QIEP (March 23, 2022)

Characterizing symmetry-protected thermal equilibrium by work extraction

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Y. M., K. Kaneko, T. Sagawa, arXiv:2103.06060, to appear in PRX

Outline of this talk

1. Introduction

2. Setup of the problem

3. Main result

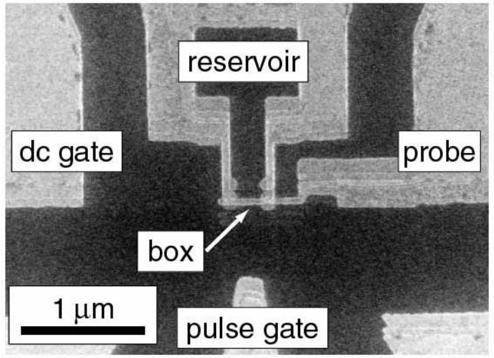
4. Result for the setup with work storage

5. Summary and discussion

Quantum thermodynamics

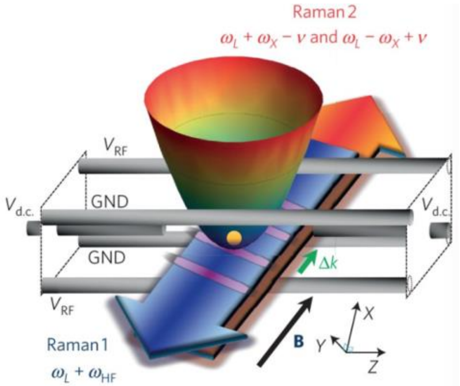
Superconducting qubit

Y. Nakamura *et al.*, Nature (1999)



Trapped ion

S. An *et al.*, Nat. Phys. (2015)



With the development of quantum control technique, quantum thermodynamics becomes more important.

➡ Application to quantum heat engines and quantum batteries

Quantum version of the 2nd law

W. Pusz and S. L. Woronowicz, Comm. Math. Phys. (1978)

Passivity: Quantum version of Kelvin's principle

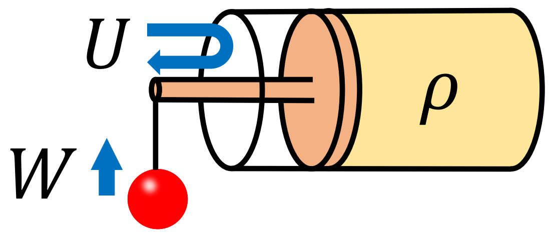
(No work can be extracted by a cycle.)

Setup

Extracted work from a state ρ by a unitary U

$$W(\rho, U) \coloneqq \mathrm{tr}(\rho H) - \mathrm{tr}\left(U\rho U^\dagger H\right)$$

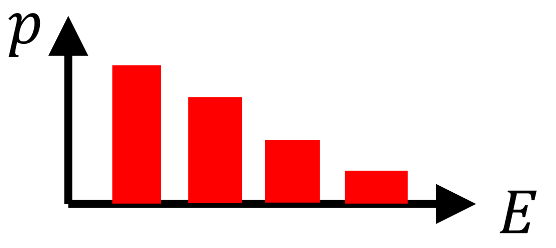
The initial and the final Hamiltonians are the same because we consider a cycle.



Definition ρ is passive $\Leftrightarrow$ For any unitary U, $W \leq 0$

Necessary and sufficient condition:

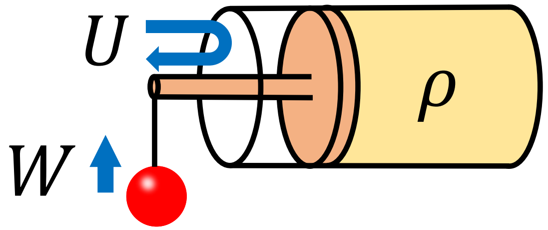
$$\rho = \sum_i p_i |E_i\rangle\langle E_i| \quad (E_1 \leq E_2 \leq \cdots, p_1 \geq p_2 \geq \cdots)$$



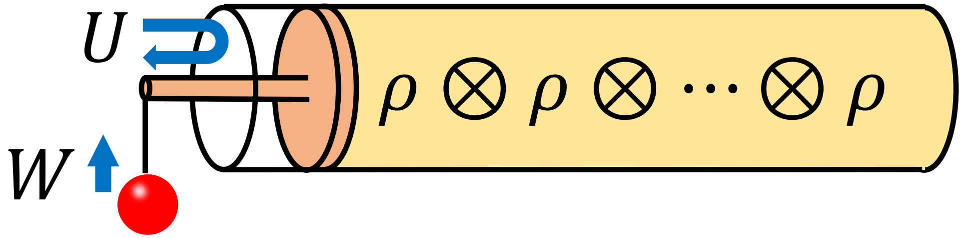
Complete passivity

W. Pusz and S. L. Woronowicz, Comm. Math. Phys. (1978)

Let ρ be passive. $W \leq 0$



What about multiple copies of ρ? $W > 0$?



No work can be extracted from multiple thermal equilibriums.

➡ Thermal equilibrium must be completely passive.

<u>Definition</u> ρ is completely passive $\Leftrightarrow \forall N \in \mathbb{N}, \rho^{\otimes N}$ is passive

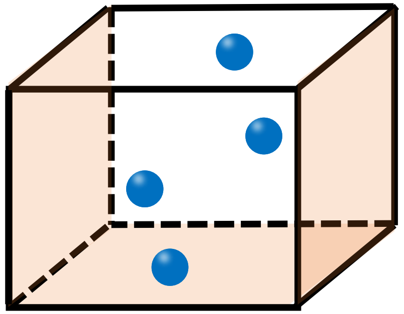
Necessary and sufficient condition: $\rho = e^{-\beta H}/Z$ Gibbs ensemble

$H_N^{\mathrm{tot}} \coloneqq \sum_{k=1}^{N} I^{\otimes k-1} \otimes H \otimes I^{\otimes N-k}$ (with no correlation).

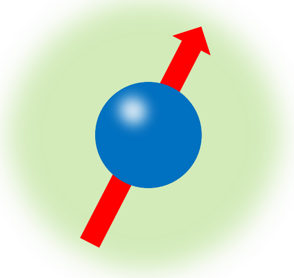
Symmetry in physics

Symmetry is ubiquitous in physics
and imposes constraints on possible operations.

1. $U(1)$ symmetry
e.g. Particle number conservation



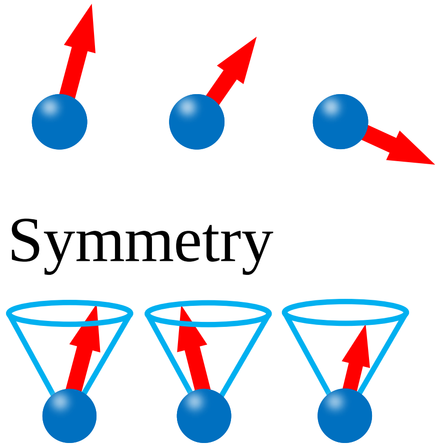
2. $SU(2)$ symmetry
e.g. x, y, z-spin conservation



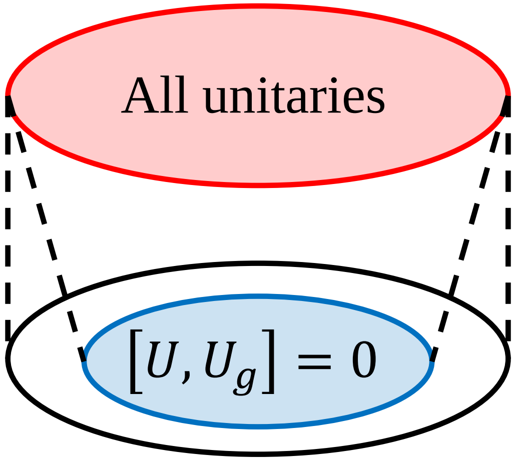
Non-commutative symmetry
(unique to quantum mechanics)

Effective thermal equilibrium under symmetry

No symmetry

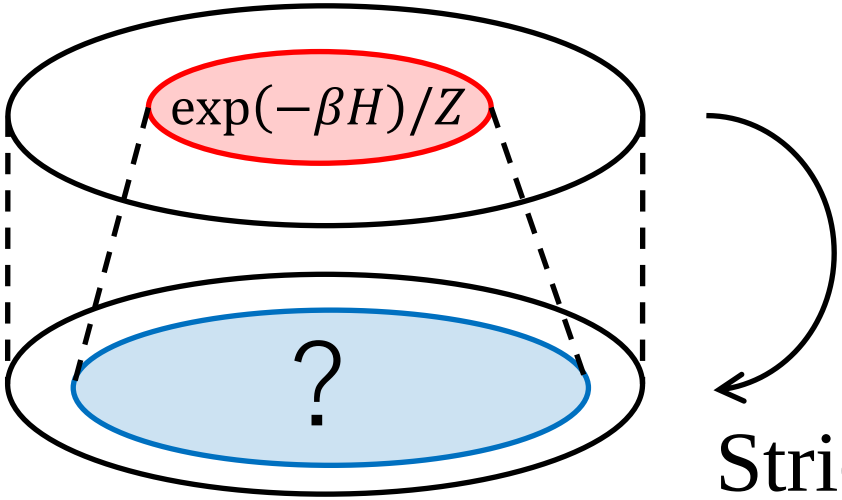


Allowed operations



Symmetry-respecting unitaries

Completely passive states



Symmetry-protected completely passive states

Strictly larger

for continuous symmetry

Main theorem

Theorem 1. arXiv:2103.06060

Under continuous symmetry（connected compact Lie group),

completely passive ⇔ generalized Gibbs ensemble (GGE)

$$e^{-\beta H-\sum_i \mu_i Q_i}/Z$$

($\beta \geq 0, \mu_i \in \mathbb{R}, Q_i$’s are charges corresponding to symmetry.)

This holds even for non-commutative charges.

Unconventional extension of the GGE.

The GGE is also investigated in the context of equilibration.

GGE and thermalization

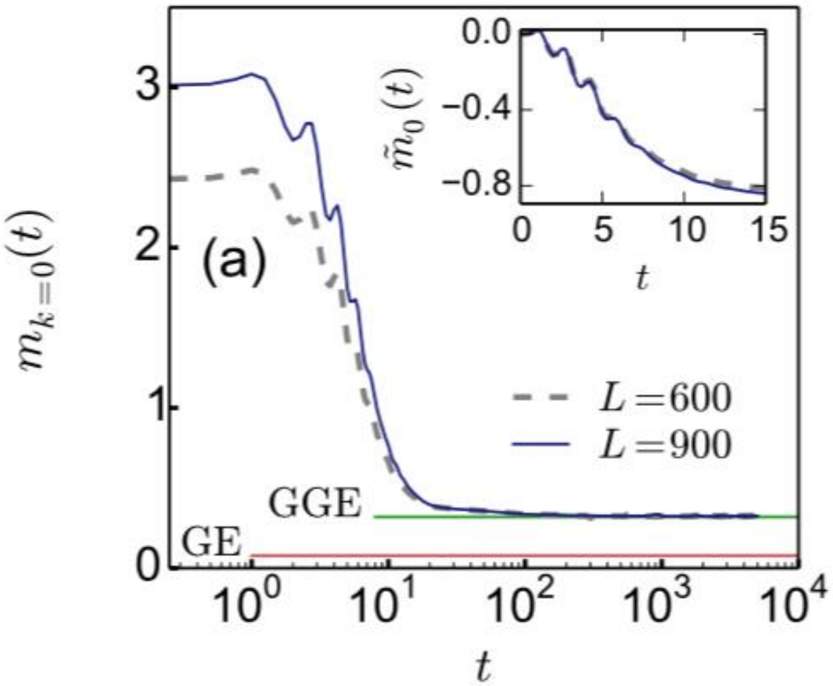
L. Vidmar and M. Rigol, J. Stat. Mech. (2016).

The GGE has been investigated in the context of thermalization of integrable systems

(Systems with an extensive number of charges)

$$e^{-\beta H-\sum_i \mu_i Q_i}/Z$$

Q_i's commute with each other.



Unconventional extension: non-commutative charges

N. Yunger Halpern *et al.*, Nat. Commun. (2016).

K. Fukai *et al.*, Phys. Rev. Research (2020).

F. Cranzl *et al.*, arXiv:2202.04652.

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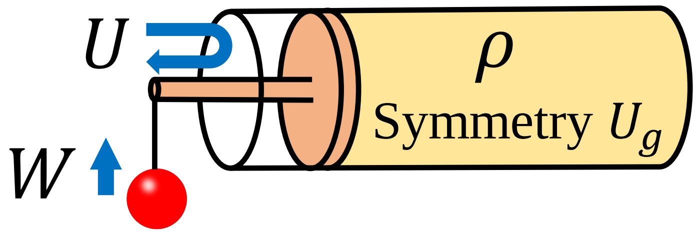
Symmetry-protected passivity

A unitary U is symmetry-respecting
$\Leftrightarrow$ U commutes with every unitary representation U_g of a group G.

$$\forall g \in G, [U, U_g] = 0$$

Definition
ρ is symmetry-protected passive
$\Leftrightarrow$ No work can be extracted by any symmetry-respecting unitary U



Symmetry-protected complete passivity

A unitary U acting on $\rho^{\otimes N}$ is symmetry-respecting
$\Leftrightarrow$ U commutes with tensor product representation.

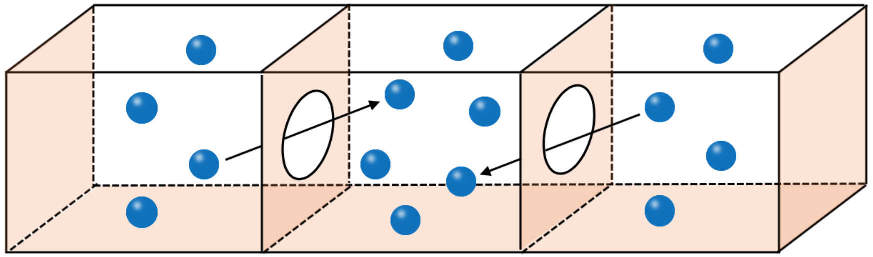
$$[U, U_g^{\otimes N}] = 0$$

Conservation of total charges

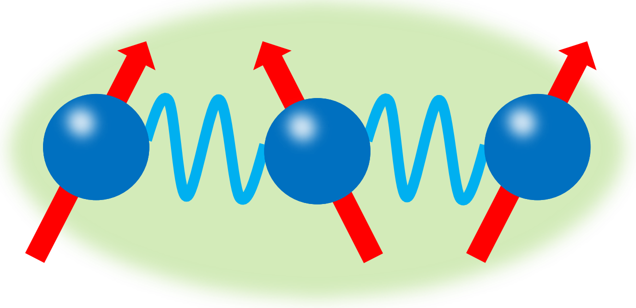
$$U_g^{\otimes N} = \exp\left(i \sum_{k=1}^{N} I^{\otimes k-1} \otimes Q \otimes I^{\otimes N-k} \right)$$

Total charge

E.g.) Conservation of total particle number

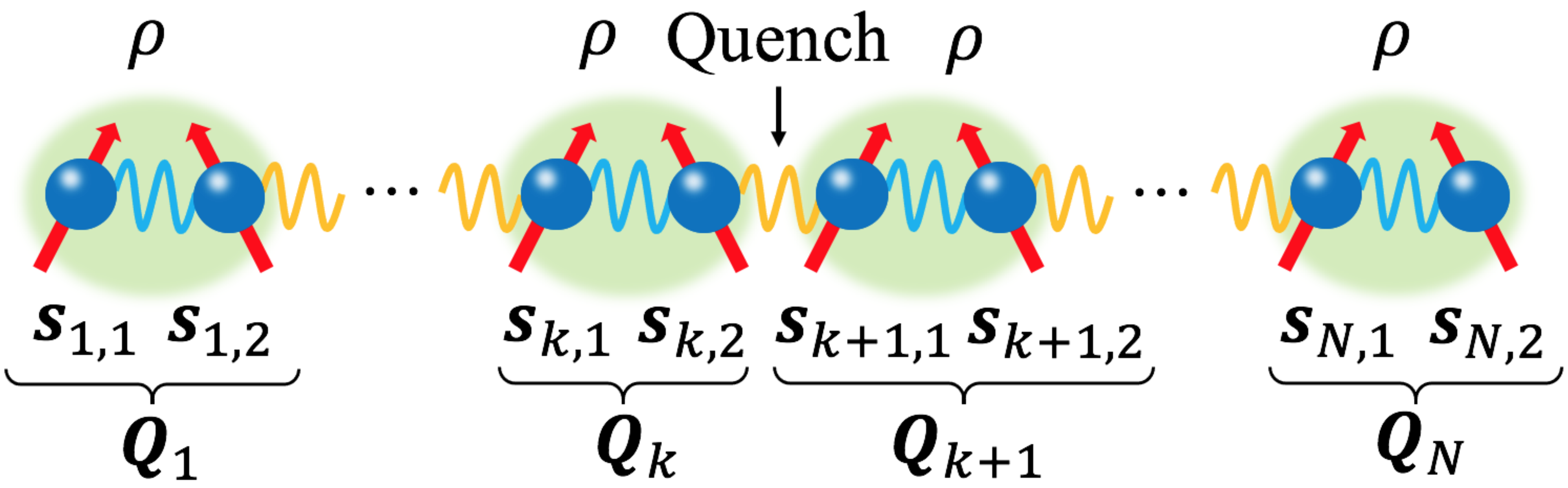


Conservation of total spin



Example: Dimer model

Example of total spin conservation: Isotropic Heisenberg-type interaction



$$H' = \sum_k \boldsymbol{s}_{k,1} \cdot \boldsymbol{s}_{k,2} + \sum_k \boldsymbol{s}_{k+1,1} \cdot \boldsymbol{s}_{k,2}$$

$$\rightarrow [H', Q^{x,\mathrm{tot}}] = [H', Q^{y,\mathrm{tot}}] = [H', Q^{z,\mathrm{tot}}] = 0$$

Setup of the problem

<u>Definition</u>

ρ is symmetry-protected completely passive

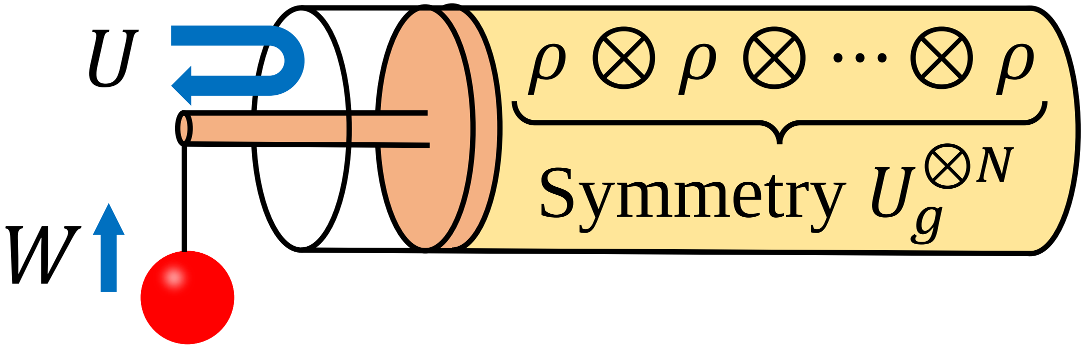
$\Leftrightarrow\ \forall N \in \mathbb{N},\ \rho^{\otimes N}$ is symmetry-protected passive

What is the necessary and sufficient condition?

Assumption:

Hamiltonian H is symmetry-respecting

Every pair of two copies has no correlation.



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Main theorem

Theorem 1.

arXiv:2103.06060

Under continuous symmetry（connected compact Lie group),

completely passive ⇔ generalized Gibbs ensemble (GGE)

$$e^{-\beta H-\sum_i \mu_i Q_i}/Z$$

($\beta \geq 0, \mu_i \in \mathbb{R}, Q_i$'s are charges corresponding to symmetry.)

✓ This holds even for non-commutative charges.

➡ Unconventional extension of the GGE.

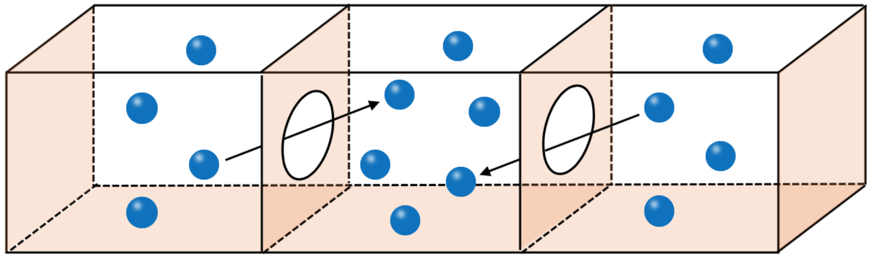
Examples

1. Case of $U(1)$ symmetry

e.g. Conservation of particle number N

$$e^{-\beta H - \mu N}/Z$$

Grand canonical ensemble

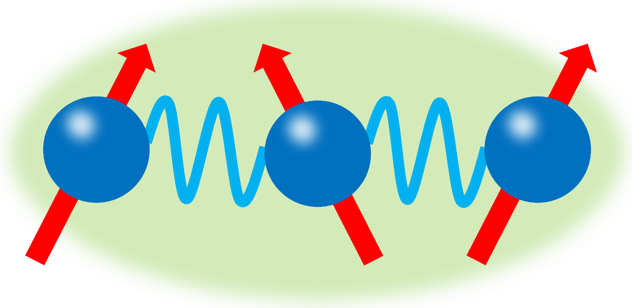


2. Case of $SU(2)$ symmetry

e.g. Conservation of spins S_x, S_y, S_z

$$e^{-\beta H - \sum_{i=x,y,z} \mu_i S_i}/Z$$

Non-commutative GGE



Proof: GGE ⇒ completely passive

Key: Positivity of quantum relative entropy

$$S(\rho||\sigma) \coloneqq \mathrm{tr}\left(\rho\left(\log(\rho) - \log(\sigma)\right)\right)$$

Suppose that ρ is the GGE $\left(\rho = e^{-\beta H - \sum_i \mu_i Q_i}/Z\right)$.

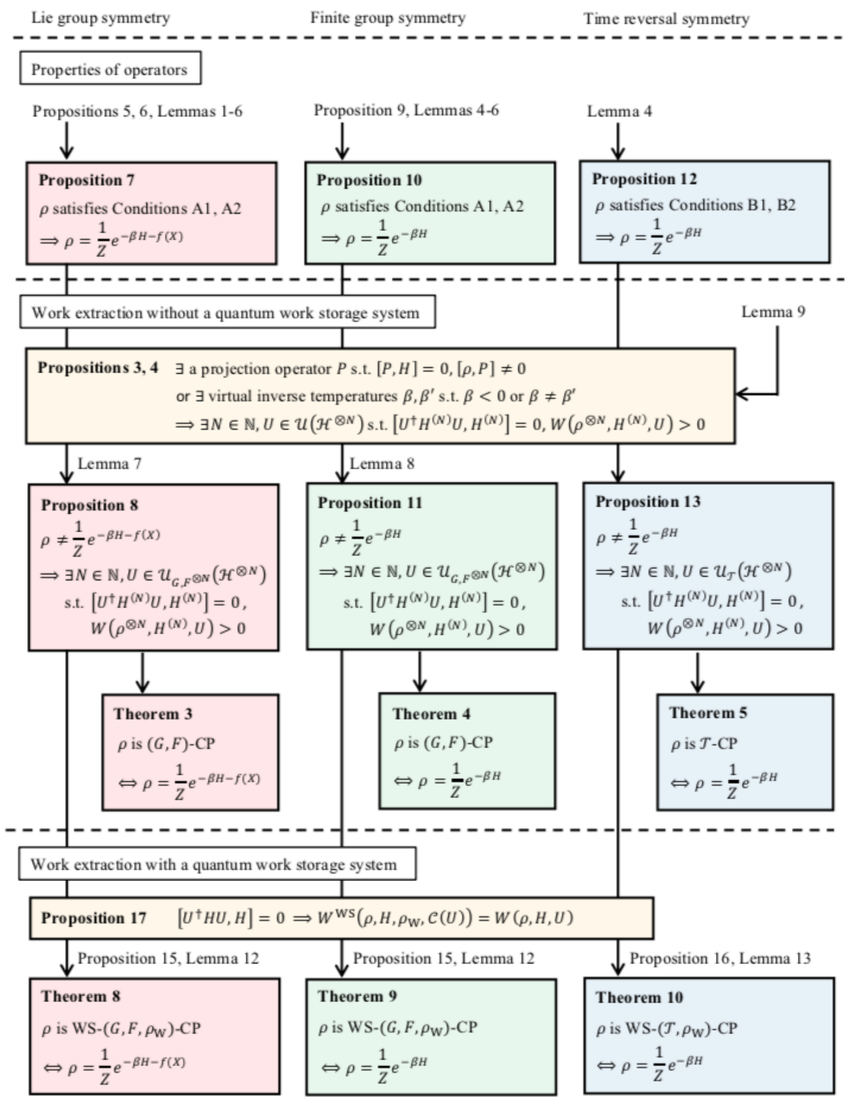
The extracted work from $\rho^{\otimes N}$ by symmetry-respecting unitary U is

$$W = -\beta^{-1} S\left(U\rho^{\otimes N}U^{\dagger}||\rho^{\otimes N}\right) \leq 0$$

➡ ρ is symmetry-protected completely passive.

The converse is far from trivial.

Proof: Completely passive ⇒ GGE



Then, $\mathcal{O}(\{A_{ij}\})$ is unitary. In addition, if $[\rho^{\otimes M}, P] \neq 0$, then $W(\rho^{\otimes 2mM+L}, H^{(2mM+L)}, \mathcal{O}(\{A_{ij}\})) > 0$ for some $m \in \mathbb{N}$.

Proof. First, we prove that $\mathcal{O}(\{A_{ij}\})$ is unitary. From Lemma S9, it is sufficient to prove that $\{A_{ij}\}$ satisfy

$$A_{ij}^\dagger = A_{ji},\ A_{ij}A_{kl} = \delta_{jk}A_{il} \tag{S20}$$

for all $i,j,k,l \in \{0,1\}$. For $i,j \in \{0,1\}$, we define $R_{ij} \in \mathcal{B}(\mathcal{H}^{\otimes 2M})$ as

$$R_{ij} := \frac{1}{2}[I-(-1)^iT][P\otimes(I-P)][I-(-1)^jT]. \tag{S21}$$

Then, $A_{ij} = R_{ij}^{\otimes m} \otimes |\Psi_i\rangle\langle\Psi_j|$. Since $R_{ij}^\dagger = R_{ji}$ and $(|\Psi_i\rangle\langle\Psi_j|)^\dagger = |\Psi_j\rangle\langle\Psi_i|$,

$$A_{ij}^\dagger = (R_{ij}^\dagger)^{\otimes m} \otimes (|\Psi_i\rangle\langle\Psi_j|)^\dagger = R_{ji}^{\otimes m} \otimes |\Psi_j\rangle\langle\Psi_i| = A_{ji}. \tag{S22}$$

Since $\frac{1}{2}[I-(-1)^iT]$ is the antisymmetrizer for $i=0$ and the symmetrizer for $i=1$,

$$\frac{1}{2}[I-(-1)^iT]\cdot\frac{1}{2}[I-(-1)^jT] = \delta_{ij}\frac{1}{2}[I-(-1)^iT]. \tag{S23}$$

From a property of the swapping operator T,

$$[P\otimes(I-P)]T[P\otimes(I-P)] = [P\otimes(I-P)][(I-P)\otimes P]T = 0. \tag{S24}$$

From Eqs. (S23) and (S24), for $i,j,k,l \in \{0,1\}$,

$$\begin{aligned} &R_{ij}R_{kl}\\ =&\frac{1}{2}[I-(-1)^iT][P\otimes(I-P)][I-(-1)^jT]\frac{1}{2}[I-(-1)^kT][P\otimes(I-P)][I-(-1)^lT]\\ =&\frac{1}{2}[I-(-1)^iT][P\otimes(I-P)]\delta_{jk}[I-(-1)^jT][P\otimes(I-P)][I-(-1)^lT]\\ =&\delta_{jk}\frac{1}{2}[I-(-1)^iT][P\otimes(I-P)][I-(-1)^lT]\\ =&\delta_{jk}R_{il}. \end{aligned} \tag{S25}$$

Since $(|\Psi_i\rangle\langle\Psi_j|)(|\Psi_k\rangle\langle\Psi_l|) = \delta_{jk}|\Psi_i\rangle\langle\Psi_l|$, we obtain

$$\begin{aligned} A_{ij}A_{kl} =&(R_{ij}^{\otimes m}\otimes|\Psi_i\rangle\langle\Psi_j|)(R_{kl}^{\otimes m}\otimes|\Psi_k\rangle\langle\Psi_l|)\\ =&(R_{ij}R_{kl})^{\otimes m}\otimes(|\Psi_i\rangle\langle\Psi_j|)(|\Psi_k\rangle\langle\Psi_l|)\\ =&\delta_{jk}(R_{il}^{\otimes m}\otimes|\Psi_i\rangle\langle\Psi_l|)\\ =&\delta_{jk}A_{il}. \end{aligned} \tag{S26}$$

Therefore, $\{A_{ij}\}$ satisfies Eq. (S20). From Lemma S9, $\mathcal{O}(\{A_{ij}\})$ is unitary.

Next, we calculate the extracted work from $\rho^{\otimes 2mM+L}$ by $\mathcal{O}(\{A_{ij}\})$. We define $\Delta\mathcal{E} := \mathcal{E}_1 - \mathcal{E}_0 (> 0)$. Since $[H^{(M)}, P] = 0$ and $[H^{(2M)}, T] = 0$, we obtain $[H^{(2M)}, R_{ij}] = 0$ and thus

$$\begin{aligned} [H^{(2mM+L)}, A_{ij}] =&[H^{(2mM)}\otimes I + I\otimes H^{(L)}, R_{ij}^{\otimes m}\otimes|\Psi_i\rangle\langle\Psi_j|]\\ =&[H^{(2mM)}, R_{ij}^{\otimes m}]\otimes|\Psi_i\rangle\langle\Psi_j| + R_{ij}^{\otimes m}\otimes[H^{(L)}, |\Psi_i\rangle\langle\Psi_j|]\\ =&R_{ij}^{\otimes m}\otimes\Delta\mathcal{E}(i-j)|\Psi_i\rangle\langle\Psi_j|\\ =&\Delta\mathcal{E}(i-j)A_{ij}. \end{aligned} \tag{S27}$$

From Lemma S9,

$$\begin{aligned} &W(\rho^{\otimes 2mM+L}, H^{(2mM+L)}, \mathcal{O}(\{A_{ij}\}))\\ =&\Delta\mathcal{E}(\mathrm{tr}(\rho^{\otimes 2mM+L}A_{11}) - \mathrm{tr}(\rho^{\otimes 2mM+L}A_{00}))\\ =&\Delta\mathcal{E}\left[(\mathrm{tr}(\rho^{\otimes 2M}R_{11}))^m\langle\Psi_1|\rho^{\otimes L}|\Psi_1\rangle - (\mathrm{tr}(\rho^{\otimes 2M}R_{00}))^m\langle\Psi_0|\rho^{\otimes L}|\Psi_0\rangle\right]. \end{aligned} \tag{S28}$$

arXiv:2103.06060

48 pages

Proof for the dimer model

Example: Dimer model

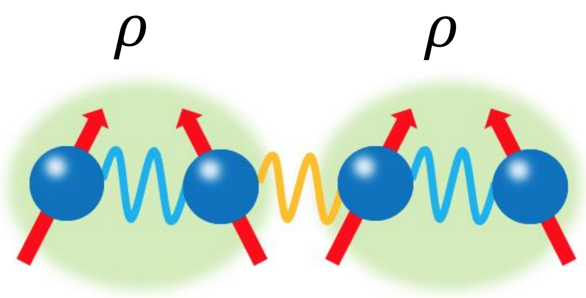
$\rho \neq e^{-\beta H - \sum_i \mu_i Q_i}/Z$ ➡ $[\rho^{\otimes 2}, C] \neq 0$ ➡ $[\rho^{\otimes 2}, P_\omega] \neq 0$

C: Spin inner product between two dimers
P_ω: Projection onto an eigenspace of C

➡ We can extract work from $\rho^{\otimes 4m+3}$ by a unitary

$$U_m = \sum_{k,l \in \{0,1\}} {R_{kl}}^{\otimes m} \otimes |\Phi_k\rangle\langle\Phi_l|$$

for sufficiently large $m \in \mathbb{N}$.



$R_{kl} = [I - (-1)^k T][P_\omega \otimes (I - P_\omega)][I - (-1)^l T]$ T: SWAP of two dimers

$|\Phi_0\rangle = |s\rangle|s\rangle|s\rangle$, $|\Phi_1\rangle = \frac{1}{\sqrt{6}} \sum_{ijk} \epsilon_{ijk} |t_i\rangle|t_j\rangle|t_k\rangle$ $|s\rangle$: Singlet, $|t_i\rangle$: Triplet

Generalization

Spin inner product → Casimir operator

Levi decomposition: Connected compact Lie group = Semisimple Lie group × Abelian Lie group

(e.g. $SU(2)$) (e.g. $U(1)$)

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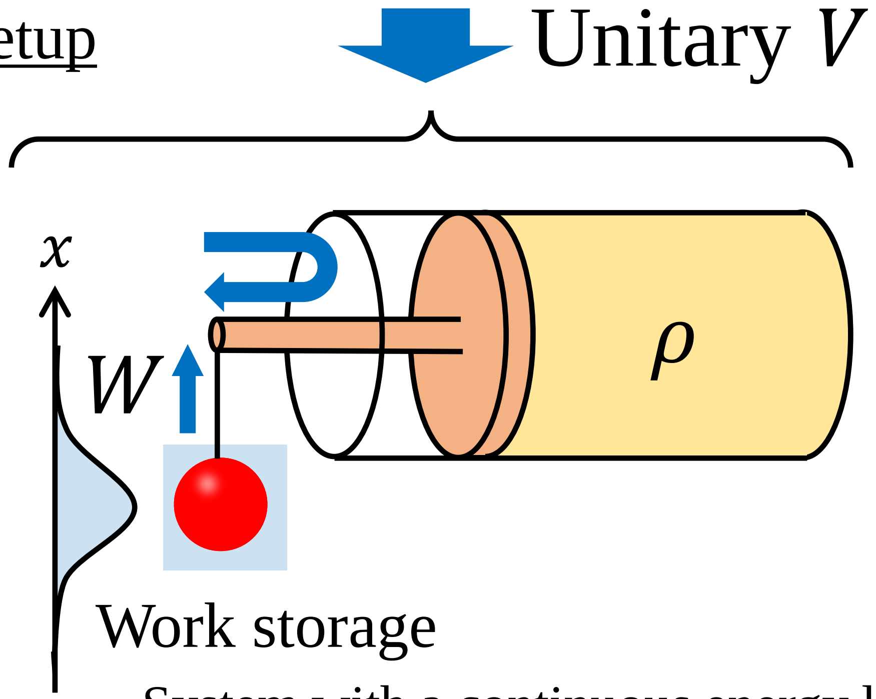
Work storage

P. Skrzypczyk *et al.*, arXiv:1302.2811

<u>Setup</u>



Work storage

System with a continuous energy level
(The Hamiltonian: the position operator)

✓ Quantum-mechanical treatment of work

✓ Energy conservation

$$[V, H \otimes I + I \otimes x] = 0$$

Assumption: $[V, I \otimes p] = 0$

Energy translation symmetry of work storage

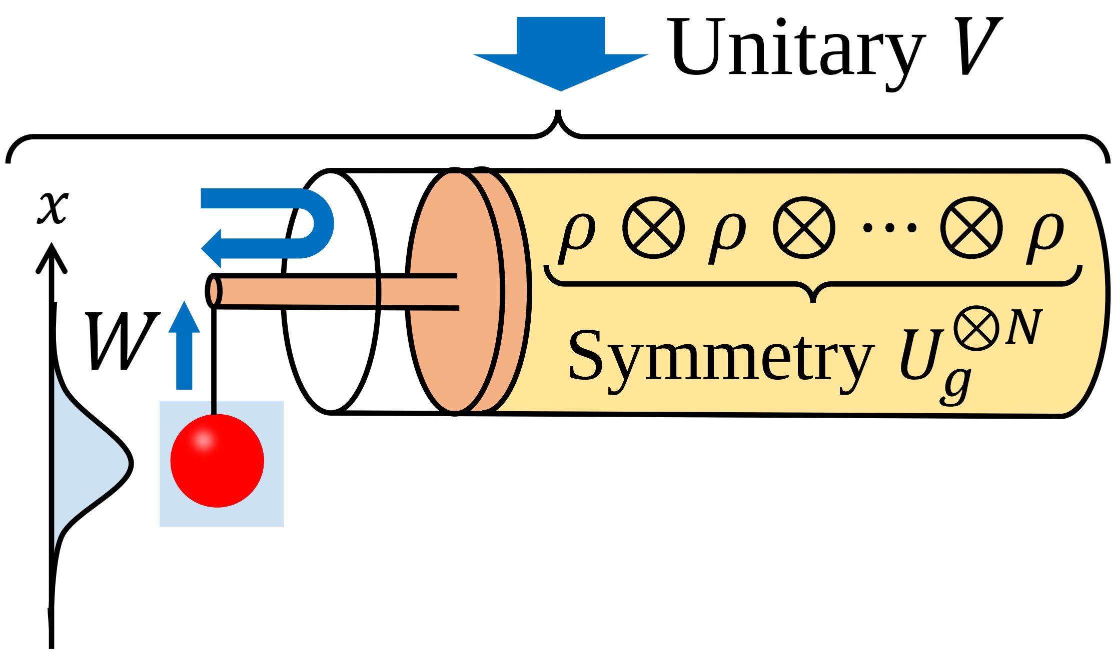
Complete passivity with work storage

Definition

A unitary V is symmetry-respecting

$$\Leftrightarrow [V, U_g^{\otimes N} \otimes I] = 0$$

Conservation of system's charge



Theorem 2. Even in this setup,

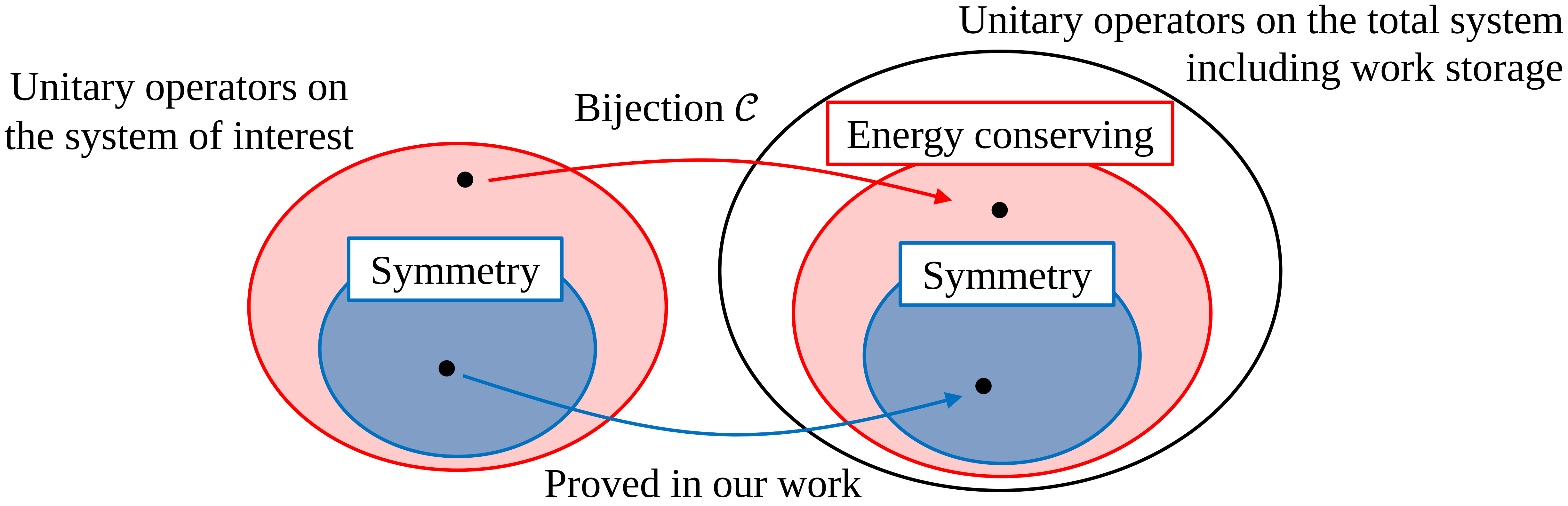
symmetry-protected completely passive $\Leftrightarrow$ GGE

independently of the initial state of the work storage.

Correspondence of unitary operators

$$\mathcal{C}(U) \coloneqq \int_{-\infty}^{\infty} dq\, e^{iqH} U e^{-iqH} \otimes |q\rangle_p \langle q|$$

A. Kitaev, *et. al*., Phys. Rev. A (2004).



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Comparison with the previous work

[1] N. Yunger Halpern *et al.*, Nat. Commun. (2016)

		Definition of work	
		Change in $\langle H \rangle$	Change in $\langle H + \sum_i \xi_i Q_i \rangle$
Charge conservation	Local (Theorem 2)	$e^{-\beta H - \sum_i \mu_i Q_i}/Z$ μ_i's are freely chosen.	$e^{-\beta H - \sum_i \mu_i Q_i}/Z$ μ_i's are freely chosen.
	Global (Ref. [1])	$e^{-\beta H}/Z$	$e^{-\beta \left(H + \sum_i \xi_i Q_i\right)}/Z$

Our result complements their result by providing a further support for the proper form of the non-commutative GGE.

Summary

We characterize thermal equilibrium by complete passivity. arXiv:2103.06060

1. Under continuous symmetry,

Completely passive ⇔ GGE $e^{-\beta H-\sum_i \mu_i Q_i}/Z$

2. We also obtained the same result when we explicitly introduce work storage.

✓ These hold even for non-commutative charges.

➡ Unconventional extension of the GGE.

Future perspectives

We have completely identified thermally stable states under symmetry.

➡ Foundation of quantum thermodynamics under symmetry

More states behave as thermally stable states under symmetry.

➡ Flexible design principles of
quantum heat engines using symmetric systems