Entanglement entropy and $T\bar{T}$

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Background

The $T\bar{T}$ deformation

Consider a family of quantum field theories along a trajectory defined as follows:

$$\partial_\mu S^{QFT}[\mu] = \int_x (T\bar{T})_\mu$$

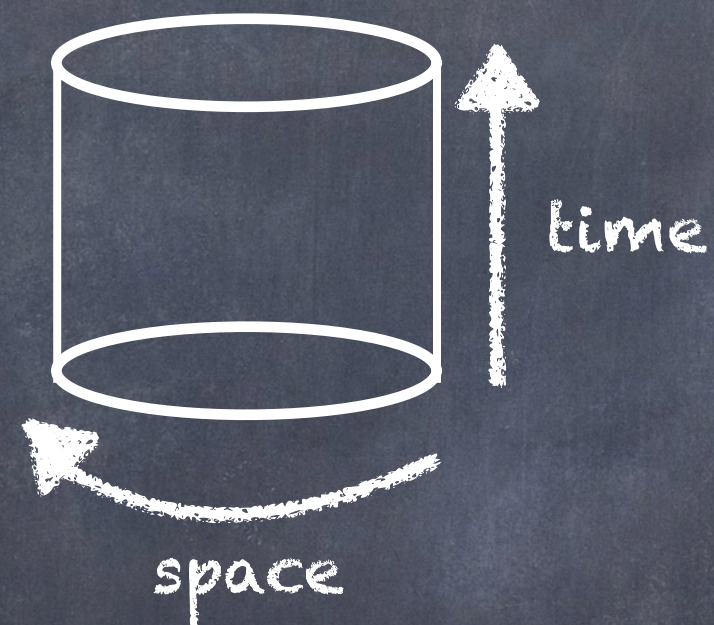
where: $$T\bar{T}(x) = \frac{1}{8}(g_{a(c}g_{d)b} - g_{ab}g_{cd})T^{ab}(x)T^{cd}(x)$$

whose expectation value on a plane or on the cylinder factories:

$$\langle T\bar{T}\rangle = \frac{1}{8}\left(\langle T^{ab}\rangle\langle T_{ab}\rangle - \langle T^a_a\rangle^2\right)$$

'Solvability': Deformed Energy levels

Consider a deformed CFT on the cylinder:



solve

$$\partial_\mu E_n[\mu, L] = L\langle n | T\bar{T} | n \rangle$$

Taking the factorisation into account:

$$\partial_\mu E_n = -\frac{1}{4}\left(E_n \partial_L E_n + \frac{P_n^2}{L^2} \right)$$

The Solution reads:

$$E_n(\mu, L) = \frac{2L^2}{\mu}\left(1 - \sqrt{1 \mp \frac{2\pi\mu}{L^2}\left(\Delta_n + \bar{\Delta}_n - \frac{c}{12} \right) + \frac{\pi^2\mu^2}{L^4}(\Delta_n - \bar{\Delta}_n)^2} \right)$$

Good and bad signs

$$E_n(\mu, L) = \frac{2L^2}{\mu}\left(1 - \sqrt{1 + \frac{2\pi\mu}{L^2}\left(\Delta_n + \bar{\Delta}_n - \frac{c}{12}\right) + \frac{\pi^2\mu^2}{L^4}(\Delta_n - \bar{\Delta}_n)^2}\right)$$

Good- Hagedorn behaviour

Bad- square root singularity at some Δ_{n*}

Idea: treat this as a UV cutoff.

The level n at which the energies hit the square root singularity is set by μ

Bad sign: Truncated Spectrum

[McGough,Mezei,Verlinde 18']

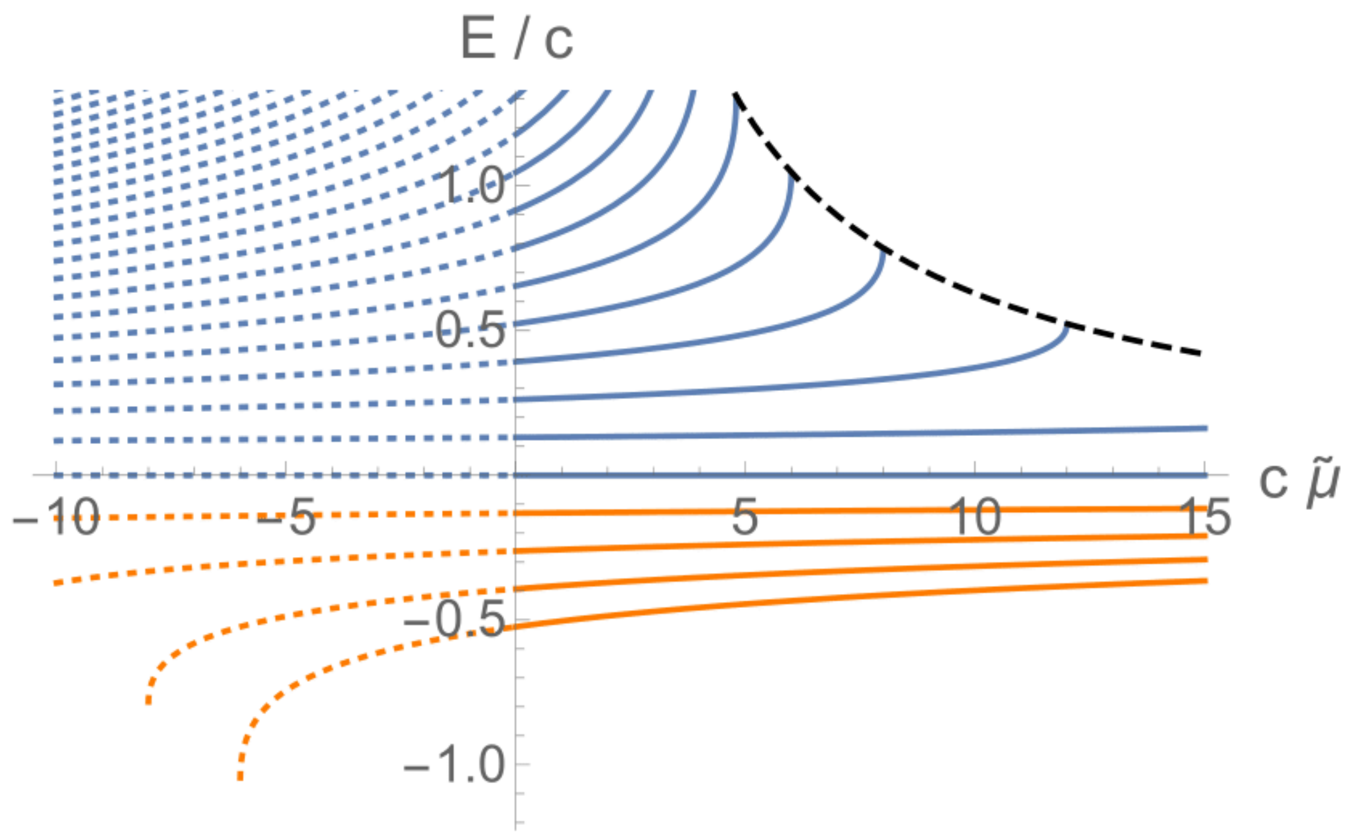
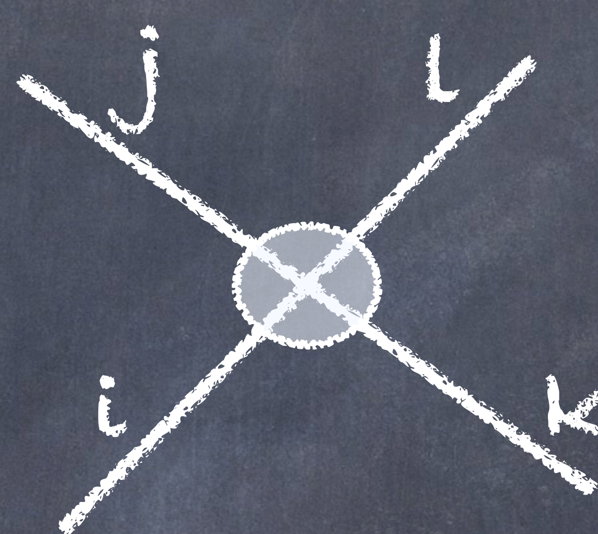


Figure 1: The energy levels E_n at $L = 2\pi$ and $J = 0$ as a function of μ for different values of $E(0) = \Delta_n + \bar{\Delta}_n - \frac{c}{12}$. States with $E(0) > 0$ that correspond to black holes in holographic CFTs are plotted in blue, while low-lying states are plotted in orange. For $\mu > 0$ that is the relevant regime in our study we used solid lines, while for $\mu < 0$ the spectrum is plotted with dotted lines. The levels exhibit a square root singularity at the critical value $\mu E(0) = 2\pi$. This indicates that, for given μ, the energy spectrum of the deformed CFT is bounded by $E < \frac{8}{\mu}$, indicated on the plot by a dashed black line.

CDD phase of the 2->2 S-matrix in IQFT

[A. Cavaglia, S. Negro, I.M. Szecsenyi, R. Tateo 16']

[F. Smirnov, A. Zamolodchikov 16']



In an integrable QFT, the 2->2 S-Matrix is fixed by crossing, analyticity, unitarity, YBE etc. up to a phase:

$$S_{ij}^{kl}(\theta) = e^{\frac{i\mu}{4} m_i m_j \sinh(\theta_i - \theta_j)} S_{ij}^{o\,kl}$$

It can be shown that turning on this phase corresponds to deforming by $T\bar{T}$

Modular Invariance and Torus Partition Function

[O. Aharony, S. Datta, A. Giveon, Y. Jiang & D. Kutasov 19']

The torus partition function reads:

$$Z(\tau, \bar{\tau}; \lambda) = \sum_{n} e^{2\pi i \tau_1 R P_n - 2\pi \tau_2 R E_n(\lambda, R)}$$

solves the Burgers' equation

Is also modular invariant in a unique sense:

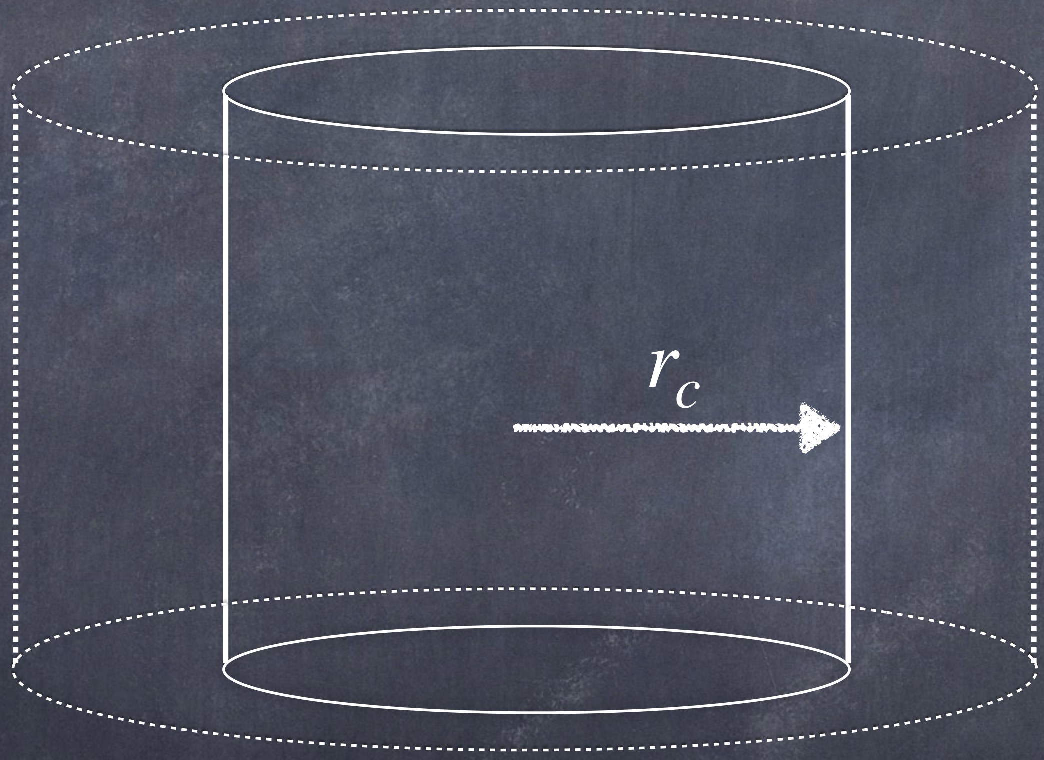
$$Z(\tau, \bar{\tau}; \lambda) = Z\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}; \frac{\lambda}{|c\tau + d|^2}\right)$$

Setting of Interest

Main motivation: Finite cutoff Holography

[McGough, Mezei, Verlinde 18']

Apply the following deformation to large c, holographic CFTs.



Dual to gravity in a truncated AdS space:

$$ds^2 = \frac{dr^2}{r^2} + r^2 g_{ab} dx^a dx^b,\ r < r_c$$

The radius of the cutoff surface is related to the deformation parameter:

$$\mu = \frac{16\pi G}{r_c^2} = \frac{24\pi}{c}\frac{1}{r_c^2} > 0$$

Check: Quasi Local Energy

The quasi local energy of a BTZ black hole with a cutoff surface at $r = r_c$ given by:

$$E = \frac{r_c}{4G}\left[1 - \sqrt{1 - \frac{8GM}{r_c^2} + \frac{16G^2J^2}{r_c^4}}\right]$$

Match with

Identify:

$$M = M_n = \Delta_n + \bar{\Delta}_n - \frac{c}{12},$$

$$J = J_n = \Delta_n - \bar{\Delta}_n$$

$$E_n(\mu, L) = \frac{2(2\pi r_c)^2}{\mu}\left(1 - \sqrt{1 - \frac{2\pi\mu}{(2\pi r_c)^2}\left(\Delta_n + \bar{\Delta}_n - \frac{c}{12}\right) + \frac{\pi^2\mu^2}{(2\pi r_c)^4}(\Delta_n - \bar{\Delta}_n)^2}\right)$$

Large-c flow equation on curved space

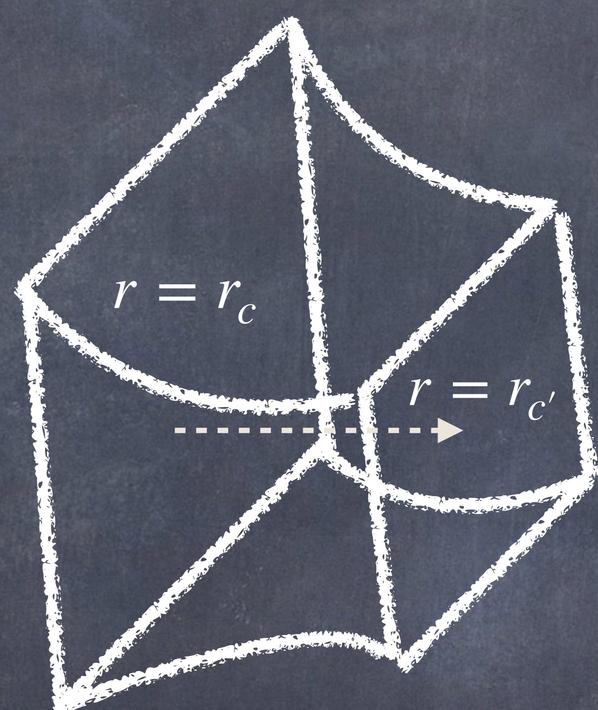
$$\boxed{\langle T^a_a \rangle = -\frac{\mu}{4}\underline{\langle T\bar{T}\rangle} - \frac{c}{24\pi}R(g)}$$

$$\langle T\bar{T}\rangle = \left(\langle T^{ab}\rangle\langle T_{ab}\rangle - \langle T^a_a\rangle^2\right)$$

This leads to the large-c flow equation:

$$\langle T^a_a \rangle = -\frac{\mu}{4}\left(\langle T^{ab}\rangle\langle T_{ab}\rangle - \langle T^e_e\rangle^2\right) - \frac{c}{24\pi}R(g)$$

The Radial Hamiltonian Constraint



Identify the momentum conjugate to the induced metric on a constant r surface:

$$\pi^{ab} = \sqrt{g}\left(\langle T^{ab}\rangle - \frac{2}{\mu} g^{ab}\right) \quad \text{where}$$

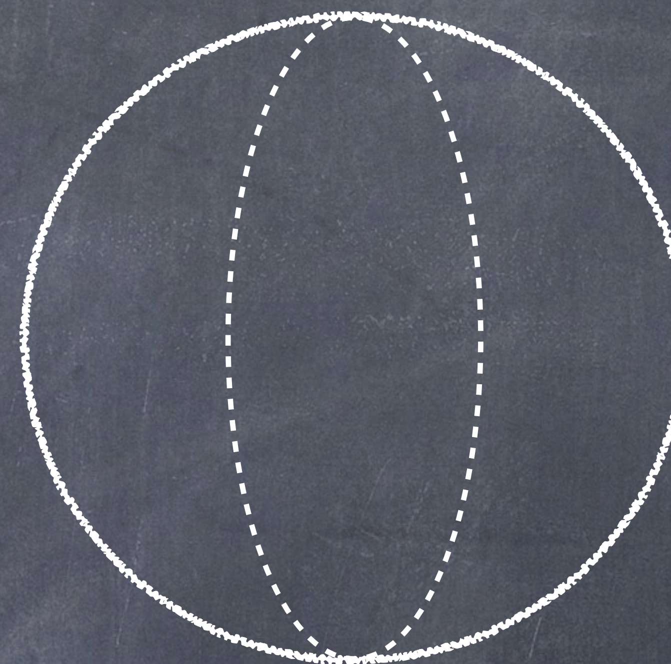
$$\frac{2N}{\sqrt{g}}\left(\pi^{ab} - g^{ab}\pi^e_e\right) = K^{ab}$$

Large c flow equation <-> Radial Hamiltonian constraint in AdS_3:

$$\frac{1}{\sqrt{g}}\left(\pi^{ab}\pi_{ab} - (\pi^e_e)^2\right) + \sqrt{g}\left(R + \frac{2}{\ell^2}\right) = 0$$

Sphere Partition function

Due to symmetry: $\langle T^{ab}\rangle = \alpha(r) g^{ab}$



$$\langle T^a_a\rangle = -\frac{\mu}{4}\left(\langle T^{ab}\rangle\langle T_{ab}\rangle - \langle T^e_e\rangle^2\right) - \frac{c}{24\pi}R(g)$$

$$\alpha(r) = \frac{2}{\mu}\left(1-\sqrt{1+\frac{c\mu}{24\pi r^2}}\right)$$

$$\frac{2}{r^2}$$

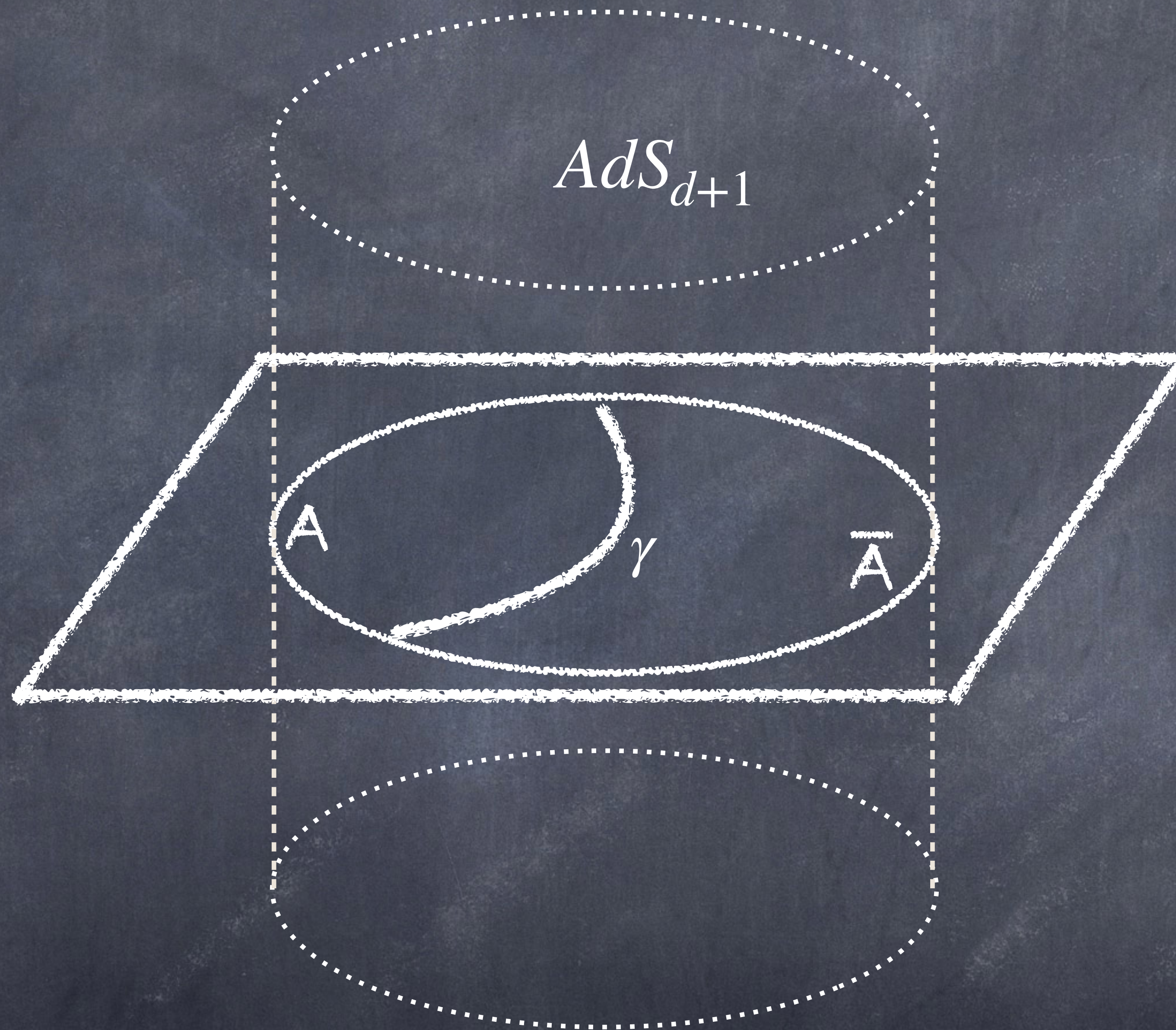
Matches on shell action in cutoff- AdS_3 [P.Caputa, S.Datta, V.Shyam 19']

$$\log Z(r_c) = -4\pi\int_0^{r_c} dr\, r\alpha(r) = \frac{c}{3}\sinh^{-1}\left(\frac{\sqrt{24\pi}r_c}{\sqrt{\mu c}}\right) + \frac{8\pi}{\mu}\left(r_c\sqrt{\frac{c\mu}{24\pi}+r_c^2} - r_c^2\right)$$

Holographic Entanglement Entropy

Ryu-Takayanagi Prescription

[S. Ryu, T. Takayanagi 06']

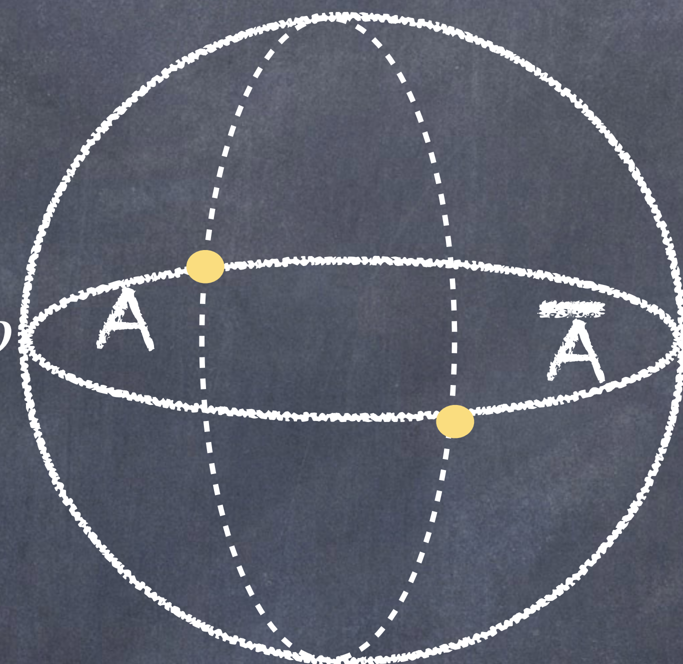


$$S_A = Tr\left(\rho_A \log \rho_A\right) = Ar(\gamma)$$

Entanglement Entropy

[W.Donnelly, V.S. 18']

$\rho_A = Tr_{\bar{A}}\rho$



$$\langle T^{\theta}_{\theta}\rangle = \frac{1}{2}\langle T^{a}_{a}\rangle$$

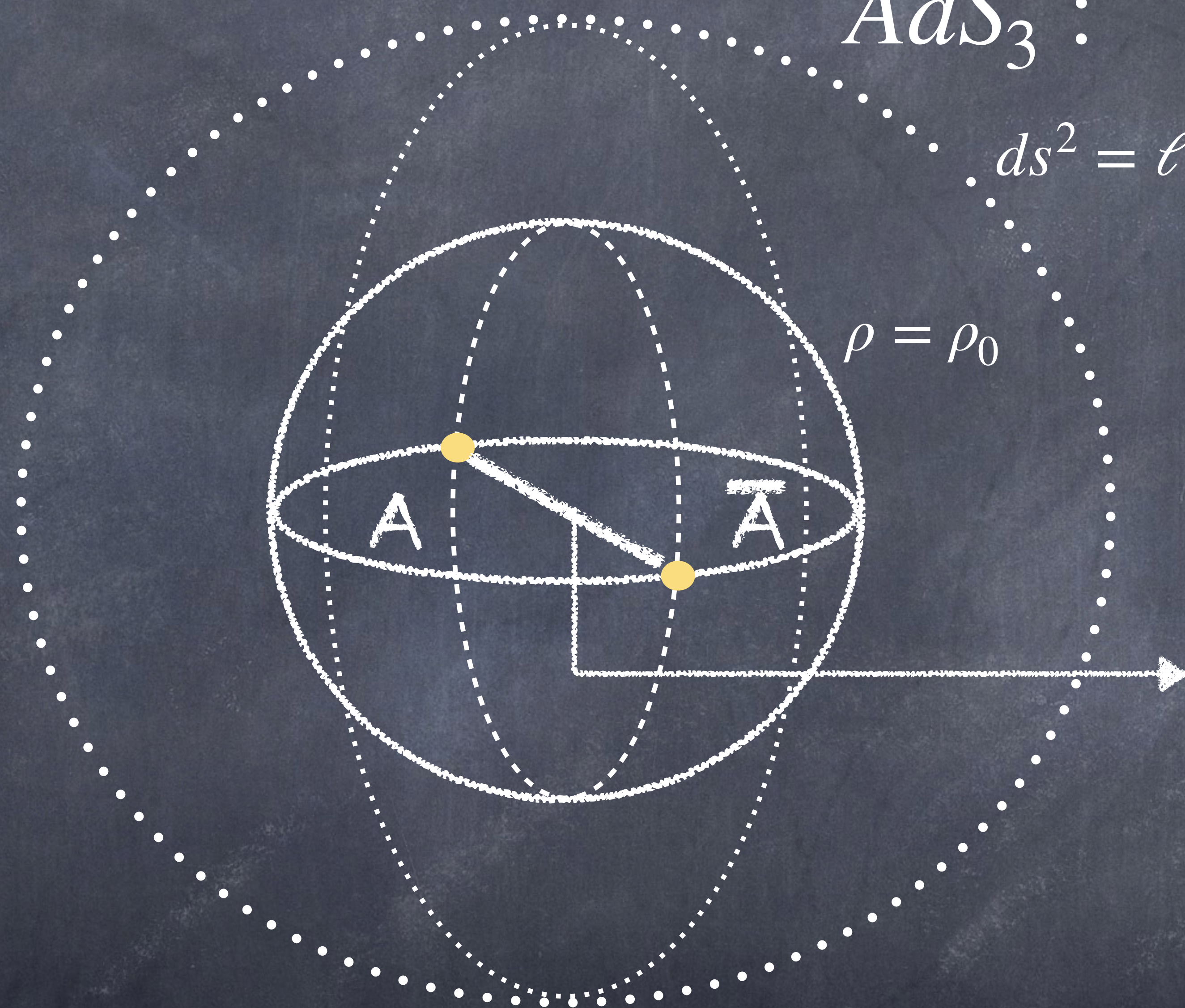
$$S_A = Tr(\rho_A \log \rho_A) = \left(1 - n\frac{d}{dn}\right) logZ_n|_{n=1} = 1 + \left.\underline{\left(n\int\sqrt{g}\langle T^{\theta}_{\theta}\rangle\right)}\right|_{n=1}$$

$$= \left(1 - \frac{r}{2}\frac{d}{dr}\right)\log Z(r) = \frac{c}{3}\sinh^{-1}\left(\frac{\sqrt{24\pi r}}{\sqrt{\mu c}}\right)$$

Bulk Minimal Surface

$AdS_3:$

$$ds^2 = \ell^2\left(d\rho^2 + \sinh^2(\rho)d\Omega_{S^2}^2\right)$$



$$L = 2\ell\rho_0$$

$$S = \frac{L}{4G} = \frac{\ell}{2G}\sinh^{-1}\left(\frac{r}{\ell}\right)$$

Entanglement Entropy in CFT vs Deformed Theory

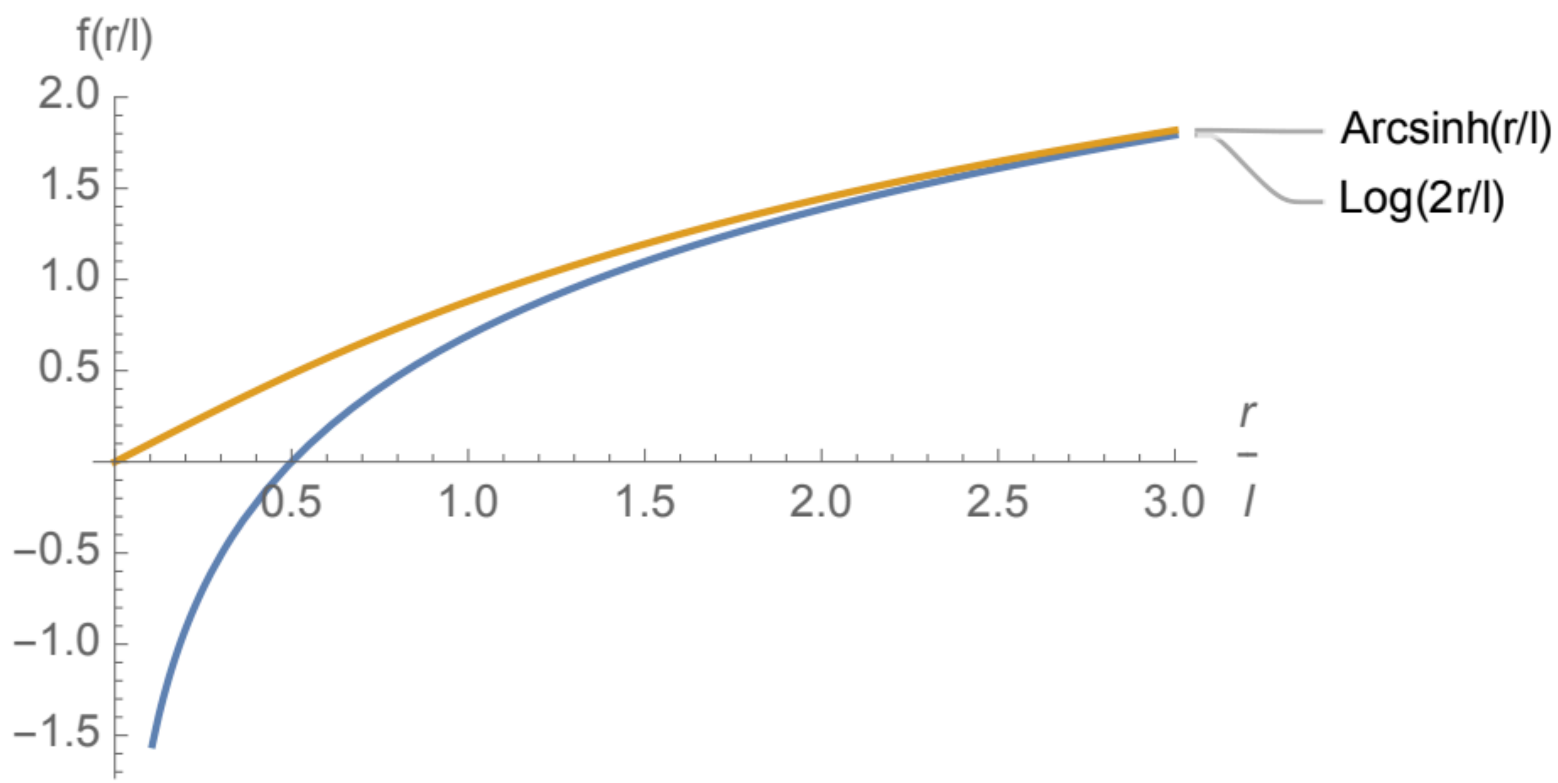


Figure 1. Entanglement entropy in the $T\overline{T}$ theory agrees with the CFT result for $r \gg \sqrt{\mu c}$, but is UV finite.

[W.Donnelly, V.S. 18']

Conical Entropies

[X.Dong 16']

Definition: $$\tilde{S}_n = \left(1 - n\frac{d}{dn}\right)\log Z_n$$

Relation to Renyi Entropy $$= n^2\partial_n\left(\frac{1-n}{n}S_n\right)$$

In holography, this quantity is the conjectured dual to the area of a cosmic brane in the bulk $AdS_{d+1}/\mathbb{Z}_n$ anchored to the entangling surface.

Tension:

$$T_n = \frac{(n-1)}{4\pi G n}$$

Back-reacts to create a conical deficit with opening angle:

$$\Delta\phi = 2\pi\frac{n-1}{n}$$

Energy momentum tensor on $\left(S^2\right)^n/\mathbb{Z}_n$

$$\langle T^\theta_\theta\rangle = \frac{2}{\mu}\left(1-\sqrt{1+\frac{c\mu}{24\pi r^2}+\frac{c\mu}{24\pi r^2}\underbrace{\left(\frac{1}{n^2}-1\right)}\frac{1}{\sin(\theta)^2}}\right),$$

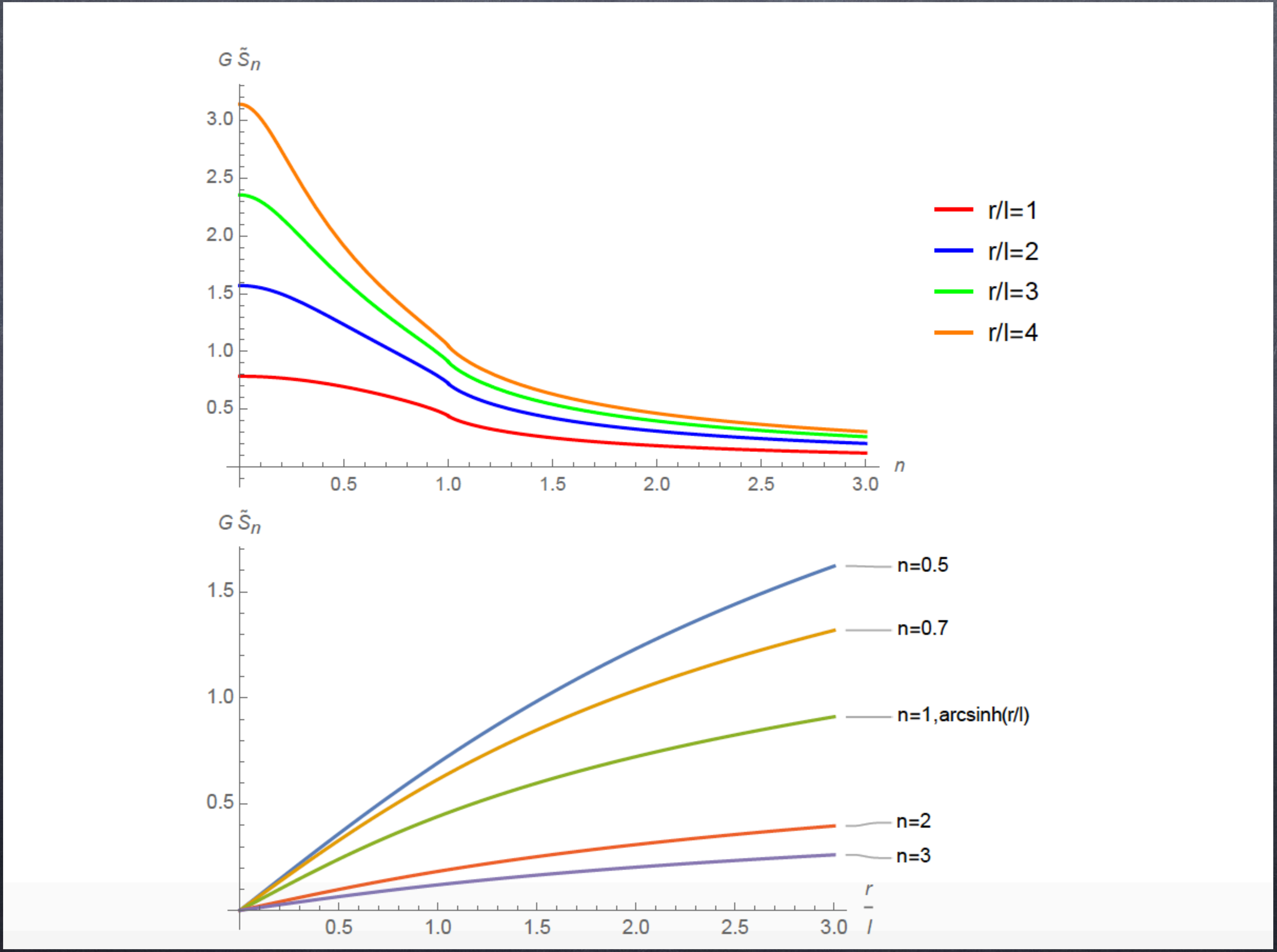
$$\langle T^\phi_\phi\rangle = \frac{2}{\mu}\left(1-\frac{1+\frac{c\mu}{24\pi r^2}}{\sqrt{1+\frac{c\mu}{24\pi r^2}+\frac{c\mu}{24\pi r^2}\underbrace{\left(\frac{1}{n^2}-1\right)}\frac{1}{\sin(\theta)^2}}}\right).$$

Reality requires: $n<1$

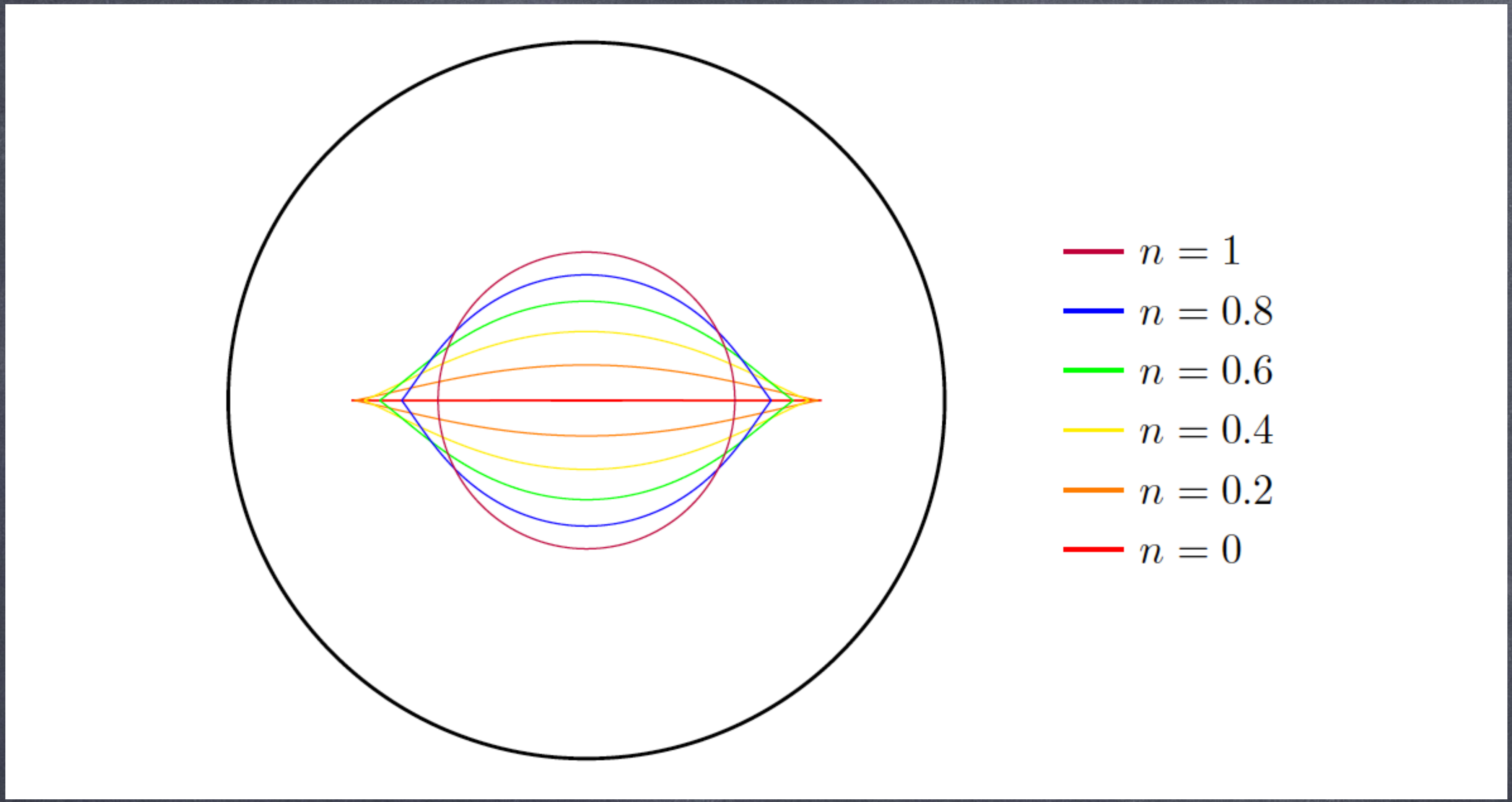
Integrate then analytically continue result to $n>1$

$$\tilde{S}_n = \frac{c}{3}\frac{(1-n^2)}{\sqrt{\frac{c\mu}{24\pi r^2}+n^2}}\Pi\left(n^2 \middle| \frac{r^2+\frac{c\mu}{24\pi}}{r^2+\frac{c\mu}{24\pi n^2}}\right).$$

[W.Donnelly, V.S. 18']



Dong's conjecture at finite radius



Rescale the angular co-ordinate to put the conical singularities on the boundary

$$\frac{L}{4G} = \frac{\ell}{2G}\frac{(1-n^2)}{\sqrt{\frac{\ell^2}{r^2}+n^2}}\Pi\left(n^2 \middle| \frac{r^2+\ell^2}{r^2+\frac{\ell^2}{n^2}}\right)$$

0th Conical Entropy

$$\tilde{S}_n \equiv -n^2 \partial_n \left(\frac{\log(Tr\rho^n)}{n} \right)$$

$$\Big\downarrow \; n \to 0$$

$$\tilde{S}_0 = \log\big(\underbrace{Tr(\mathbf{1})}\big) = \sqrt{\frac{2\pi c}{3\mu}}\pi r$$

Rank of reduced density matrix

c.f. $\sqrt{\frac{2\pi c}{3\mu}}L$ for a theory on a lattice of size L

Discussion:

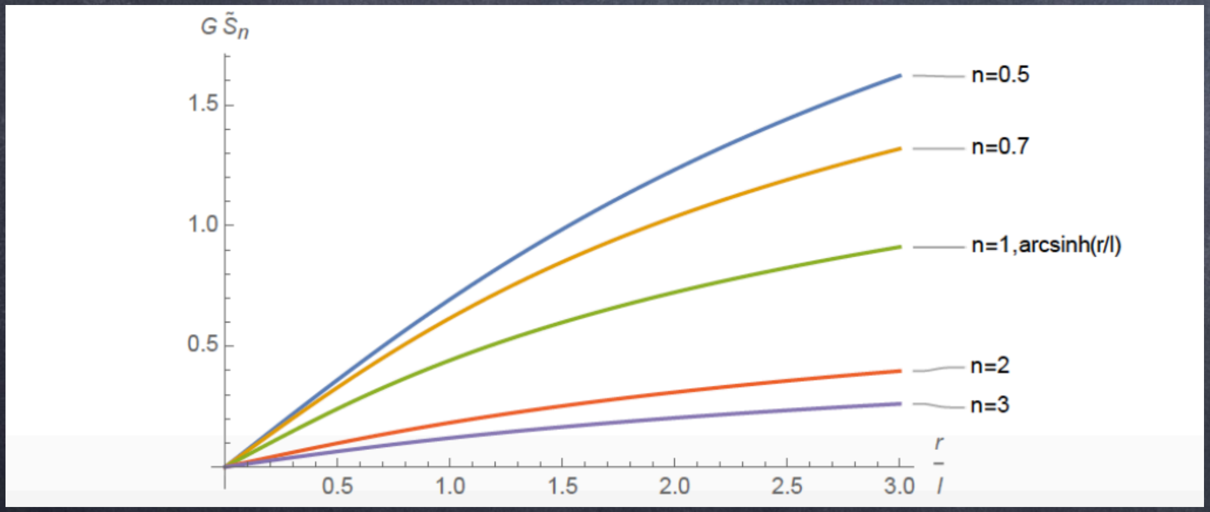
Similarity with Entanglement Renormalization

Finiteness of EE at Short distances

Notice that at small r, both Von Neumann entropy:

$$S_A = \frac{c}{3}\sinh^{-1}\left(\frac{\sqrt{24\pi r}}{\sqrt{\mu c}}\right)$$

and Conical entropies at fixed n:



Conical entropies at fixed n

$$\tilde{S}_n = \frac{c}{3}\frac{(1-n^2)}{\sqrt{\frac{c\mu}{24\pi r^2}+n^2}}\Pi\left(n^2 \,\middle|\, \frac{r^2+\frac{c\mu}{24\pi}}{r^2+\frac{c\mu}{24\pi n^2}}\right).$$

do NOT diverge, instead go to a constant.

Choice of boundary conditions

In order to obtain this behaviour of the entanglement entropy, we need to impose the boundary condition:

$$Z(r_c = 0) = 0$$

However, one could also fix boundary conditions at large radius to match with a CFT:

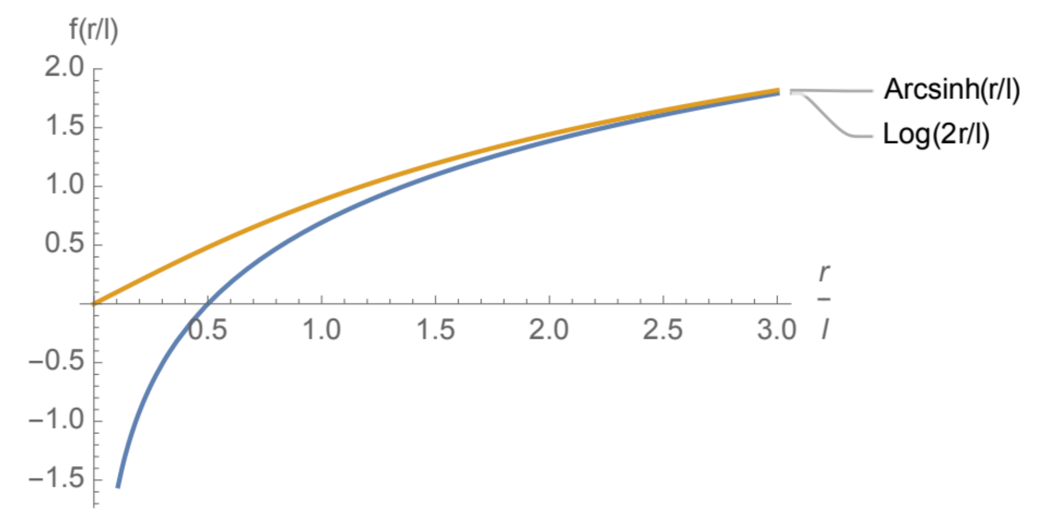
$$Z(r_c)\,|_{r_c >> 1} = \left(\frac{r_c}{\epsilon}\right)^{\frac{c}{3}}$$

[V. Gorbenko, E.Silverstein, G.Torroba 18']

So that the partition function is given by:

$$Z(r_c) = \exp\left(\frac{c}{3}\log\left(\frac{r_c}{\epsilon}\left(1+\sqrt{\frac{\mu c}{24\pi r_c^2}+1}\right)\right)+\frac{8\pi}{\mu}\left(r_c\sqrt{\frac{c\mu}{24\pi}+r_c^2}-r_c^2\right)\right).$$

Interpolation between trivial theory and CFT



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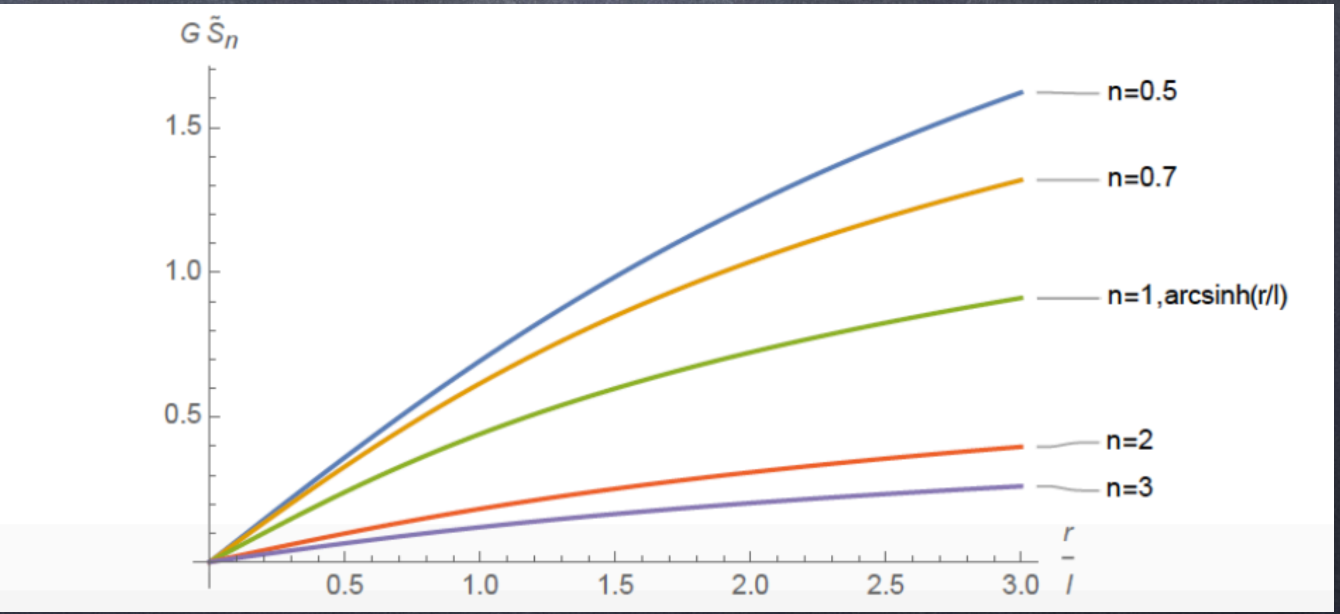
Von Neumann Entropy



Conical Entropy

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