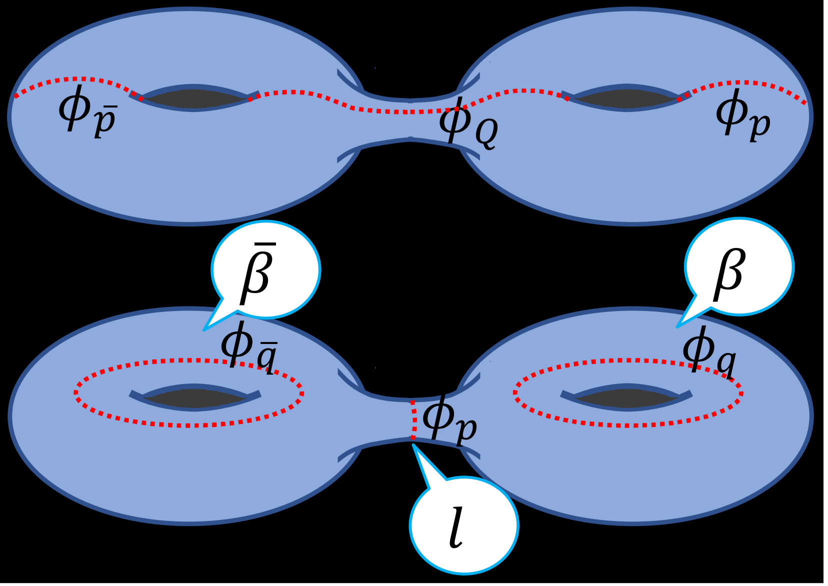
Lightcone Bootstrap in BCFT



$$\int \mathrm{d}\alpha_p \int \mathrm{d}\bar{\alpha}_p \int \mathrm{d}\alpha_Q \, \rho(\alpha_p, \bar{\alpha}_p) \rho^{bdy}(\alpha_Q) \overline{(C_{pQ})^2}$$

$$=$$

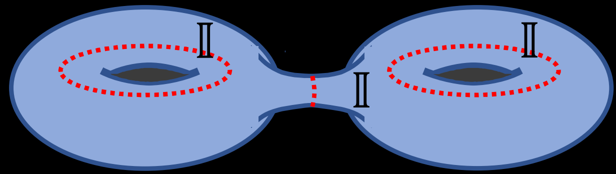
$$\int \mathrm{d}\alpha_p \int \mathrm{d}\alpha_q \int \mathrm{d}\bar{\alpha}_q \, \rho(\alpha_p) \rho(\alpha_q, \bar{\alpha}_q) \overline{C_{pqq} C_{p0}}$$



Lightcone B

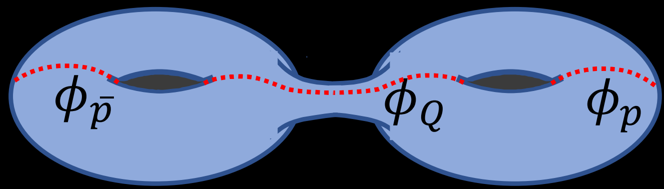
Compact $c > 1$ CFT without other currents
$\rho(0, \bar{\alpha}_p) = 1$ only if $\bar{\alpha}_p = 0$

Step 1. vacuum approximation in $l \to \infty$ & $\beta \to \infty$ with $\bar{\beta}$ fixed



Step 2. modular & fusion transformation

$$= \int \mathrm{d}\alpha_p \int \mathrm{d}\bar{\alpha}_p \int \mathrm{d}\alpha_Q \, S_{0p} S_{0\bar{p}} F_{0\alpha_Q} \begin{bmatrix} \alpha_p & \alpha_p \\ \bar{\alpha}_p & \bar{\alpha}_p \end{bmatrix}$$



Step3. coefficient comparison

$$\overline{(C_{pQ})^2} \simeq g^{-2} {S_{0Q}}^{-1} F_{0\alpha_Q} \begin{bmatrix} \alpha_p & \alpha_p \\ \bar{\alpha}_p & \bar{\alpha}_p \end{bmatrix}$$

large spin limit

$(h_p \to \infty, \bar{h}_p \to \infty \; h_Q \to \infty)$

Virasoro bloc

Complete set

$$\sum_{h,\bar{h}\in\text{primaries}} \sum_{a,\bar{a}} \frac{|\phi_{h,\bar{h}}^{a,\bar{a}}\rangle\langle\phi_{h,\bar{h}}^{a,\bar{a}}|}{\langle\phi_{h,\bar{h}}^{a,\bar{a}}|\phi_{h,\bar{h}}^{a,\bar{a}}\rangle}$$

$$\langle\phi_{h,\bar{h}}(z,1)\phi_{h,\bar{h}}(0,0)\rangle_{bdy}$$

$$= \sum_{h_p,\overline{h_p}\in\text{primaries}} \sum_{a,\bar{a}} \frac{1}{\langle\phi_{h_p,\overline{h_p}}^{a,\bar{a}}|\phi_{h_p,\overline{h_p}}^{a,\bar{a}}\rangle} \left\langle\phi_{h_p,\overline{h_p}}^{a,\bar{a}}(0,0)\right\rangle_{bdy}$$

$$\times\left\langle\phi_{h_p,\overline{h_p}}^{a,\bar{a}}(\infty,\infty)\phi_{h,\bar{h}}(z,1)\phi_{h,\bar{h}}(0,0)\right\rangle$$

$$= \sum_{h_p}\sum_{a} \frac{1}{\langle\phi_{h_p}^{a}|\phi_{h_p}^{a}\rangle}\left\langle\phi_{\bar{h}}(\infty)\phi_{\bar{h}}(1)\phi_{h_p}^{a}(0)\right\rangle\left\langle\phi_{h_p}^{a}(\infty)\phi_h(z)\phi_h(0)\right\rangle$$

$$= \sum_{h_p} C_{hhh_p}C_{\bar{h}\bar{h}h_p}\mathcal{F}_{\bar{h}\bar{h}}^{hh}(h_p|z)$$

Reminder

$$\mathcal{F}_{kl}^{ji}(h_p|z) \equiv \sum_a \frac{1}{\langle\phi_p^a|\phi_p^a\rangle}\frac{\langle\phi_l(\infty)\phi_k(1)\phi_p^a(0)\rangle\langle\phi_p^a(\infty)\phi_j(z)\phi_i(0)\rangle}{C_{ijp}C_{klp}}$$

Virasoro bloc

Complete set

$$\sum_{h}\sum_{a}\frac{|\psi_h^a\rangle\langle\psi_h^a|}{\langle\psi_h^a|\psi_h^a\rangle}$$

$$\langle\phi_{h,\bar{h}}(\infty,1)\phi_{h,\bar{h}}(z,0)\rangle_{bdy}$$

$$=\sum_{h_p}\sum_{a}\frac{1}{\langle\psi_h^a|\psi_h^a\rangle}\left\langle\phi_{h,\bar{h}}(\infty,1)\psi_{h_p}^a(0)\right\rangle_{bdy}\left\langle\psi_{h_p}^a(\infty)\phi_{h,\bar{h}}(z,0)\right\rangle_{bdy}$$

$$=\sum_{h_p}\sum_{a}\frac{1}{\langle\psi_h^a|\psi_h^a\rangle}\left\langle\phi_h(\infty)\phi_{\bar{h}}(1)\psi_{h_p}^a(0)\right\rangle\left\langle\psi_{h_p}^a(\infty)\phi_{\bar{h}}(z)\phi_h(0)\right\rangle$$

$$=\sum_{h_p}C_{h\bar{h}h_p}^2\mathcal{F}(h_p|z)$$

Reminder

$$\mathcal{F}_{kl}^{ji}(h_p|z)\equiv\sum_{a}\frac{1}{\langle\phi_p^a|\phi_p^a\rangle}\frac{\langle\phi_l(\infty)\phi_k(1)\phi_p^a(0)\rangle\langle\phi_p^a(\infty)\phi_j(z)\phi_i(0)\rangle}{C_{ijp}C_{klp}}$$