Universal Hamiltonians for Exponentially Long Simulation: Exploring Susskind’s Conjecture

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Joint work with Fernando Brandão

What do I have for you?

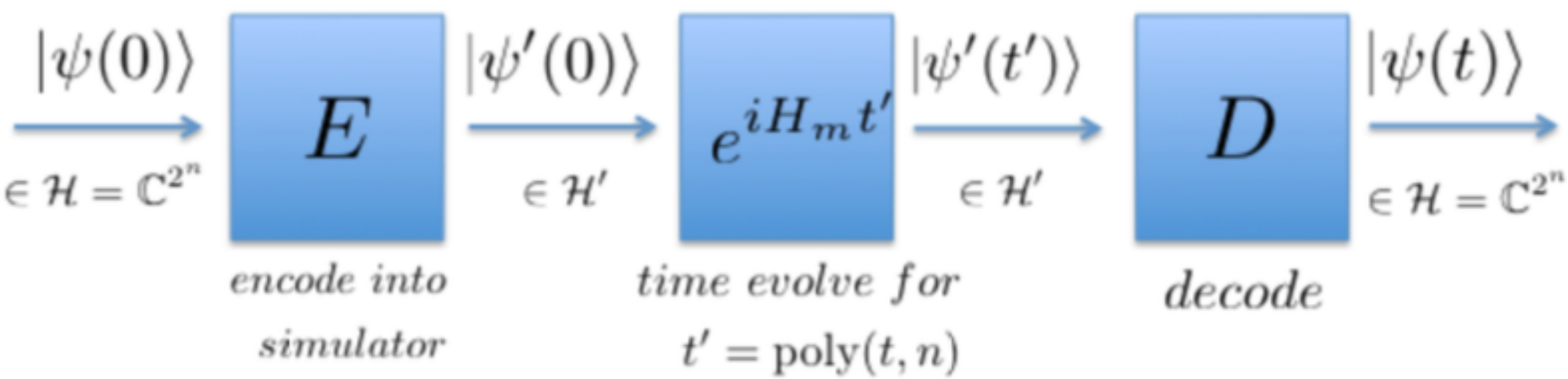
- A new construction and result in simulation of Hamiltonian dynamics
- Progress towards a conjecture by Susskind (Complexity + Holography)
- The **most complex** Hamiltonian

Hamiltonian Simulation

- What does this mean?
- Analogue simulation reproduces all possible physics of a Hamiltonian: eigenstates, spectrum, observables, thermal properties, dynamics, etc. within tolerable error
- Cubitt et. al. have very nice universality results for analogue simulation: 2D Heisenberg with tunable couplings can do anything! (arXiv: 1701.05182)

Hamiltonian Simulation

- In this work, we are concerned with universality for a very restricted notion of simulation: the simulation of Hamiltonian dynamics



Universality

- Here, universality of our simulation scheme refers to the ability to simulate the dynamics of *any* time-independent Hamiltonian

State of the Art

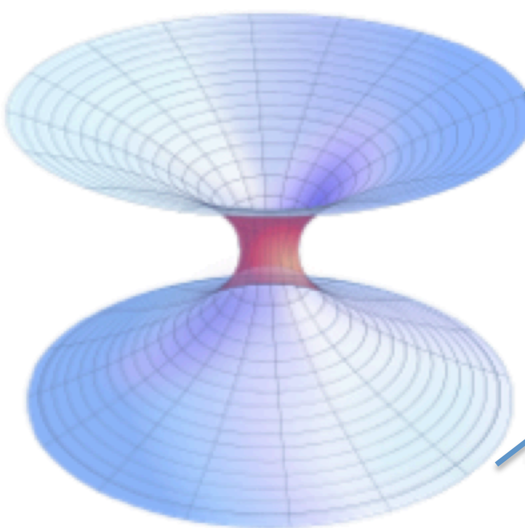
- No known simulation schemes can faithfully simulate quantum dynamics for times up to exponentially large in the system size (without exponential space resources)
- Ours can!

Circuit Complexity

- The **Circuit Complexity** of a **state** $|\psi\rangle$ is the minimum number of two-qubit gates from a fixed gate set that is required in order to build a quantum circuit that creates that state from the trivial reference state $|0\rangle^{\otimes n}$
- The **circuit complexity** of a **unitary** U is the minimum number of two-qubit gates from a fixed gate set required to build a circuit that implements U

Why Might You Care: Holography and Complexity

- Consider a non-traversable AdS wormhole connecting two black holes, whose dual/ boundary theory is a pair of entangled CFTs



$$|TDS\rangle = \sum_{i=1}^{2^n} |i\rangle_{CFT1} \otimes |i\rangle_{CFT2}$$

Holography and Complexity

- Classical gravity dictates that the volume of the wormhole increases linearly in time up until it saturates at a time exponentially large in system size, and hits recurrences at doubly exponential times

Holography and Complexity

- AdS/CFT duality suggests that there should be an analogous physical quantity in the boundary CFT that has similar qualitative behavior
- Dynamical quantities in quantum field theories tend to saturate quickly
- So… what kind of quantity in the CFT could be dual to the ever-growing AdS wormhole volume?

Susskind's Proposal

- Susskind has proposed that it should be the circuit complexity of the CFT thermofield double state that behaves this way!

$$|TDS(t)\rangle = \sum_{i=0}^{2^n} e^{iHt}|i\rangle \otimes e^{iHt}|i\rangle$$

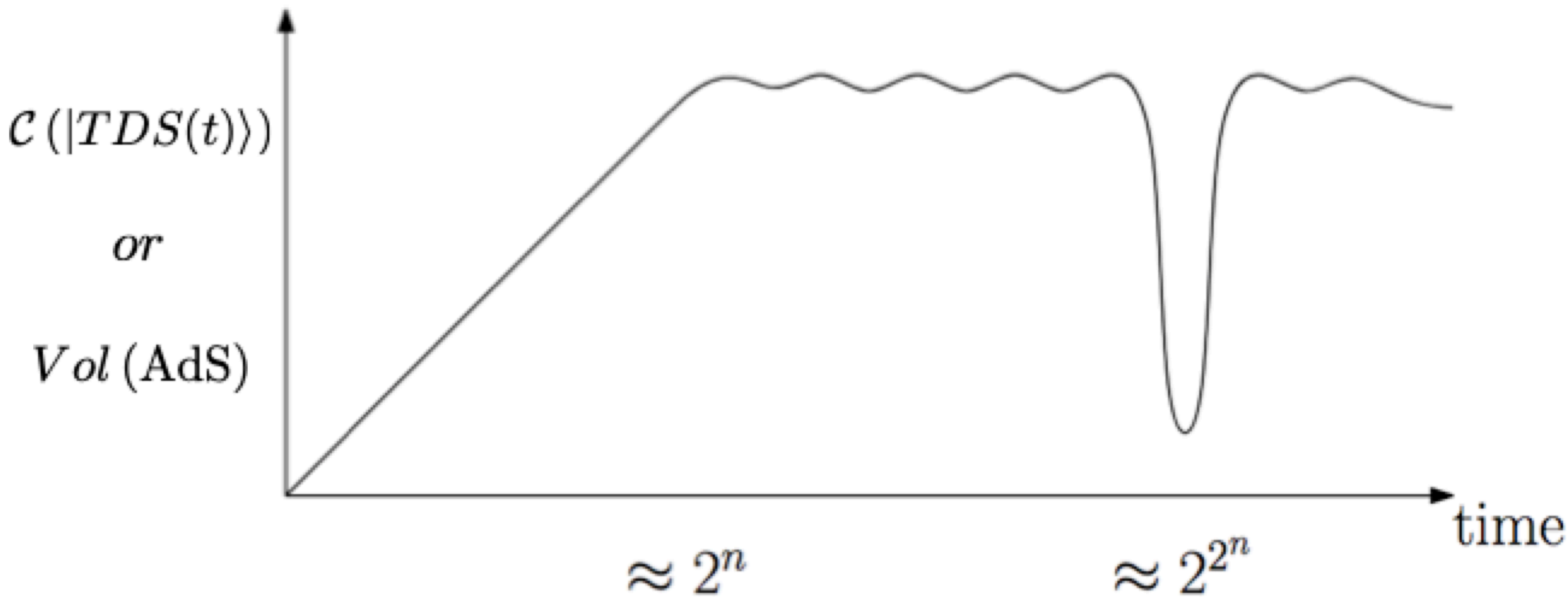
Susskind’s Proposal

- Starting with a standard maximally entangled TFD state (which has trivial complexity), time evolution under the CFT’s Hamiltonian should generate a state whose complexity is increasing **linearly** in time up to **exponentially long times**

$$\mathcal{C}\left(|TDS(t)\rangle\right) = \Theta(t)$$

$$t \geq 2^n \implies \mathcal{C}\left(|TDS(t)\rangle\right) \sim 2^n$$

Susskind's Proposal



Susskind's Proposal

- Aaronson and Susskind (arXiv: 1607.05256) have proved the following: Assuming that PSPACE is not contained in PP/poly, then there exists a time t=c^n and a polynomial size unitary U such that

$$\mathcal{C}\left(U^t|TDS(0)\rangle\right) \sim 2^n$$

Wishlist

- Would be better if it were a physically reasonable time evolution from a CFT Hamiltonian that generated the exponentially complex state
- Would also be better if linear growth were explicit

Two Questions

- Question 1: Is there a physically reasonable Hamiltonians we could write down whose time evolution generates a circuit whose complexity is exponentially large after exponentially long time evolutions?
- Question 2: Can one faithfully simulate the dynamics of an n-qubit system for times exponential in n using polynomial resources?

Two Birds With One Stone

- Motivated by the Aaronson/Susskind problem, we built a family of Hamiltonians that actually addresses both!
- Specifically: we have a family of geometrically local, translation invariant, time independent Hamiltonians whose dynamics can faithfully simulate the dynamics of any Hamiltonian for times up to exponential in the system size

And?

- We can show that under suitable conditions, it can generate a state of exponentially large complexity after an exponentially long time evolution

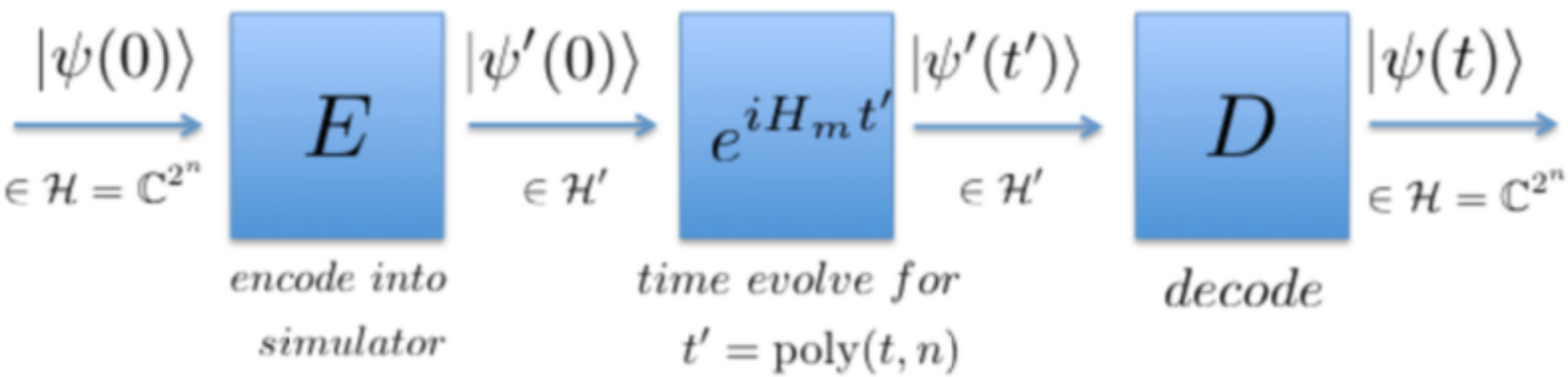
Technical Statement of Main Results

Definition 1 *A family of Hamiltonians* $\{H_m\}_{m\in\mathbb{N}}$, *indexed by the number of qudits* m *they act on, is called a* Circuit Universal Hamiltonian Family *if for every* poly(n)-*sized circuit* U *on* n *qubits and time* t, *there are* poly(n)-*sized quantum circuits* D *and* E, $m = \mathrm{poly}(n, \log t)$, *and* $t' = \mathrm{poly}(t, n)$ *such that*

$$\|U^t - (I^{\otimes n} \otimes \langle 0^{m-n}|) D e^{iH_m t'} E (I^{\otimes n} \otimes |0^{m-n}\rangle)\| < 1/\mathrm{poly(n)}.$$

Theorem 2 *There is a* Circuit Universal Hamiltonian Family *in one spatial dimension with translation-invariance and local spin dimension of 14580.*

Unpacking Definition 1



How?

- Our construction uses the concepts of Hamiltonian computation (as explored by Nagaj) and cellular automata to build a Hamiltonian whose local terms are a set of 54 carefully chosen local cellular automaton transition rules acting on a spin chain of local dimension 14580

Construction Overview

- We build what is called a Hamiltonian Quantum Cellular Automaton (HQCA)
- Basically: take a classical reversible cellular automaton (state space and reversible transition rules)
- Encode these transition rules into local Hamiltonian terms for H
- Time evolution under H will produce quantum superpositions of states of your classical CA state space!

HQCA?

Definition 3. *A Hamiltonian Quantum Cellular Automaton (HQCA) is a local, time-independent, translation-invariant Hamiltonian H on a lattice of qudits which carries out quantum computation via the following sequence of steps:*

1. *The input of the computation as well as information describing the computation to be performed (which is described by some unitary operator U) is encoded in the state of the qudits.*
2. *The qudits undergo continuous time evolution under H for some time t.*
3. *A simple basis state measurement on a subset of the qudits collapses, with high probability, the state of the whole system to one where an appropriate subset of the qudits contains the desired output of the quantum computation.*

What should our HQCA do?

- Well, what I promised you is a single Hamiltonian that can simulate *all* possible dynamics
- To do this, there has to be a way of specifying *which* dynamics you want to simulate. That is, what is the unitary U that we want to apply?
- This is specified as input to the simulation protocol

But…

- If we’re interested in simulating dynamics for a long and complicated time evolution, this means we need to describe a long and complicated circuit! So, naively, the simulator would need to be exponentially large for exponentially long time evolution
- However, since the Hamiltonians we’re simulating are time independent…

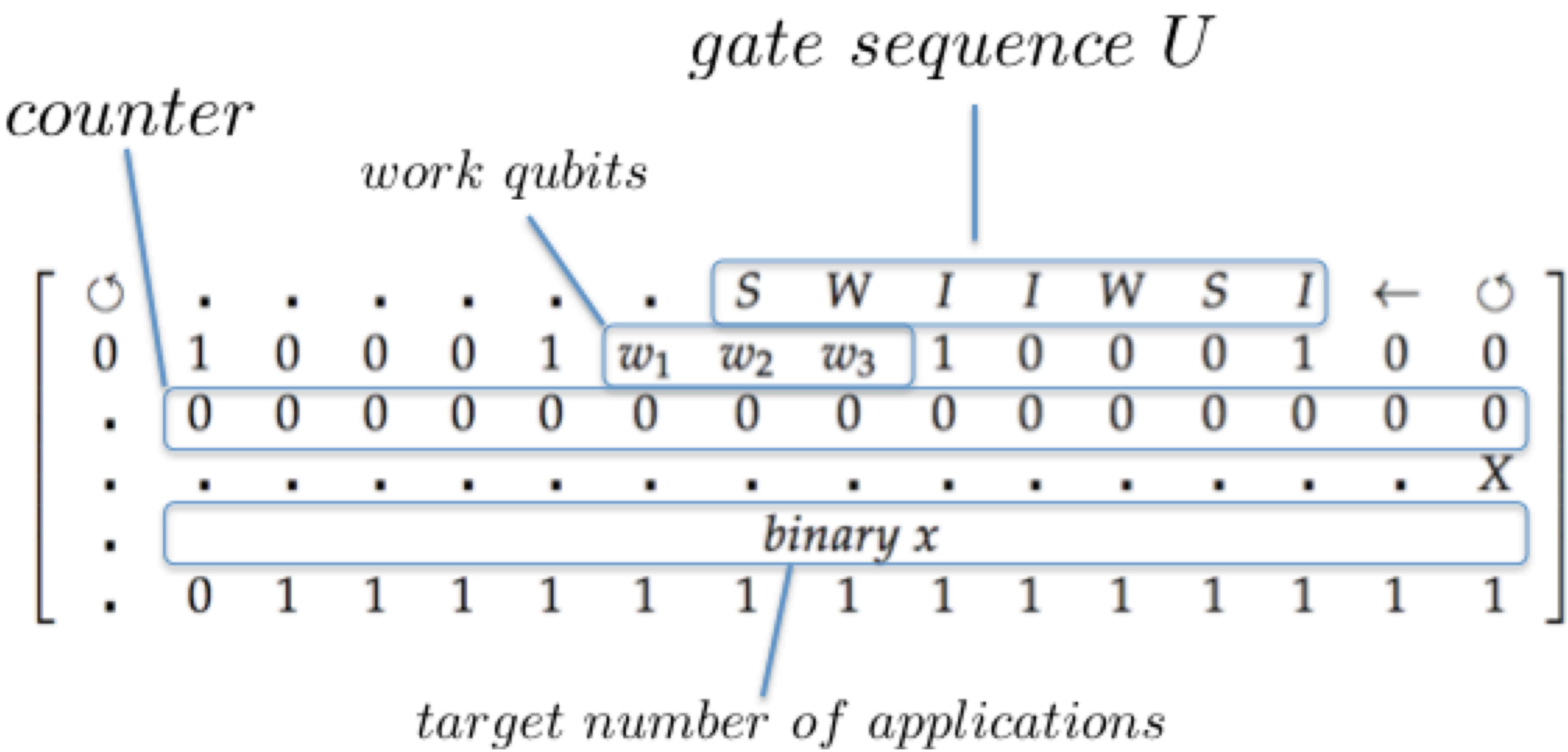
$$t \sim \mathrm{poly}(n) \implies \mathcal{C}\left(e^{iHt}\right) = \mathrm{poly}(n)$$

$$t \sim 2^n \implies \begin{array}{l} e^{iHt} = U^t \\ \mathcal{C}(U) = \mathrm{poly}(n) \end{array}$$

So then:

- Our simulator is an HQCA that takes an input state for some n-qubit system, a description of a poly(n) circuit U whose repeated application generates our desired time evolution, and then simply goes through the motions of applying U gate by gate to the system over and over!

Here it is…


gate sequence U
counter
work qubits
S W I I W S I
w_1 w_2 w_3
X
binary x
target number of applications

Why does it work?

$$H = \sum H_i$$

$$e^{iHt} = I + itH - \frac{t^2}{2}H^2 - \frac{it^3}{6}H^3 + \dots$$

$$e^{itH}|\psi_0\rangle = |\psi_0\rangle + itH|\psi_0\rangle - \frac{t^2}{2}H^2|\psi_0\rangle - i\frac{t^3}{6}H^3|\psi_0\rangle + \dots$$

So?

- Thanks to carefully engineered local transition rules making up our simulator Hamiltonian, the problem ends up looking the same as a quantum particle hopping on a 1D line
- Just need to wait for the particle to hop far enough!

The Simulation in a nutshell

- Come up with a poly(n) U that will generate the dynamics you want
- Feed its description into the simulator, wait long enough for most of the amplitudes concentrate on the particle having diffused "far enough"
- Measure the counter to collapse the state of the work qubits to the desired one with high probability

(Overly) Technical Details

- I'm not going to describe the full state space and transition rules − read the paper
- Length of chain: m=poly(n, log(t))
- Number of discrete time steps before U is applied k times: T=poly(n,k)

Complexity Growth of Dynamics

- The simulation Hamiltonian H is time-independent, translation invariant, local
- Run it with U from Aaronson and Susskind's argument (U is the step function of a universal classical cellular automaton that can solve PSPACE-complete problems)

Most Complex Hamiltonian

- The circuit complexity of our Hamiltonian's evolution must (asymptotically) be as complex as any other time independent Hamiltonian
- This is because it generates the time evolution of any other TI Hamiltonian with only polynomial overhead!

In Conclusion

- Simulation scheme that allows exponentially long simulation time
- Hamiltonians that generate the most complex time evolutions possible
- A physical Hamiltonian whose time evolution supports Susskind conjecture

Thank you!!