On 2d CFT with One Critical Exponent

Sunil Mukhi

IISER PUNE

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- **Based on:**

  “Towards a classification of two-character rational conformal field theories”,
  A. Ramesh Chandra and Sunil Mukhi,
  JHEP 1904 (2019) 153, arXiv:1810.09472.

  “Curiosities above $c = 24$”,
  A. Ramesh Chandra and Sunil Mukhi,
  SciPost 6 (2019), 053, arXiv:1812.05109.

- **And previous work:**

  “On 2d conformal field theories with two characters”,
  Harsha Hampapura and Sunil Mukhi,
  JHEP 1601 (2016) 005, arXiv: 1510.04478.

  “Cosets of meromorphic CFTs and modular differential equations”,
  Matthias Gaberdiel, Harsha Hampapura and Sunil Mukhi,
  JHEP 1604 (2016) 156, arXiv: 1602.01022.

- **And older work:**

  “On the classification of rational conformal field theories”,
  Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
  Phys. Lett. B213 (1988) 303.

  “Reconstruction of CFT from modular geometry on the torus”,
  Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
  Nucl. Phys. B318 (1989) 483.

Outline

1 Introduction

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3 Two-character CFT

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- 2d CFTs play multiple roles in Physics:
  - Critical statistical systems
  - String world-sheet theory
  - Boundary theory dual to bulk gravity
  - Topological quantum computing
- Their spectrum has the following structure:

$$\text{primaries } \phi_i, \quad \text{dimensions } (h_i, \bar{h}_i)$$
$$\text{secondaries } \mathcal{W}_{-n,-\bar{n}}\,\phi_i, \quad \text{dimensions } (h_i + n, \bar{h}_i + \bar{n})$$

  where $\mathcal{W}_{-n,-\bar{n}}$ stands for arbitrary products of negative modes of the spin-1, spin-2, spin-3 $\cdots$ chiral fields that generate the symmetry algebra.
- Defining $q = e^{2\pi i\tau}$, the partition function:

$$Z(\tau, \bar{\tau}) = \operatorname{tr} q^{L_0 - \frac{c}{24}}\,\bar{q}^{\bar{L}_0 - \frac{c}{24}}$$

  counts the number of primaries and secondaries.

- For consistency, the partition function must be modular invariant:

$$Z(\gamma\tau, \gamma\bar{\tau}) = Z(\tau, \bar{\tau})$$

where:

$$\gamma\tau \equiv \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL(2,Z)}$$

- The modern modular bootstrap programme [Hellerman 2009, Friedan-Keller 2013 etc] proposes to constrain possible 2d CFT by just imposing the above condition. These works focus on CFT's with a semi-classical AdS dual (large c, sparse spectrum).
- The modular bootstrap in fact originated much earlier in [Mathur-Mukhi-Sen, 1988] where the goal was to classify and construct CFT's with a small number of critical exponents (primary fields).

- Modern-day physics motivations for such theories:
  - Interesting for statistical physics: very few primary deformations, and if $(h_i, \bar{h}_i) > 1$ then theory tends to be more stable (perfect metals, [Plamadeala-Mulligan-Nayak 2014]).
  - Useful for string compactifications because potentially have smaller number of moduli (e.g. Gepner models).
  - Relevant for topological quantum computing (e.g. [Freedman-Kitaev-Larsen-Wang 2003, Tener-Wang 2017]). The relation involves non-Abelian anyons, fractional quantum Hall systems and unitary modular tensor categories.
  - Still might be relevant for a quantum/stringy version of $\mathrm{AdS}_3/\mathrm{CFT}_2$.
- They are also extremely interesting to mathematicians.

- In this talk I will deal with Rational CFT having one critical exponent h. They can have one or more non-trivial primary fields ϕ with the same conformal dimension.
- Using the MMS approach to modular bootstrap, one can classify and construct (not just constrain) theories.
- Recently, in [arXiv:1810.09472] we have classified all possible characters for such theories, for the first time.
- Thereafter, in [arXiv:1812:05109] we showed that large numbers of such characters actually do correspond to CFT's.
- We explicitly constructed several completely new CFT's with a single critical exponent.

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- Theories with a finite number of primaries are called Rational Conformal Field Theories (RCFT):

$$Z(\tau,\bar\tau)=\sum_{i=0}^{p-1}|\chi_i(\tau)|^2$$

- $\chi_i(\tau)$ is the character for a given primary ϕ_i:

$$\chi_i(q)=\mathrm{tr}_i\, q^{L_0-\frac{c}{24}}$$

  where tr_i is over all holomorphic descendants $\mathcal{W}_{-n}\phi_i$.
- The characters take the form:

$$\chi_i(q)=q^{-\frac{c}{24}+h_i}(a_0^i+a_1^i q+a_2^i q^2+\cdots)$$

  where the a_n^i are non-negative integer degeneracies.
- Characters are holomorphic in the interior of moduli space but can diverge on the boundary $\tau\to i\infty$.

- For the partition function to be modular-invariant, the characters must be vector-valued modular functions:

$$\chi_i\,(\gamma\tau) = \sum_{j=0}^{p-1} M_{ij}(\gamma)\chi_j(\tau), \quad \gamma \in \mathrm{SL(2,Z)}$$

  with $M^\dagger M = 1$.
- From the work of [Belavin-Polyakov-Zamolodchikov (1984)] and generalisations, we know many examples of such RCFT's including their characters and correlation functions. They possess null vectors and fall into minimal series.
- In this approach we have to first define the chiral algebra. Also, in each minimal series the number of critical exponents quickly grows, so the theories may be less physically interesting.
- As alternate approach is to classify CFT by their number of characters (= number of exponents +1). This has already yielded many novel insights.

- To classify RCFT by their characters, one must first fix a number ≥ 1 of characters.
- Then, there are two problems to be solved:
    - Problem (I): Find all possible characters with modular invariance and positive integrality of the q-series ("admissible").
    - Problem (II): Find which of these really corresponds to a CFT.
- If we want to be fashionable we could say that those characters satisfying (I) could lie in the swampland unless they are shown to satisfy (II)! (Analogy not to be taken too seriously.)
- I will now describe how each of these problems is addressed, first very briefly for one character (= meromorphic CFT) and then for two characters (= one critical exponent).

- In the one-character case, the partition function has the form:

$$Z(\tau, \bar{\tau}) = |\chi(\tau)|^2$$

For this to be modular-invariant, $\chi(\tau)$ has to be modular invariant upto a phase.

- It is a well-known mathematical fact that this is only possible if χ is a function of the Klein j-invariant:

$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

- Requiring non-negative integer coefficients puts strong restrictions: we must have specific fractional powers of j times a polynomial. This implies $c = 8n$ for some integer n.
- For example:

$$
\begin{array}{lll}
c = 8: & \chi = j^{\frac{1}{3}} & E_8 \quad \text{(unique)} \\
c = 16: & \chi = j^{\frac{2}{3}} & E_8 \times E_8,\ \mathrm{Spin}_{32}/\mathrm{Z}_2 \\
c = 24: & \chi = j + \mathcal{N} & \text{free boson, Niemeier lattice} \\
c = 32: & \chi = j^{\frac{1}{3}}(j + \mathcal{N}) & \text{free boson, even unimodular 32d lattice}
\end{array}
$$

- All these examples correspond to c free bosons compactified on a torus $\mathbb{R}^c/\Gamma$, where Γ is an even, unimodular lattice – but there are more general possibilities when $c \geq 24$.
- In 1988, Peter Goddard labelled such theories as "meromorphic CFT".

- We see that from $c = 24$ onwards, there are undetermined integer parameters consistent with modular invariance.
- However not all values lead to genuine CFT.
- For example at $c = 24$, there are only 24 even unimodular lattices and a finite number of generalisations involving orbifolding etc [Schellekens (1992)], bringing the total number of theories to 71.
- The characters of these 71 theories are all of the form $j + \mathcal{N}$ with just 30 distinct values of $\mathcal{N}$. For all other values of $\mathcal{N}$ there seem to be no consistent CFT.

- Thus the status of Problems (I) and (II) for one-character (meromorphic) CFT is as follows.
- Problem (I) was effectively solved by Klein in the 19th century by discovering the j-invariant.
- But to this day, Problem (II) is solved only for $c \leq 24$.
- At $c = 32$ there are already around 10^{10} even unimodular lattices. By compactifying free bosons on the associated torus, each of these determines a meromorphic CFT.
- But there is very likely a larger number of orbifold and other generalised theories.

- A hypothetical class of one-character theories ("extremal") was famously proposed in [Witten (2007)] to be dual to pure gravity in AdS_3.
- This led to a controversy (still not settled as far as I know) about the existence of "extremal" one-character CFT at large central charge. I will return to one of the arguments below.
- It now seems that Witten's original motivation (to find RCFT dual to semi-classical Einstein gravity) may not be in the right direction.
- Still, understanding the space of one-character CFT at $c > 24$ is a difficult and interesting open problem.

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- For two-character theories, we need to classify all pairs:

$$\chi_0(q) = q^{-\frac{c}{24}}\left(1 + a_1^0\, q + a_2^0\, q^2 + \cdots\right)$$
$$\chi_1(q) = q^{-\frac{c}{24}+h}\left(a_0^1 + a_1^1\, q + a_2^1\, q^2 + \cdots\right)$$

  that transform into a linear combination of themselves under modular transformations. Here h is the critical exponent and $a_n^{(i)} \in \mathbb{Z}^* \equiv \mathbb{Z}^+ \cup \{0\}$.
- This was first addressed in [Mathur-Mukhi-Sen (1988)].
- Key insight:
  - The partition function is modular invariant, but not holomorphic.
  - The characters are holomorphic, but not modular invariant.
  - However they solve a **modular linear differential equation (MLDE)** that is **both** holomorphic **and** modular invariant. This is very restrictive.

- Here is a proof. If χ_0, χ_1 are two characters and χ is an arbitrary linear combination of them, then:

$$\begin{vmatrix} \chi_0 & \chi_1 & \chi \\ \mathcal{D}\chi_0 & \mathcal{D}\chi_1 & \mathcal{D}\chi \\ \mathcal{D}^2\chi_0 & \mathcal{D}^2\chi_1 & \mathcal{D}^2\chi \end{vmatrix} = 0, \qquad \text{where } \mathcal{D} \equiv \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{k}{12}E_2(\tau)$$

- Expanding by the last column gives a 2nd order linear differential equation for χ:

$$\begin{vmatrix} \chi_0 & \chi_1 \\ \mathcal{D}\chi_0 & \mathcal{D}\chi_1 \end{vmatrix} \mathcal{D}^2\chi - \begin{vmatrix} \chi_0 & \chi_1 \\ \mathcal{D}^2\chi_0 & \mathcal{D}^2\chi_1 \end{vmatrix} \mathcal{D}\chi + \begin{vmatrix} \mathcal{D}\chi_0 & \mathcal{D}\chi_1 \\ \mathcal{D}^2\chi_0 & \mathcal{D}^2\chi_1 \end{vmatrix} \chi = 0$$

- This can be rewritten in monic form:

$$\mathcal{D}^2\chi + \phi_2(\tau)\mathcal{D}\chi + \phi_4(\tau)\chi = 0$$

where ϕ_2, ϕ_4 are meromorphic in τ (due to possible zeroes of the first det) and of modular weight $2, 4$ respectively.

- For two-character theories it can be shown that the number of zeroes is $\frac{\ell}{6}$ where $\ell = 0, 2, 4, \cdots$. The fractional number is due to the orbifold nature of the torus moduli space.
- For any fixed number of poles $\frac{\ell}{6}$, there is a finitely generated ring of modular functions.
- So without knowing χ_0, χ_1, we can parametrise these functions in terms of known modular forms (Eisenstein series) with arbitrary real coefficients.
- For example, at $\ell = 0$ the most general MLDE is given by:

$$\begin{aligned} \ell = 0: \qquad & \phi_2(\tau) = 0 \\ & \phi_4(\tau) = \mu E_4(\tau) \end{aligned}$$

$$\implies \quad \mathcal{D}^2\chi + \mu E_4\,\chi = 0 \qquad \text{(MMS equation)}$$

- For higher values of ℓ the MLDE has more and more free parameters. For example at $\ell = 2$ we have:

$$\ell = 2: \qquad \phi_2(\tau) = \mu_1 \frac{E_6(\tau)}{E_4(\tau)}$$
$$\phi_4(\tau) = \mu_2 E_4(\tau)$$
$$\Longrightarrow \qquad \mathcal{D}^2\chi + \mu_1 \frac{E_6}{E_4} \mathcal{D}\chi + \mu_2 E_4\, \chi = 0$$

- Note that, if we assume an MLDE that is holomorphic when expressed in monic form, then we are assuming $\ell = 0$. This has caused some confusion in the literature.

- The Riemann-Roch theorem gives an important relation between the central charge c, the conformal dimension h and the integer ℓ labelling singularities of the equation:

$$-\frac{c}{12} + h = \frac{1-\ell}{6}$$

- For a unitary theory with positive c, h this implies that:

$$c + 2 > 2\ell$$

so theories with large ℓ must have a large central charge.

- For any values of the coefficients μ_i, solutions of the differential equation are vector-valued modular functions, and have an expansion of the form:

$$\chi_i(\tau) = q^{-\frac{c}{24}+h_i}(a_0^i + a_1^i q + a_2^i q^2 + \cdots)$$

where we identify $h_0 = 0, h_1 = h$.

- But we want admissible characters, i.e. those that have non-negative integer coefficients a_n^i.
- The a_n^i are rational functions of the parameters in the equation (e.g. μ). The methodology to find admissible characters is then:

  (i) Vary the parameters μ_i of the equation until the first few coefficients a_n^i are non-negative integers.

  (ii) Verify that the a_n^i continue to be non-negative integers to very high orders in q. Then we have an "admissible character".

- Thus, Problem (I) for two-character CFT becomes: what are all the admissible characters for $\ell = 0, 2, 4, 6, \cdots$?
- After solving this, we can turn to Problem (II) – to find out which ones correspond to actual CFT.
- Until 2018, the only studied cases were:
  - $\ell = 0$ [Mathur-Mukhi-Sen (1988)],
  - $\ell = 2$ [Naculich (1989), Hampapura-Mukhi (2015), Gaberdiel-Hampapura-Mukhi (2016)],
  - $\ell = 4$ [Tener-Wang (2016)].

	$\ell = 0$ (WZW)				$\ell = 2$ (KM, but not WZW)			
No.	c	h	a_1^0	KM Algebra	$\tilde{c}$	$\tilde{h}$	$\tilde{a}_1^0$	KM Algebra
1	1	$\frac{1}{4}$	3	$A_{1,1}$	23	$\frac{7}{4}$	69	$(A_{1,1})^{23}, \cdots$
2	2	$\frac{1}{3}$	8	$A_{2,1}$	22	$\frac{5}{3}$	88	$(A_{2,1})^{11}, \cdots$
3	$\frac{14}{5}$	$\frac{2}{5}$	14	$G_{2,1}$	$\frac{106}{5}$	$\frac{8}{5}$	106	$E_{6,3} \oplus G_{2,1}, \cdots$
4	4	$\frac{1}{2}$	28	$D_{4,1}$	20	$\frac{3}{2}$	140	$(D_{4,1})^5, \cdots$
5	$\frac{26}{5}$	$\frac{3}{5}$	52	$F_{4,1}$	$\frac{94}{5}$	$\frac{7}{5}$	188	$C_{8,1} \oplus F_{4,1}, \cdots$
6	6	$\frac{2}{3}$	78	$E_{6,1}$	18	$\frac{4}{3}$	234	$(E_{6,1})^3, \cdots$
7	7	$\frac{3}{4}$	133	$E_{7,1}$	17	$\frac{5}{4}$	323	$D_{10,1} \oplus E_{7,1}, \cdots$
8	8	–	248	$E_{8,1}$	16	–	496	$E_{8,1} \oplus E_{8,1}$

Table: CFT with $\ell = 0$ and $\ell = 2$.

C. R. Acad. Sci. Paris, t. 322, Série I, p. 321-326, 1996
Algèbres de Lie/*Lie Algebra*

La série exceptionnelle de groupes de Lie

Pierre DELIGNE

School of Mathematics, Institute for Advanced Study,
Princeton, NJ 08540, USA.
e-mail: deligne@math.ias.edu

Résumé. Numérologie des groupes exceptionnels et une interprétation conjecturale.

The exceptional series of Lie groups

Abstract. *Numerology of exceptional Lie groups and a conjectural explanation.*

Soit G^0 le groupe déployé adjoint de l'un des types suivants : A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8. On fixe un épinglage de G^0. On note G le groupe des automorphismes de G^0. Pour Γ le groupe des

Remarkably the Kac-Moody algebras appearing in the $\ell = 0$ series are in 1-1 correspondence with a special set of Lie algebras whose properties were noted by [Deligne (1996)].

- In each of these cases there is a finite set of admissible characters.
- For $\ell = 0, 2$ each set has been completely identified with actual RCFT.
- Also, we found a novel coset relation between each $\ell = 0$ theory and a corresponding $\ell = 2$ theory, with $c + \tilde{c} = 24, h + \tilde{h} = 2$.
- Thus both Problems (I) and (II) are solved for $\ell = 0, 2$.
- Only Problem I is solved for $\ell = 4$. There are just three irreducible new sets of characters, but so far no one has been able to associate them to CFT.
- But until recently, nothing was known about $\ell \geq 6$.

- The literature has had some suggestions/claims (and one "proof") that only $\ell = 0$ is allowed, or only low values of ℓ are allowed (other than tensor products).
- But it was shown in [Harvey-Wu (2018)], using Hecke operators, that it is quite easy to construct admissible pairs of characters for generically large ℓ. Their method is rather complicated and they made no claim of completeness.
- In [Chandra-Mukhi (2018)] we have shown by a different method that, starting from every $\ell \geq 6$, there are infinitely many admissible pairs of characters, and we have provided a complete construction of all of them.
- This solves Problem (I) for all 2d CFT with one critical exponent.

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Quasi-characters and $\ell \geq 6$

- The strategy we used to solve Problem (I) for all even $\ell \geq 6$, is based on a series of works by mathematicians: [Kaneko, Zagier, Koike].
- [Kaneko-Zagier (1998)] studied a 2nd order MLDE which, after a simple transformation, is the same as the MMS equation for $\ell = 0$ CFT:

$$\Big(\mathcal{D}^2 + \mu E_4(\tau)\Big)\chi = 0$$

- When this equation was studied by MMS, only solutions with non-negative integer q-series were retained. There are finitely many, all lying in the range $0 < c < 8$.
- Remarkably, if we relax the assumption of non-negativity then we get infinitely many integral solutions.

- To see this, note first that all possible fusion classes were classified for two-character theories in [Christe-Ravanini (1989), Mathur-Sen (1989)] and they are of four types: Lee-Yang, A_1, A_2, D_4.
- Now choosing the parametrisation $\mu = -\frac{c(c+4)}{576}$ in the MMS equation, Kaneko et al studied the following rational values of c, where n is an integer:

$$
\begin{array}{ll}
c = 6n+1, & A_1 \text{ class} \\
c = 4n+2, \ n \neq 2 \bmod 3 & A_2 \text{ class} \\
c = 8n+4 & D_4 \text{ class} \\
c = \frac{2(6n+1)}{5}, \ n \neq 4 \bmod 5 & \text{Lee-Yang class}
\end{array}
$$

- For those values of c in the above list that also satisfy $0 \leq c \leq 8$, the solutions are precisely the ones of MMS. They are admissible characters that correspond to genuine CFT's.

- For all the remaining (infinitely many) values of c in the above list one still finds integer degeneracies, but some of them are negative.
- We call such solutions quasi-characters. There is precisely one for each c in the list.
- Example: for the $c = 6n + 1$ series with $n = 4$, the "identity" quasi-character looks like:

$$\chi_0 = q^{-\frac{25}{24}}(1 - 245q + 142640q^2 + 18615395q^3 + 837384535q^4 + \cdots)$$

  and all higher coefficients are positive.
- Using the works of [Kaneko et al] we were able to classify all quasi-characters with $\ell = 0$. They exhibit two types of behaviour depending on the value of c:
  - Type I have finitely many negative signs, and then asymptote to positive integers.
  - Type II have finitely many positive signs, and then asymptote to negative integers.

- Such quasi-characters cannot directly describe a CFT since they are not admissible: what sense does a degeneracy of -245 make?
- However we showed that quasi-characters with $\ell = 0$ are building blocks for all admissible characters with $\ell = 6p$ for every positive integer p. The latter are obtained as linear combinations with integer coefficients.
- We also constructed quasi-characters for $\ell = 2, 4$ and showed that these are building blocks for admissible characters with $\ell = 6p + 2, 6p + 4$ respectively, thus exhausting all even ℓ.
- Due to time constraints I will only discuss the $\ell = 6p$ cases in this talk.

- Let us see how this works in a simple example. We add a pair of quasi-characters in a given fusion class to each other, chosen such that their value of c differs by 24.
- Such addition is consistent, because when c jumps by 24, the quasi-characters transform in the same way under modular transformations.
- By the Riemann-Roch theorem:

$$-\frac{c}{12}+h=\frac{1-\ell}{6}$$

  the h value of these two will differ by 2 units.
- Thus, if one of them is labelled by (c,h) then the other is labelled by $(c+24,h+2)$.
- Let us choose the former character to be admissible and the latter to be a Type I quasi-character with a single negative coefficient.

- Thus the behaviour of the sum is given by:

$$\chi_0 = q^{-\frac{c}{24}-1}(1 - \cdots) + \mathcal{N}_1\, q^{-\frac{c}{24}}(1 + \cdots)$$
$$\chi_1 = q^{-\frac{c}{24}+h+1}(1 + \cdots) + \mathcal{N}_1\, q^{-\frac{c}{24}+h}(1 + \cdots)$$

- From the leading power of q in each of these, we find that these characters correspond to a central charge $c + 24$ and dimension $h + 1$.
- Applying Riemann-Roch again, we find that the added quasi-characters have $\ell = 6$.
- Moreover, choosing $\mathcal{N}_1$ suitably we can cancel the negative term, leading to an admissible character.
- If we start with a Type I quasi-character having multiple negative values then we need to add several terms to get an admissible character.

- The algorithm to construct an admissible character is then:
  - (i) First pick a quasi-character for a particular central charge and having finitely many negative degeneracies.
  - (ii) To it, add some more quasi-characters in the same class. Adjust coefficients such that the result is admissible (all negative signs cancelled).
- We have proved that this procedure is complete: every set of characters with $\ell = 6$ is obtained as a sum of $\ell = 0$ quasi-characters.

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- We now turn to Problem (II): given these new infinite families of admissible characters, which of them are actual CFT?
- We address the case of $\ell = 6$. This is the first value for which an infinite family of admissible characters arose.
- This is somewhat reminiscent of the meromorphic case at $c = 24$ (which also has $\ell = 6$, in fact).
- A complete list of admissible characters for $\ell = 6$ is given on the next page. They all have:

$$24 < c < 32$$

No.	c	h	Character sum
1	$\frac{122}{5}$	$\frac{6}{5}$	$\chi_{LY}^{n=10} + \mathcal{N}_1 \chi_{LY}^{n=0}$
2	25	$\frac{5}{4}$	$\chi_{A_1}^{n=4} + \mathcal{N}_1 \chi_{A_1}^{n=0}$
3	26	$\frac{4}{3}$	$\chi_{A_2}^{n=6} + \mathcal{N}_1 \chi_{A_2}^{n=0}$
4	$\frac{134}{5}$	$\frac{7}{5}$	$\chi_{LY}^{n=11} + \mathcal{N}_1 \chi_{LY}^{n=1}$
5	28	$\frac{3}{2}$	$\chi_{D_4}^{n=2} + \mathcal{N}_1 \chi_{D_4}^{n=0}$
6	$\frac{146}{5}$	$\frac{8}{5}$	$\chi_{LY}^{n=12} + \mathcal{N}_1 \chi_{LY}^{n=2}$
7	30	$\frac{5}{3}$	$\chi_{A_2}^{n=7} + \mathcal{N}_1 \chi_{A_2}^{n=1}$
8	31	$\frac{7}{4}$	$\chi_{A_1}^{n=5} + \mathcal{N}_1 \chi_{A_1}^{n=1}$
9	$\frac{158}{5}$	$\frac{9}{5}$	$\chi_{LY}^{n=13} + \mathcal{N}_1 \chi_{LY}^{n=3}$

Table: $\ell = 6$ pairs obtained by addition of quasi-characters

- Though there are only 9 rows in the table, each one has infinitely many pairs of characters due to the free integer $\mathcal{N}_1$. Do any of these correspond to actual CFT?
- Our proposed method to construct CFT's starts by looking at even, unimodular lattices with $c = 32$ [Chandra-Mukhi (2018)].
- As mentioned earlier, there are more than 10^{10} of them. But 132 of these are special. They have complete root systems and are called Kervaire lattices.
- Now in [Gaberdiel-Hampapura-Mukhi (2016)] we discovered a novel coset construction where, in particular, one can divide a meromorphic CFT by a class of WZW models at level 1.
- Such WZW models have $\ell = 0$. If they also have two characters then one can show that the quotient is a two-character CFT with:

$$\ell = \frac{c}{2} - 10$$

- Thus if $c = 32$ then the coset theory has $\ell = 6$.
- So we take the coset of a Kervaire lattice CFT, having $c = 32$, by any of the WZW theories falling in the MMS series, which all have $\ell = 0$.
- The result has $\ell = 6$, and moreover has a definite value of $\mathcal{N}_1$ for its characters.
- Thus each coset gives a fixed value of the coefficient $\mathcal{N}_1$ in the table and assures that a CFT exists for that $\mathcal{N}_1$.
- In this way we can find one or more CFT's for every Kervaire lattice.

- Let us illustrate this using a simple example: a 32-dimensional lattice having the complete root system A_2^{16}.
- Its root lattice is not unimodular, but one can extend it to an even unimodular lattice Γ by adding in a few vectors from the dual lattice of A_2^{16}.
- Scalar field theory on the torus $\mathbf{C}^{32}/\Gamma$ defines a unique $c = 32$ meromorphic CFT with $A_{2,1}^{16}$ as its Kac-Moody algebra.
- The number of spin-1 currents is the dimension of the algebra, which is 128.

- We can write the single character of this theory as a non-diagonal modular invariant combination of the affine characters of $A_{2,1}^{16}$.
- These are of the form $\chi_0^p \chi_1^{16-p}$ where χ_0, χ_1 are the $A_{2,1}$ characters. They have conformal dimensions

$$m_i = \tfrac{16-p}{3} = 0, \tfrac{1}{3}, \tfrac{2}{3}, 1, \cdots, \tfrac{14}{3}, 5, \tfrac{16}{3}$$

- Denoting these by $\boldsymbol{\chi}_{m_i}$, the modular invariant (upto a phase) combination of these characters is easily found to be:

$$\begin{aligned}\chi(\tau) &= \boldsymbol{\chi}_0 + 224\boldsymbol{\chi}_2 + 2720\boldsymbol{\chi}_3 + 3360\boldsymbol{\chi}_4 + 256\boldsymbol{\chi}_5 \\ &= j(\tau)^{\frac{1}{3}}(j(\tau) - 864)\end{aligned}$$

- Since this $c = 32$ meromorphic theory has $A_{2,1}^{16}$ as its Kac-Moody algebra, we can coset it by the $\ell = 0$ two-character $A_{2,1}$ affine theory, to get a new $\ell = 6$ two-character CFT with $A_{2,1}^{15}$ as its symmetry.
- The affine $A_{2,1}$ theory has $c = 2,\ h = \frac{1}{3}$.
- Hence the coset theory has $\tilde{c} = 30$ and $\tilde{h} = \frac{5}{3}$.
- Its characters must be linear combinations of $\chi_0^p \chi_1^{15-p}$ whose dimensions are $m_i = \frac{15-p}{3}$. These combinations turn out to be:

$$\begin{aligned}
\tilde{\chi}_0(\tau) &= \boldsymbol{\chi}_0 + 140\boldsymbol{\chi}_2 + 1190\boldsymbol{\chi}_3 + 840\boldsymbol{\chi}_4 + 16\boldsymbol{\chi}_5 \\
\tilde{\chi}_1(\tau) &= 42\boldsymbol{\chi}_{\frac{5}{3}} + 765\boldsymbol{\chi}_{\frac{8}{3}} + 1260\boldsymbol{\chi}_{\frac{11}{3}} + 120\boldsymbol{\chi}_{\frac{14}{3}}
\end{aligned}$$

- Now we know more than just the characters and partition function! In fact for all such theories we can use methods of [Mathur-Mukhi-Sen (1989)] to compute correlation functions on the plane and torus. So the CFT is fully defined.

- One can construct many more (over 100) two-character CFT's with $\ell = 6$ in this way.
- But we do not have a complete list of $\ell = 6$ CFT, and we never will because there is no complete list of $c = 32$ meromorphic CFT.
- Still, given a lattice CFT with a complete root system, we can coset it in one or more ways by an $\ell = 0$ CFT and obtain large classes of theories with various ℓ.
- For lattices with incomplete root systems, things are more complicated and not yet worked out.

1. Introduction
2. RCFT basics
3. Two-character CFT
4. Quasi-characters and $\ell \geq 6$
5. $\ell = 6$ CFT
6. Conclusions and Outlook

- A long-standing problem, to find all admissible vector-valued modular forms of rank p, has now been solved for $p = 2$.
- Previously it had been solved only for $p = 1$, with rather striking consequences for theoretical physics related to Monster symmetry, 3d gravity etc.
- We saw that for both $p = 1$ and 2, $\ell < 6$ turns out to be extremely non-generic and gives rise to finite families of admissible characters. Infinite families start to appear from $\ell = 6$ onwards.

- We did not actually use MLDE to classify $\ell \geq 6$ characters! Our method just uses $\ell = 0$ MLDE to construct quasi-characters and then builds characters from them.
- We have settled the debate about whether two-character CFT with $\ell \geq 6$ do exist, and provided a method to construct examples of such theories for $\ell = 6$ using cosets of even, unimodular lattices.
- Our method can be extended to $\ell \geq 6$.

- For rank 3, the $\ell = 0$ case was studied in [Mathur-Mukhi-Sen (1989)], but virtually nothing is known about admissible characters or actual CFT's with $\ell > 0$. The methods discussed here can very likely be applied to that case.
- Since c is bounded below by ℓ, theories with arbitrarily large ℓ have large c. This might be interesting for holography.
- Few-character CFT with superconformal invariance might provide interesting (and solvable) world-sheet theories for superstrings.