Towards a C-theorem in defect CFT

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Introduction

Let us consider a RG flow triggered by a relevant operator $\mathcal{O}$ of dimension $\Delta < d$,

$$I_{\text{CFT}} + \lambda \int \mathrm{d}^d x \sqrt{g}\, \mathcal{O}(x)$$

We are interested in a monotonic decreasing function under the RG flow.

C-function

- The C-function counts the effective degrees of freedom.
- The monotonicity provides nonperturbative constraints on the RG dynamics.

C-theorem

We want to generalize a C-theorem by adding boundary or defect.

Even dimensions

C-function $=$ the type A central charge of the conformal anomaly

- $d = 2$: Zamolodchikov's c-theorem
- $d = 4$: a-theorem

Odd dimensions (no conformal anomalies)

C-function $=$ the sphere free energy $F \equiv (-1)^{\frac{d-1}{2}} \log Z[\mathbb{S}^d]$

The conjecture has been extended to continuous d dimensions.

Generalized F-theorem

$\tilde{F} \equiv \sin\left(\frac{\pi\, d}{2}\right) \log Z[\mathbb{S}^d]$ is positive and does not increase along any RG flow

This is the most general C-theorems proposed in arbitrary dimensions.

Entanglement entropy

$$S^{(\mathrm{CFT})} = \frac{A_{d-2}}{\epsilon^{d-2}} + \frac{A_{d-4}}{\epsilon^{d-4}} + \cdots + \begin{cases} a_{\log} \log\left(\frac{R}{\epsilon}\right) \ , & (d = \text{even}) \\ a_0 \ , & (d = \text{odd}) \end{cases}$$

(R : typical length scale, ϵ : UV cutoff scale)

- Universal terms $a_{\log}, a_0$ are conjectured to be C-functions.
- This entropic version of the C-theorem looks different from the generalized F-theorem based on the sphere free energy.
- But the two C-function are the same at the fixed points due to the relation

$$S^{(\mathrm{CFT})} = \log Z^{(\mathrm{CFT})} \qquad \text{for spherical entangling surface}$$

Free energy and EE are C-functions!

without boundary or boundary

C-theorem in BCFTs & DCFTs

g-theorem (C-theorem in BCFT$_2$)

$$S_{\text{thermal}} = \frac{c\pi}{3}\frac{L}{\beta} + \boxed{\log g} \quad g\text{-function}$$

(c : central charge, L : size, β : inverse temperature)

- $\log g$ monotonically decreases under a boundary RG flow.
- The g-theorem can also be proved by using the equivalence of the g-function and the boundary entropy $S_{\text{bdy}} := S^{(\text{BCFT})} - \frac{1}{2}S^{(\text{CFT})}$.

b-theorem (C-theorem in BCFT$_3$ & DCFT$_d$ with 2-dim defect)

$$\langle\, t^{\mu}{}_{\mu}\,\rangle = -\frac{1}{24\pi}\left[\boxed{b}\,\hat{\mathcal{R}} + d_1\,\tilde{\mathcal{K}}^{(\alpha)}_{ab}\tilde{\mathcal{K}}^{(\alpha)\,ab} + d_2\,W_{abcd}\,\hat{g}^{ac}\,\hat{g}^{bd}\right]\delta^{d-2}(x_\perp)$$

- When $d = 3$, the b-theorem implies the g-theorem in BCFT$_3$.
- For $d > 3$ it yields a class of C-theorems in DCFTs.

Towards a C-theorem in DCFT

We want to establish a C-function under a defect RG flow,

$$I = I_{\mathrm{DCFT}} + \hat{\lambda} \int \mathrm{d}^p \hat{x} \, \sqrt{\hat{g}} \, \hat{\mathcal{O}}(\hat{x})$$

Two possibilities

1. Defect free energy : $\log \langle \mathcal{D}^{(p)} \rangle = \log Z^{(\mathrm{DCFT})} - \log Z^{(\mathrm{CFT})}$

   $=$ additional contribution to the sphere free energy from the spherical defect

2. Defect entropy : $S_{\mathrm{defect}} = S^{(\mathrm{DCFT})} - S^{(\mathrm{CFT})}$

   $=$ increment of the EE across a sphere due to the planer defect

Question

Which is a C-function?

Plan of my talk

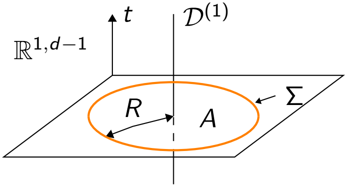
1. Introduction
2. Proposal for a C-theorem in DCFT
3. Wilson loop as a defect operator
4. Conclusion

Sphere partition function and EE in DCFT

Let us clarify a relation between $\log \langle \mathcal{D}^{(p)} \rangle$ and S_{defect}.

A *conformal defect* $\mathcal{D}^{(p)}$ respects $\mathrm{SO}(2,p) \times \mathrm{SO}(d-p)$ of the conformal group $\mathrm{SO}(2,d)$.

$\mathcal{D}^{(p)} = \{X^p = \cdots = X^{d-1} = 0\}$



- Using CHM map, $\mathbb{R}^{1,d-1} \to \mathbb{S}^1 \times \mathbb{H}^{d-1}$
- The Rényi entropies are

$$S_n^{((\mathrm{D})\mathrm{CFT})} = \frac{1}{1-n} \log \frac{Z^{((\mathrm{D})\mathrm{CFT})}[\mathbb{S}^1_n \times \mathbb{H}^{d-1}]}{\left(Z^{((\mathrm{D})\mathrm{CFT})}[\mathbb{S}^1 \times \mathbb{H}^{d-1}]\right)^n}$$

- Around $n = 1$, the free energy can be expanded,

$$\log Z^{(\mathrm{DCFT})}[\mathbb{S}^1_n \times \mathbb{H}^{d-1}] = \log Z^{(\mathrm{DCFT})}[\mathbb{S}^1 \times \mathbb{H}^{d-1}] - \frac{1}{2} \int_{\mathbb{S}^1 \times \mathbb{H}^{d-1}} \delta g_{\tau\tau} \langle (T_{\mathrm{DCFT}})^{\tau\tau} \rangle^{(\mathrm{DCFT})}_{\mathbb{S}^1 \times \mathbb{H}^{d-1}} + \cdots$$

- The difference of the entanglement entropies becomes

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(p)} \rangle + \int_{\mathbb{S}^1 \times \mathbb{H}^{d-1}} \langle (T_{\text{DCFT}})^{\tau}{}_{\tau} \rangle^{(\text{DCFT})}_{\mathbb{S}^1 \times \mathbb{H}^{d-1}}$$

with the stress-energy tensor

$$\begin{aligned} &\langle (T_{\text{DCFT}})_{\mu\nu} \rangle^{(\text{DCFT})}_{\mathbb{S}^1 \times \mathbb{H}^{d-1}} \, \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} \\ &= \frac{a_T}{\sinh^d x} \left[\frac{d-p-1}{d} \left(\mathrm{d}\tau^2 + \mathrm{d}x^2 + \cosh^2 x \, \mathrm{d}s^2_{\mathbb{H}^{p-1}} \right) - \frac{p+1}{d} \sinh^2 x \, \mathrm{d}s^2_{\mathbb{S}^{d-p-1}} \right] \end{aligned}$$

Main result

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(p)} \rangle - \frac{2(d-p-1)\, \pi^{d/2+1}}{\sin(\pi p/2)\; d\, \Gamma(p/2+1)\; \Gamma((d-p)/2)}\, a_T$$

- This is a generalization of the result for $p = 1$ [Lewkowycz-Maldacena '13].
- For $p = d - 1$, the defect entropy is given by

$$S_{\text{defect}} = \log \langle \mathcal{D}^{(d-1)} \rangle$$

Proposal for a C-theorem in DCFT

Two candidates for a C-function in DCFT

1. defect free energy $\log \langle \mathcal{D}^{(p)} \rangle = \log Z^{(\mathrm{DCFT})}[\mathbb{S}^d]/Z^{(\mathrm{CFT})}[\mathbb{S}^d]$

$$\log \langle \mathcal{D}^{(p)} \rangle = \frac{c_p}{\epsilon^p} + \frac{c_{p-2}}{\epsilon^{p-2}} + \cdots + \begin{cases} (-1)^{p/2}\, B \, \log \epsilon + \cdots \,, & (p : \text{even}) \,, \\ (-1)^{(p-1)/2}\, D \,, & (p : \text{odd}) \,. \end{cases}$$

2. defect entropy S_{defect}

$$S_{\text{defect}} = \frac{c'_{p-2}}{\epsilon^{p-2}} + \frac{c'_{p-4}}{\epsilon^{p-4}} + \cdots + \begin{cases} (-1)^{p/2}\, B' \, \log \epsilon + \cdots \,, & (p : \text{even}) \,, \\ (-1)^{(p-1)/2}\, D' \,, & (p : \text{odd}) \,. \end{cases}$$

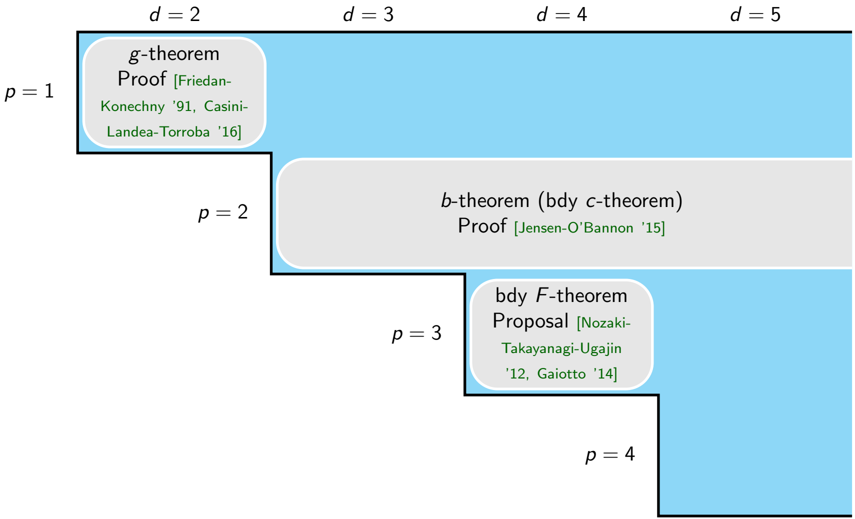
Our proposal

In DCFT_d with a defect of dimension p, the universal part of the defect free energy

$$\tilde{D} \equiv \sin\left(\frac{\pi p}{2}\right) \log |\langle \mathcal{D}^{(p)} \rangle|$$

does not increase along any defect RG flow $\tilde{D}_{\mathrm{UV}} \geq \tilde{D}_{\mathrm{IR}}$.

Summary of the conjectured and proved C-theorems



Our proposal reduces to the known ones in the shaded regions & provides new ones in the region colored in blue.

Wilson loop as a defect operator

- We test our proposal for $p = 1$ using a circular Wilson loop operators

$$W_{\mathfrak{R}}[A] = \mathrm{Tr}_{\mathfrak{R}} \exp\left[\mathrm{i} \int \mathrm{d}x^{\mu} A_{\mu}\right]$$

- The Wilson loop can be regarded as an action localized on the defect.

[Gomis-Passerini '06, Tong-Wong '14]

$$W_{\mathfrak{R}}[A] = \frac{Z_q[A]}{Z_q[0]} \quad \text{with}$$

$$Z_q[A] \equiv \frac{1}{q!} \int \mathcal{D}\chi^{\dagger} \mathcal{D}\chi \, \chi_{a_1}(+\infty) \cdots \chi_{a_q}(+\infty) \, \chi^{\dagger, a_1}(-\infty) \cdots \chi^{\dagger, a_q}(-\infty) \, \mathrm{e}^{-I_{\chi}}$$

$$I_{\chi} = \int \mathrm{d}t \, \chi^{\dagger} \left(\mathrm{i}\,\partial_t - A(t)\right) \chi$$

- The defect theory can flow to the trivial theory without fermions,

$$W_{\mathfrak{R}}[A] \to 1$$

under the mass deformation

$$I_M = -\int \mathrm{d}t \, M \, \chi^{\dagger} \chi \,, \qquad M \to \infty$$

U(1) gauge theory in 4d

$$W = \exp\left[\mathrm{i}\, e \oint \mathrm{d}x^\mu A_\mu\right] , \qquad e \in \mathbb{R}$$

[Lewkowycz-Maldacena '13]

- The defect free energy $\log \langle W \rangle = e^2/4$
- The defect entropy $S_{\text{defect}} = 0$
- The Wilson loop becomes trivial under a defect RG flow, $\log \langle W \rangle \to 0$.

  $\Longrightarrow$ This is consistent with our conjecture.
- On the other hand, the defect entropy vanishes at both the UV and IR fixed points.

  $\Longrightarrow$ The defect entropy doesn't capture degrees of freedom on the defect.

Free scalar field in 4d

$$W = \exp\left[\lambda \oint \mathrm{d}t\, \phi\left(x^\mu(t)\right)\right] \, , \qquad \lambda \in \mathbb{C}$$

- The defect free energy

$$\log \langle\, W \,\rangle = 0$$

  This result does not contradict with our assertion.

- The defect entropy

$$S_{\text{defect}} = -\frac{\lambda^2}{12}$$

  It can be negative for real λ at the UV fixed point.

  But it is supposed to be zero at the IR fixed point.

$\Longrightarrow$ This is a counterexample for the defect entropy being a C-function.

Conclusion

- We examine the defect free energy $\log \langle \mathcal{D}^{(p)} \rangle$ and the defect entropy S_{defect} as a candidate C-function.

- We find the relation with them

$$S_{\mathrm{defect}} = \log \langle \mathcal{D}^{(p)} \rangle - \frac{2(d-p-1)\,\pi^{d/2+1}}{\sin(\pi p/2)\; d\,\Gamma(p/2+1)\,\Gamma((d-p)/2)}\, a_T$$

- We propose a C-theorem in DCFTs.

The defect free energy does not increase under any defect RG flow.

- We find in Wilson loop examples that
  - the sphere free energy decreases
  - but the EE increases along a certain RG flow triggered by a defect localized perturbation which is assumed to have a trivial IR fixed point without defects.

- We also checked more field theoretic examples,
  - Conformal perturbation theory on defect
  - Wilson loop
    1. Chern-Simons theory
    2. 1/2-BPS Wilson loop in 4d $\mathcal{N} = 4$ SYM
    3. 1/6-BPS Wilson loop in ABJM
    4. $U(N)$ $\mathcal{N} = 4$ SYM with N_f hypermultiplets in 3d

- We also provide a proof of our proposal in several holographic models of defect RG flows.
  1. Domain wall defect RG flow
  2. Probe brane model
  3. Holographic model of defect RG flow [Yamaguchi '02]
  4. AdS/BCFT model [Takayanagi '11]

Thank you for your attention!

Stress tensor

- Defenition : $T^{\mu\nu}_{\text{DCFT}} = -\frac{2}{\sqrt{g}} \frac{\delta \log Z^{(\text{DCFT})}[g_{\mu\nu}]}{\delta g_{\mu\nu}} = T^{\mu\nu}_{\text{CFT}} + t^{\mu\nu}$
- $T^{\mu\nu}_{\text{DCFT}}$ is traceless and partially conserved

$$\partial_\mu T^{\mu a}_{\text{DCFT}} = 0 \ , \quad \partial_\mu T^{\mu i}_{\text{DCFT}} = -\delta_{\mathcal{D}}(x_\perp)\, \mathrm{D}^i \ , \quad (T_{\text{DCFT}})^\mu{}_\mu = 0$$

- The one-point function of T_{CFT}

$$\langle\, T^{ab}_{\text{CFT}}(x)\,\rangle = \frac{d-p-1}{d} \frac{a_T}{|x_\perp|^d}\, \delta^{ab} \ , \quad \langle\, T^{ai}_{\text{CFT}}(x)\,\rangle = 0 \, , \quad \langle\, T^{ij}_{\text{CFT}}(x)\,\rangle = -\frac{a_T}{|x_\perp|^d} \left(\frac{p+1}{d} \delta^{ij} - \frac{x^i_\perp x^j_\perp}{|x_\perp|^2} \right)$$

  Note that $\langle\, T^{\mu\nu}_{\text{CFT}}\,\rangle = 0$ for $p = d-1$.

- $\langle\, t^{\mu\nu}(x)\,\rangle = 0$ since $t^{\mu\nu}$ is localized on the defect,

$$t^{\mu\nu}(x) = \delta_{\mathcal{D}}(x_\perp)\, \frac{\partial x^\mu}{\partial \hat{x}^a} \frac{\partial x^\nu}{\partial \hat{x}^b}\, \hat{t}^{ab}(\hat{x})$$

  $\hat{t}^{ab}(\hat{x})$ is a defect local operator of dimension p and invariant under the translation, rotation and scale transformation on the defect.