Typicality and thermality in 2d CFT

Shouvik Datta

UCLA

This talk is based on work with
Mert Besken, Per Kraus and Ben Michel.
arXiv:1904.00668 + ongoing work

Statistical mechanics & thermodynamics

Statistical mechanics provides a successful framework to describe thermodynamic behaviour.

However, precise relation of the macroscopic quantities/phenomena to microscopic details is often subtle.

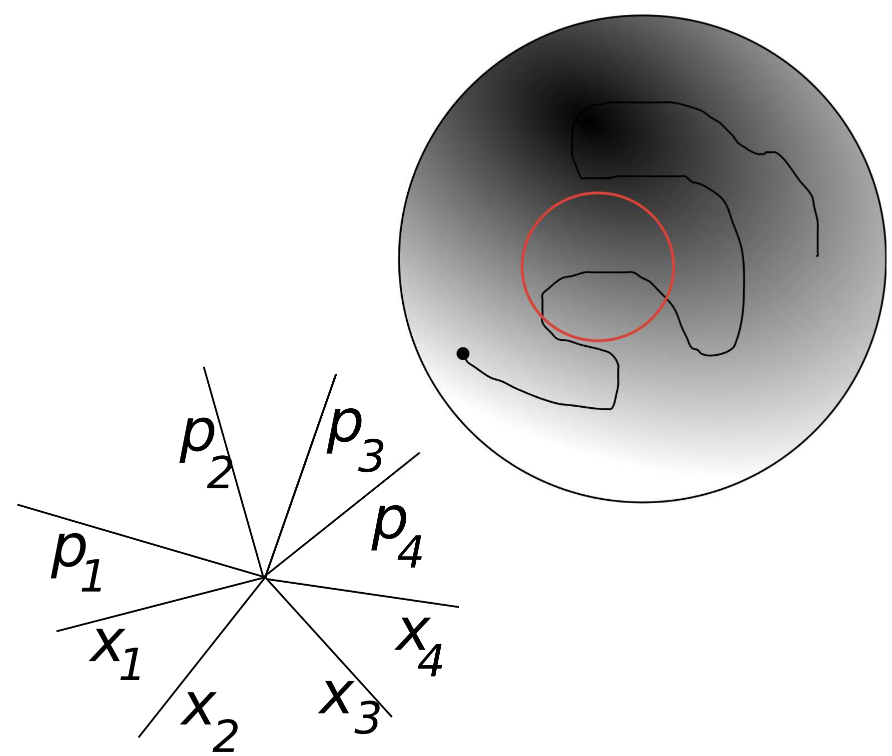
For instance, at the microscopic level a number of physical laws are reversible while thermodynamic laws aren't.

These issues are important while trying to understand thermalization microscopically.

Microstates & thermodynamics

There are a number of possible approaches to understand the emergence of macroscopic behaviour from microscopics.

The most conventional way invokes the ergodic hypothesis. This states that ensemble averages approximate long-time averages.



$$\langle O \rangle_t = \frac{\int_S O(\Gamma) d\Gamma}{\int_S d\Gamma}$$

review by J. Deutsch — 1805.01616

Microstates & thermodynamics

Another approach is to consider typicality of states.
An overwhelmingly large number of microstates
reproduce the same macroscopic behaviour.

A single typical state may be good enough
to reproduce thermodynamics.

Eigenstate thermalization

The eigenstate thermalization hypothesis (ETH) states that thermal expectation values can be reproduced by a single typical microstate of finite energy density.

$$\langle\psi|\mathbb{O}|\psi\rangle = \frac{\mathrm{tr}[e^{-\beta H}\mathbb{O}]}{\mathrm{tr}[e^{-\beta H}]}$$

[Deutsch; Srednicki]

A stronger version of ETH states that all finite energy microstates reproduce thermal expectation values.

There are of course violations of ETH.

Eigenstate thermalization

$$\langle\psi|\mathbb{O}|\psi\rangle = \frac{\mathrm{tr}[e^{-\beta H}\mathbb{O}]}{\mathrm{tr}[e^{-\beta H}]}$$

The notion of temperature arises when the operator $\mathbb{O}$ is chosen to be the Hamiltonian H.

ETH proposes an ansatz for all matrix elements of the operator

[Srednicki; ...]

$$\langle m|\mathbb{O}|n\rangle = g_{\mathbb{O}}(E_m)\delta_{mn} + e^{-S(\bar{E})/2} f_{\mathbb{O}}(\bar{E},\omega) R_{mn}.$$

Eigenstate thermalization

We can pose these questions in 2d CFTs
which offers an arena of analytic tractability.

Does a typical high-energy microstate appear thermal?

What do we mean by 'typical'?

What is the microscopic/CFT realisation of
the BTZ black hole?

Is the ETH ansatz for matrix elements obeyed?

ETH for primaries?

Quasi-primary expectation values in a heavy primary state disagree with thermal ones.

$$\langle T \rangle_\beta = -\frac{\pi^2 c}{6\beta^2} \qquad \langle h_p | T | h_p \rangle = -\left(\frac{2\pi}{L}\right)^2 \left(h_p - \frac{c}{24}\right) \qquad \frac{h_p}{L^2} = \frac{c}{24\beta^2} .$$

For the level-4 quasi-primary, $\Lambda^{(4)} =: TT : -\frac{3}{10}\partial^2 T$

$$\langle \Lambda^{(4)} \rangle_\beta = \left(\frac{\pi^2 c}{6\beta^2}\right)^2 + \frac{11}{90}\frac{\pi^4 c}{\beta^4} \qquad \langle h_p | \Lambda^{(4)} | h_p \rangle = \left(\frac{\pi^2 c}{6\beta^2}\right)^2$$

[Basu-Das-SD-Pal;…]

There is a disagreement beyond the leading order in a large central charge limit.

ETH for primaries?

One possible resolution to this may be offered by the generalised Gibbs ensemble.

[Maloney-Ng-Ross-Tsiares; Dymarsky-Pavlenko; Brehm-Das]

But the reason behind this discrepancy is that primaries are not typical states.

$$\rho(h) \simeq \exp\left[2\pi\sqrt{\frac{c}{6}h}\right] \qquad \rho(h_p) \simeq \exp\left[2\pi\sqrt{\frac{c-1}{6}h_p}\right]$$

[Cardy; Kraus-Maloney]

For fixed central charge, the growth of the number of primaries is exponentially smaller than the growth of all states.

Typical states

Consider a descendant at level (h-h_p)
of a primary with conformal dimension h_p.

Growth of primaries $$\rho(h_p) \simeq \exp\left[2\pi\sqrt{\frac{c-1}{6}h_p}\right]$$

[Cardy; Kraus-Maloney]

Growth of descendants $$\rho(h-h_p) \simeq \exp\left[2\pi\sqrt{\frac{h-h_p}{6}}\right]$$

[Hardy-Ramanujan]

Typical states maximize $\rho(h_p)\rho(h-h_p)$ with respect to h_p.

$$h = \frac{c}{c-1}h_p = h_p + \underbrace{\frac{h_p}{c-1}}_{\text{descendant contribution}}$$

The punchline

We focus on stress tensor correlators in c>1 CFTs
with Virasoro symmetry.

States in the CFT Hilbert space are

$$|h_p, M, \{N_j\}\rangle \equiv L_{-1}^{N_1} L_{-2}^{N_2} L_{-3}^{N_3} \cdots |h_p\rangle$$

primary conformal dim

descendant level

partitions of the integer M.

action of Virasoro generators

primary state

Typical states which reproduce stress tensor correlators are

$$h_p = (c-1)\frac{L^2}{24\beta^2} \qquad M = \frac{L^2}{24\beta^2} \qquad \langle N_j \rangle = \frac{1}{e^{\frac{2\pi\beta j}{L}} - 1}$$

$\{N_j\}$ are Boltzmann distributed with a Bose-Einstein mean.

The punchline

$$|\mathrm{typ}\rangle \equiv |h_p, M, \{N_j\}\rangle$$

$$\langle N_j \rangle = \frac{1}{e^{\frac{2\pi\beta j}{L}} - 1}$$

$$M = \frac{L^2}{24\beta^2}$$

$$h_p = (c-1)\frac{L^2}{24\beta^2}$$

$$\langle \mathrm{typ}|T(w_1)T(w_2)\cdots T(w_n)|\mathrm{typ}\rangle = \langle T(w_1)T(w_2)\cdots T(w_n)\rangle_\beta$$

[SD-Kraus-Michel]

Current correlators

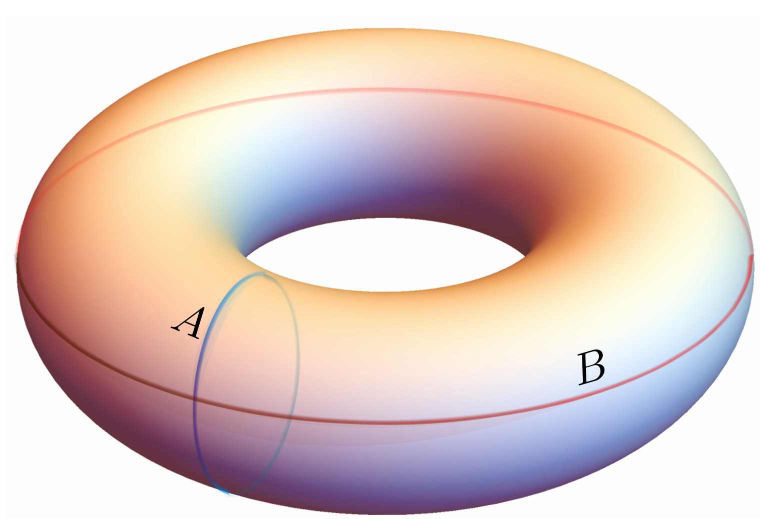
Thermal current correlators in 2d CFT

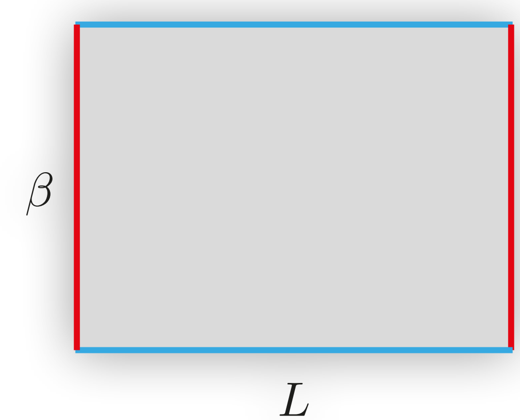
We consider the 2-point function of U(1) currents
on a torus as a simple example first.

This is fixed by modular properties and OPEs.

$$\langle J(w_1)J(w_2)\rangle_{L,\beta} = -\frac{1}{L^2}\left(\wp(w/L,\tau) + \frac{\pi^2}{3}E_2(\tau) - \frac{\pi}{\mathrm{Im}(\tau)}\right)$$

[Eguchi-Ooguri]





Thermal current correlators

We consider the 2-point function of U(1) currents
on a torus as a simple example first.

This is fixed by modular properties and OPEs.

$$\langle J(w_1)J(w_2)\rangle_{L,\beta} = -\frac{1}{L^2}\left(\wp(w/L,\tau) + \frac{\pi^2}{3}E_2(\tau) - \frac{\pi}{\mathrm{Im}(\tau)}\right)$$

[Eguchi-Ooguri]

Weierstrass-P function

$$\wp(w,\tau) = \frac{1}{w^2} + \sum_{(m,n)\neq(0,0)}\left[\frac{1}{(w+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right]$$

Eisenstein series

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$

$q = e^{2\pi i\tau} \qquad \tau = i\beta/L$

Thermal current correlators

The U(1) current can be realised as a free boson.

$$J(w) = \partial_w \phi(w, \bar{w}) \qquad J(w) = -\frac{2\pi}{L}\sum_n \alpha_n e^{\frac{2\pi i n w}{L}} \qquad [\alpha_m, \alpha_n] = m\delta_{m+n,0}$$

The thermal 2-point function is, by definition

$$\langle J(w_1)J(w_2)\rangle_{L,\beta} = \frac{1}{Z(\tau)}\mathrm{Tr}\left[q^{L_0-\frac{1}{24}}\bar{q}^{\tilde{L}_0-\frac{1}{24}}J(w_1)J(w_2)\right]$$

this and the mode-expansion gives…

$$\langle J(w)J(0)\rangle_{L,\beta} = \left(\frac{2\pi}{L}\right)^2\left[\sum_{n>0} n e^{\frac{2\pi i n w}{L}} + \langle\alpha_0^2\rangle_{L,\beta} + 2\sum_{n>0}\langle\alpha_{-n}\alpha_n\rangle_{L,\beta}\cos\left(\frac{2\pi n w}{L}\right)\right]$$

Thermal current correlators

$$\langle J(w)J(0)\rangle_{L,\beta} = \left(\frac{2\pi}{L}\right)^2 \left[\sum_{n>0} n e^{\frac{2\pi i n w}{L}} + \langle \alpha_0^2\rangle_{L,\beta} + 2\sum_{n>0}\langle \alpha_{-n}\alpha_n\rangle_{L,\beta}\cos\left(\frac{2\pi n w}{L}\right)\right]$$

We work in the occupation number eigenbasis

$\alpha_{-n}\alpha_n$ has eigenvalues nN_n

In the canonical ensemble the probability distribution of the occupation number is a Boltzmann distribution

$$P(N_n) = \frac{e^{2\pi i \tau N_n n}}{\sum_{N_n=0}^{\infty} e^{2\pi i \tau N_n n}} \qquad \langle N_n\rangle_{L,\beta} = \sum_{N_n=0}^{\infty} P(N_n) N_n = \frac{1}{e^{-2\pi i \tau n} - 1}$$

The mean is given by a Bose-Einstein function.

$\tau = i\beta/L$

Thermal current correlators

The final result is

$$\langle J(w)J(0)\rangle_{L,\beta} \;=\; \left(\frac{2\pi}{L}\right)^2 \left[-\frac{1}{4\sin^2\left(\frac{\pi w}{L}\right)} + \frac{L}{4\pi\beta} + 2\sum_{n>0} \frac{n}{e^{-2\pi i\tau n} - 1} \cos\left(\frac{2\pi n w}{L}\right)\right]$$

and this agrees with

$$\langle J(w_1)J(w_2)\rangle_{L,\beta} = -\frac{1}{L^2}\left(\wp(w/L,\tau) + \frac{\pi^2}{3}E_2(\tau) - \frac{\pi}{\mathrm{Im}(\tau)}\right)$$

The details of this computation gives insight into what might be the typical microstates which reproduce this thermal result.

Microstate current correlators

Using the mode expansion for the current again, the correlator in a microstate is given by

$$\langle\psi|J(w)J(0)|\psi\rangle = \left(\frac{2\pi}{L}\right)^2 \left[-\frac{1}{4\sin^2\left(\frac{\pi w}{L}\right)} + \langle\psi|\alpha_0^2|\psi\rangle + 2\sum_{n>0} N_n n \cos\left(\frac{2\pi n w}{L}\right)\right]$$

$$\langle\psi|\alpha_{-n}\alpha_n|\psi\rangle = N_n n$$

Total energy

$$E = \frac{2\pi}{L}\sum_n N_n n$$

microstate is an eigenstate of $\alpha_{-n}\alpha_n$.

Effective temperature

$$\beta = \sqrt{\frac{\pi L}{12E}}$$

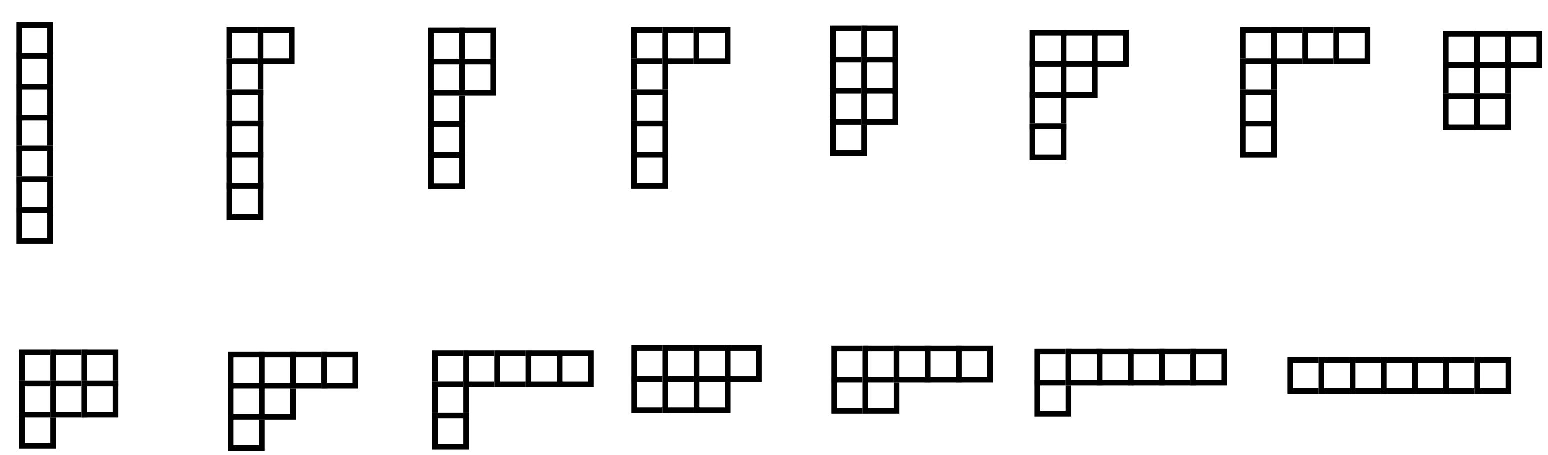
Analogy with partitions of integers

The total energy is given by $E = \frac{2\pi}{L} \sum_n N_n n$

The situation here is similar to partitioning a integer M.

$$M = \sum_{n=1}^{\infty} n N_n$$

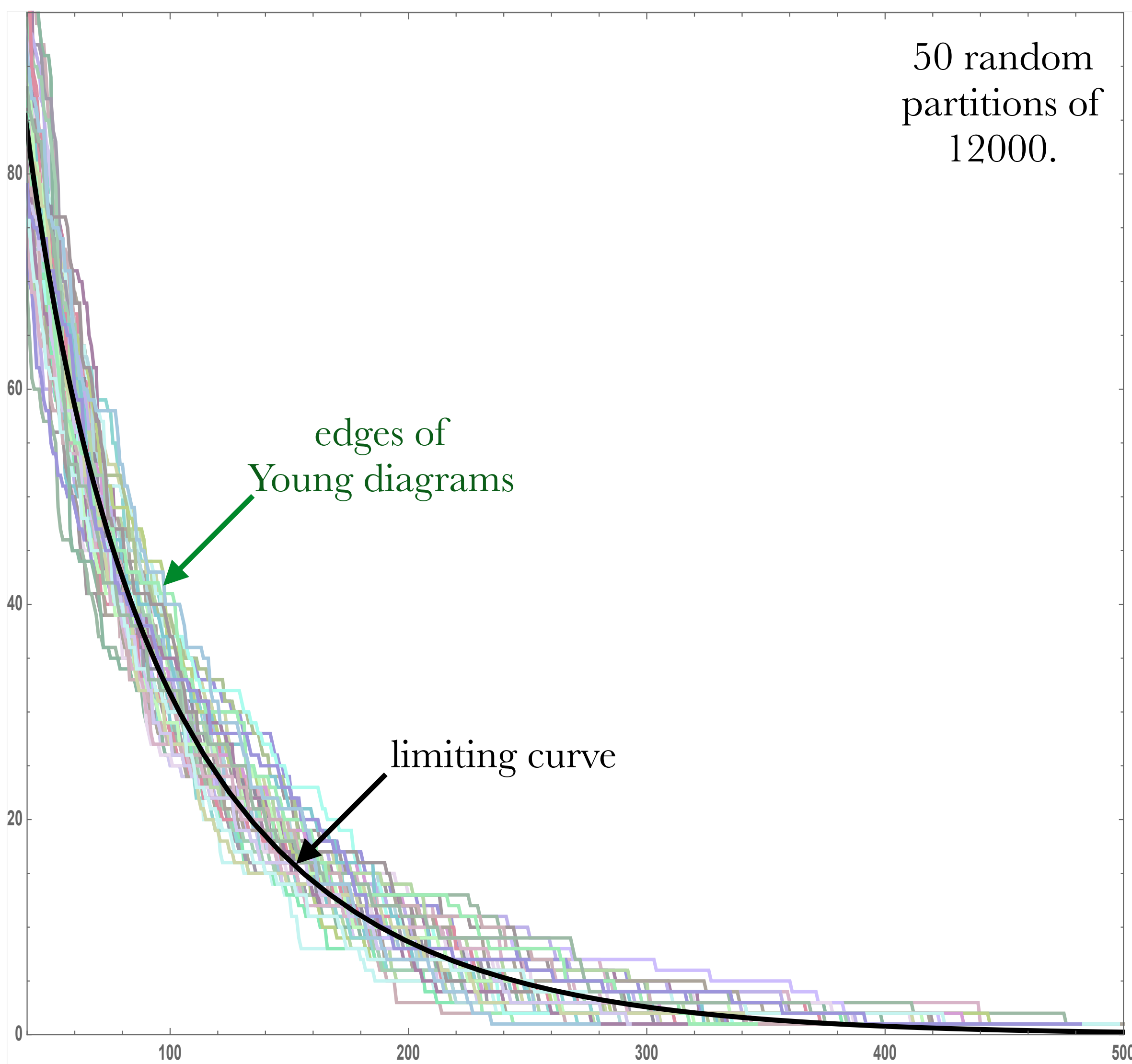
Partitions can be conveniently represented by Young diagrams.



Statistics of partitions

For large integers the most Young diagrams acquire a limiting shape.

[Vershik]



$$N(x) = \frac{1}{e^x - 1}, \qquad x = \pi j/\sqrt{6M}$$

The 'typical' profile is Bose-Einstein.

also [Nekrasov-Okounkov; Balasubramanian-deBoer-Jejjala-Simon]

Microstate current correlators

$$\langle\psi|J(w)J(0)|\psi\rangle = \left(\frac{2\pi}{L}\right)^2\left[-\frac{1}{4\sin^2\left(\frac{\pi w}{L}\right)} + \langle\psi|\alpha_0^2|\psi\rangle + 2\sum_{n>0} N_n n\cos\left(\frac{2\pi n w}{L}\right)\right]$$

Total energy

$$E = \frac{2\pi}{L}\sum_n N_n n$$

Effective temperature

$$\beta = \sqrt{\frac{\pi L}{12E}}$$

$$\langle\psi|\alpha_{-n}\alpha_n|\psi\rangle = N_n n$$

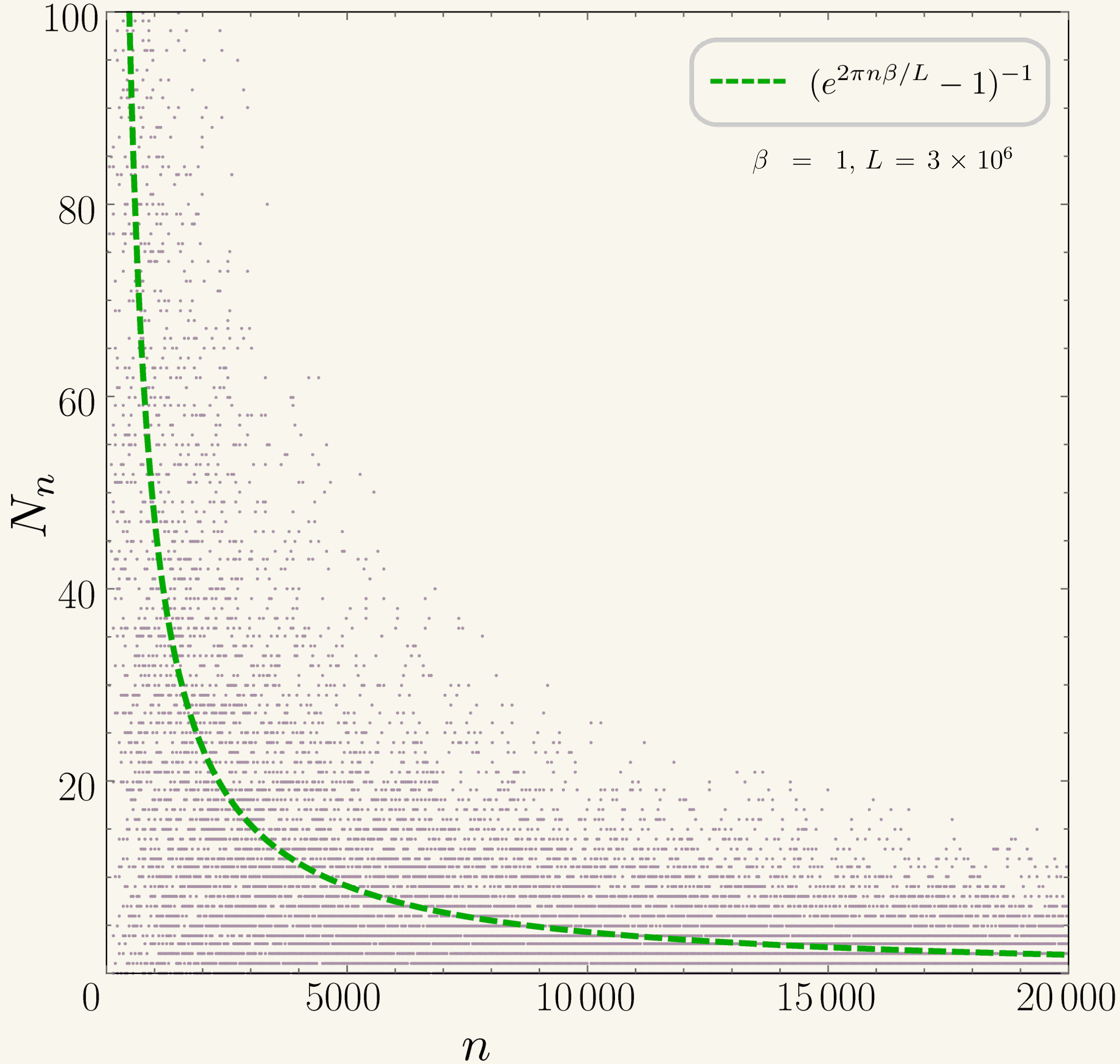
microstate is an eigenstate of $\alpha_{-n}\alpha_n$.

If states having the above properties are chosen at random, then the occupation number is again chosen from a Boltzmann distribution.

$$P(N_n) = \frac{e^{2\pi i\tau N_n n}}{\sum_{N_n=0}^{\infty} e^{2\pi i\tau N_n n}}$$

Typicality ~ randomly choosing N_n from $P(N_n)$.

Occupation numbers

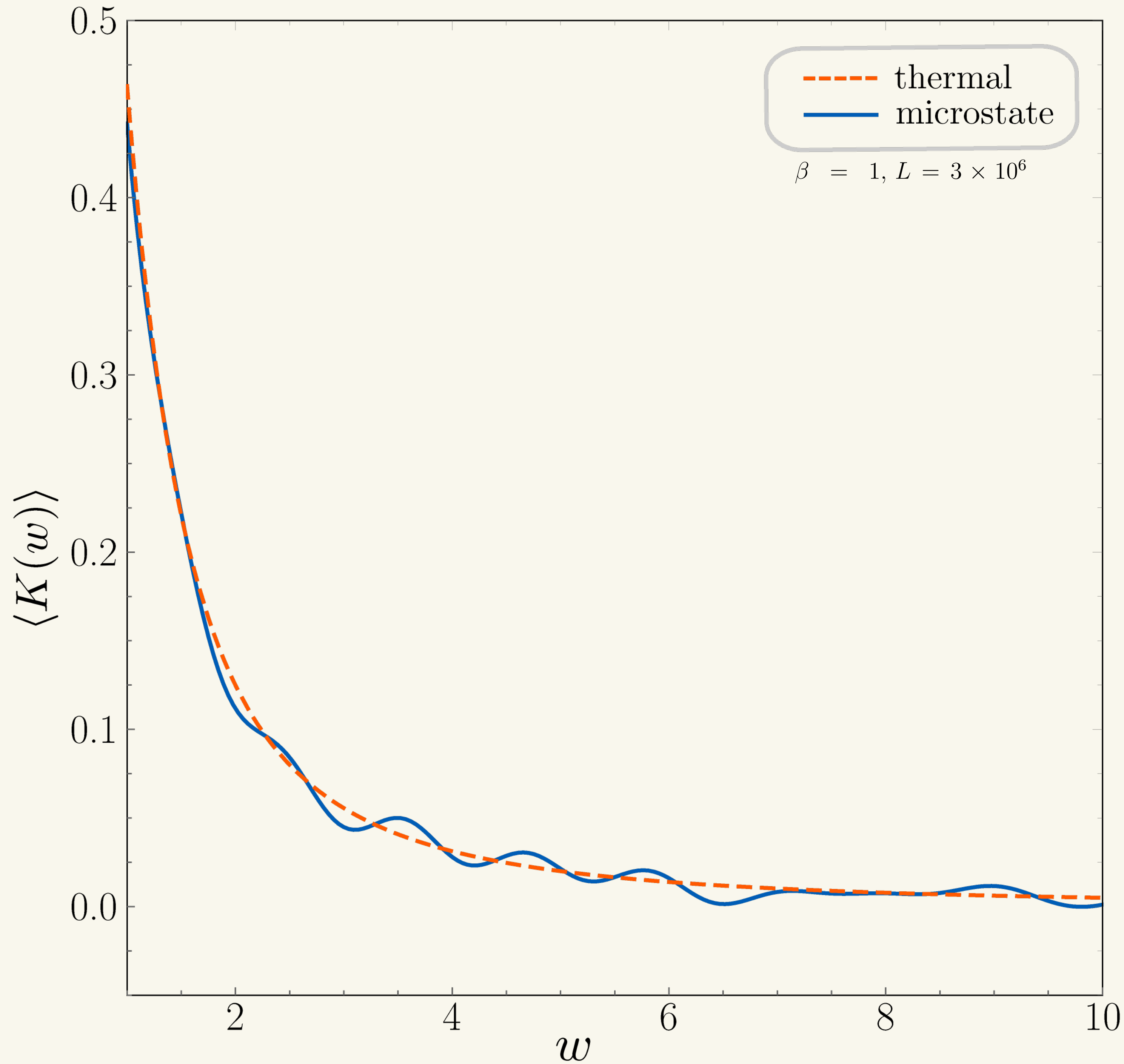


A sample of Boltzmann distributed occupation numbers. The mean is given by the Bose-Einstein function.

The variance of N_n is itself large but it can be shown that the correlator has small variance.

$$\delta\langle J(w)J(0)\rangle_{L,\beta} \sim \frac{1}{\sqrt{L}} \text{ as } L \to \infty$$

Microstate v/s thermal correlator



Stress-tensor correlators

Stress tensor correlators

A similar analysis can be performed for stress tensor correlators.

We wish to establish that

$$\langle \text{typ}|T(w)T(0)|\text{typ}\rangle = \langle T(w)T(0)\rangle_\beta$$

The thermal 2-point function is

$$\langle T(w')T(w)\rangle_\beta = \left(\frac{\pi^2 c}{6\beta^2}\right)^2 + \frac{c}{32}\left(\frac{2\pi}{\beta}\right)^4 \frac{1}{\sinh^4(\frac{\pi}{\beta}(w'-w))}$$

Microstate stress tensor correlators

Once again we use the mode expansion

$$T(w) = -\left(\frac{2\pi}{L}\right)^2 \left(L_0 - \frac{c}{24}\right) - \left(\frac{2\pi}{L}\right)^2 \sum_{n\neq 0} L_n e^{\frac{2\pi i n w}{L}}$$

to get the microstate correlator

$$\begin{aligned}\langle\psi_h|T(w)T(0)|\psi_h\rangle &= \left(\frac{2\pi}{L}\right)^4 \left(h - \frac{c}{24}\right)^2 - \left(\frac{2\pi}{L}\right)^4 \left(\frac{h}{2\sin^2(\frac{\pi w}{L})} - \frac{c}{32\sin^4(\frac{\pi w}{L})}\right) \\ &\quad +2\left(\frac{2\pi}{L}\right)^4 \sum_{n>0} \langle\psi_h|L_{-n}L_n|\psi_h\rangle \cos\left(\frac{2\pi n w}{L}\right) .\end{aligned}$$

Microstate stress tensor correlators

$$
\begin{aligned}
\langle\psi_h|T(w)T(0)|\psi_h\rangle \;=\; & \left(\frac{2\pi}{L}\right)^4\left(h-\frac{c}{24}\right)^2-\left(\frac{2\pi}{L}\right)^4\left(\frac{h}{2\sin^2(\frac{\pi w}{L})}-\frac{c}{32\sin^4(\frac{\pi w}{L})}\right)\\
& +2\left(\frac{2\pi}{L}\right)^4\sum_{n>0}\boxed{\langle\psi_h|L_{-n}L_n|\psi_h\rangle}\cos\left(\frac{2\pi n w}{L}\right)\;.
\end{aligned}
$$

need to evaluate this
for a typical state

The typical state is a descendant. We can replace $\langle\psi_h|L_{-n}L_n|\psi_h\rangle$ by the thermal average, provided the variance is small.

Microstate stress tensor correlators

$$\begin{aligned}\langle\psi_h|T(w)T(0)|\psi_h\rangle \;=\;& \left(\frac{2\pi}{L}\right)^4\left(h-\frac{c}{24}\right)^2-\left(\frac{2\pi}{L}\right)^4\left(\frac{h}{2\sin^2(\frac{\pi w}{L})}-\frac{c}{32\sin^4(\frac{\pi w}{L})}\right)\\ &+2\left(\frac{2\pi}{L}\right)^4\sum_{n>0}\boxed{\langle\psi_h|L_{-n}L_n|\psi_h\rangle}\cos\left(\frac{2\pi n w}{L}\right)\ .\end{aligned}$$

The thermal 2-point function is indeed recovered by the replacing $\langle\psi_h|L_{-n}L_n|\psi_h\rangle$ above by the thermal expectation value of the operator $L_{-n}L_n$ of a single Verma module.

$$\langle L_{-n}L_n\rangle_{h_p,\beta}=\frac{1}{Z_{h_p}(q)}\mathrm{Tr}_{h_p}\left[L_{-n}L_n q^{L_0-c/24}\right]\approx\frac{1}{e^{\frac{2\pi\beta n}{L}}-1}\left[\frac{c}{12}n^3+\left(h_p+\frac{L^2}{24\beta^2}\right)2n\right]$$

[Maloney-Ng-Ross-Tsiares]

Agreement only for real w. For complex w, there is agreement only within the strip $|\mathrm{Im}(w)|<\beta$.

Small variance

$$\langle\psi_h|T(w)T(0)|\psi_h\rangle = \left(\frac{2\pi}{L}\right)^4\left(h-\frac{c}{24}\right)^2 - \left(\frac{2\pi}{L}\right)^4\left(\frac{h}{2\sin^2(\frac{\pi w}{L})} - \frac{c}{32\sin^4(\frac{\pi w}{L})}\right)$$
$$+2\left(\frac{2\pi}{L}\right)^4\sum_{n>0}\boxed{\langle\psi_h|L_{-n}L_n|\psi_h\rangle}\cos\left(\frac{2\pi n w}{L}\right) .$$

The replacement

$$2\left(\frac{2\pi}{L}\right)^4\sum_{n>0}\langle\psi_h|L_{-n}L_n|\psi_h\rangle\cos\left(\frac{2\pi n w}{L}\right) \to 2\left(\frac{2\pi}{L}\right)^4\sum_{n>0}\langle L_{-n}L_n\rangle_{h_p,\beta}\cos\left(\frac{2\pi n w}{L}\right)$$

is allowed only if the variance of the following operator is small

$$X(w)\equiv\sum_{n>0}X_n\cos\left(\frac{2\pi n w}{L}\right)\equiv\left(\frac{2\pi}{L}\right)^4\sum_{n>0}L_{-n}L_n\cos\left(\frac{2\pi n w}{L}\right)$$

Small variance

$$X(w) \equiv \sum_{n>0} X_n \cos\left(\frac{2\pi n w}{L}\right) \equiv \left(\frac{2\pi}{L}\right)^4 \sum_{n>0} L_{-n} L_n \cos\left(\frac{2\pi n w}{L}\right)$$

It can be shown by using the Virasoro algebra that the dominant modes contributing to the sum have

$$\langle L_{-m} L_m L_{-n} L_n \rangle_{h_p,\beta} \approx \langle L_{-m} L_m \rangle_{h_p,\beta} \langle L_{-n} L_n \rangle_{h_p,\beta}$$

which implies

$$\delta X^2 = \langle X^2 \rangle - \langle X \rangle^2 \sim \frac{1}{L}$$

Sample computation

$$L_n q^{L_0} = q^{L_0+n} L_n \qquad \mathrm{Tr}[L_0 q^{L_0}] = q\partial_q \mathrm{Tr}[q^{L_0}]$$

$$\begin{aligned}\mathrm{Tr}[L_{-n} L_n q^{L_0}] = q^n\, \mathrm{Tr}[L_{-n} q^{L_0} L_n] &= q^n \mathrm{Tr}[L_n L_{-n} q^{L_0}] \\ &= q^n \mathrm{Tr}[\underbrace{[L_n, L_{-n}]}_{2nL_0 + \frac{c}{12}(n^3 - n)} q^{L_0}] + q^n \mathrm{Tr}[L_{-n} L_n q^{L_0}]\end{aligned}$$

$$\mathrm{Tr}[L_{-n} L_n q^{L_0}] = \frac{q^n}{1-q^n} \left[2nq\partial_q Z + \frac{c}{12} n^3 Z\right]$$

[Maloney-Ng-Ross-Tsiares]

Sample computation

$$\langle L_m L_n L_p \rangle = \frac{1}{1-q^p}\left[(p-n)\langle L_m L_{-m}\rangle + (p-m)\langle L_{-n} L_n\rangle\right]$$

$$\begin{aligned}\langle L_{-n} L_n L_{-m} L_m \rangle \quad = \quad & \frac{q^n}{1-q^n}\left(\langle L_n L_{-m}[L_m, L_{-n}]\rangle + \langle L_n [L_{-m}, L_{-n}] L_m\rangle\right. \\ & \left. + \langle [L_n, L_{-n}] L_{-m} L_m\rangle\right).\end{aligned}$$

Higher point functions

This analysis can be extended to higher point correlators

$$\langle\psi_h|\,T(w_1)\dots T(w_n)\,|\psi_h\rangle = (-1)^n\left(\frac{2\pi}{L}\right)^{2n}\sum_{\substack{i_1\dots i_n\\ \sum i_k=0}}\langle\psi_h|\,L_{i_1}\dots L_{i_n}\,|\psi_h\rangle\,e^{\frac{2\pi i}{L}\sum_p i_p w_p}$$

The fluctuations can again be shown to be $1/L$.

This is true when the number of stress tensors in the correlator is held fixed as $L/\beta\to\infty$.

Thank you