Universality of Squashed-Sphere Partition Functions

Robie A. Hennigar

Memorial University of Newfoundland

Main focus:

1. P. Bueno, P. A. Cano, R. A. Hennigar, R. B. Mann, *NUTs and Bolts beyond Lovelock*, JHEP 1810 (2018) 095; [arXiv: 1808.01671]

2. P. Bueno, P. A. Cano, R. A. Hennigar, R. B. Mann, *Universality of Squashed-Sphere Partition Functions*, PRL 122 (2019); [arXiv: 1808.02052]

Setup

- Setting: odd-dimensional Euclidean CFTs; main results for $d = 3$

- Question: CFT on some curved background; how does the partition function depend on this background geometry? Can any general results be obtained?

- Basic Idea: Place CFT on round sphere and then turn on a squashing deformation, study (using holography) partition function for small squashing parameter

$$\bar{g}_{\mu\nu} \to \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu} \qquad Z = \int \mathcal{D}\varphi e^{-I[\varphi, \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}]} \qquad F_{\mathbb{S}^d_\varepsilon} := -\log Z$$

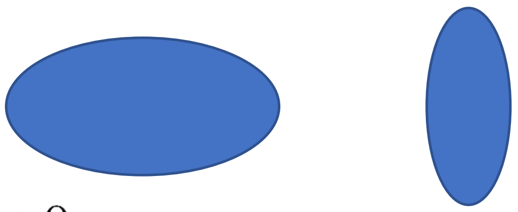
Setup

- Our focus will be particular class of squashed-spheres, written as Hopf fibrations over $\mathbb{S}^2$ (more generally $\mathbb{CP}^{(d-1)/2}$)

$$ds^2_{\mathbb{S}^3_\varepsilon} = (1+\varepsilon)\left(d\psi + \cos\theta d\phi\right)^2 + d\theta^2 + \sin^2\theta d\phi^2$$

- Squashing parameter $\varepsilon \in [-1, +\infty)$, $\varepsilon = 0 \Rightarrow$ round sphere

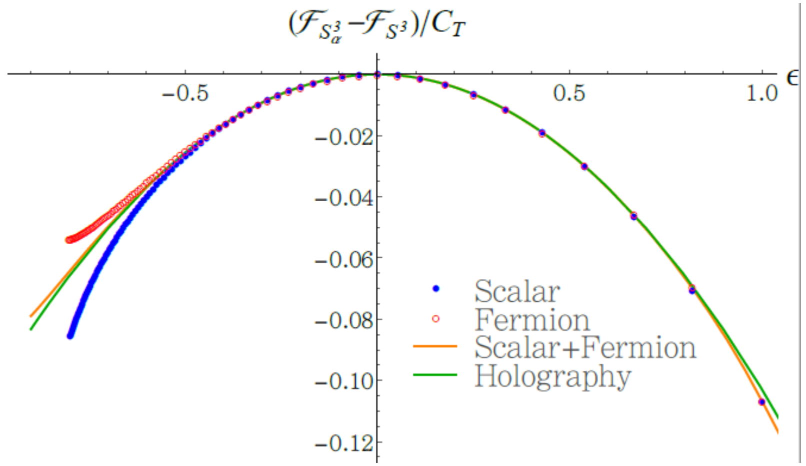
- "Easy" to get holographic input: dual geometries AdS-Taub-NUT/Bolt



$\varepsilon < 0$ $\qquad$ $\varepsilon > 0$

Known results

- Appears in different contexts
  - SUSY, localization results [eg: Imamura, Tokoyama; Closset, Dumitrescu, Festucia, Komargodski; Nishioka, Yonekura]
  - Round sphere, F-theorem for 3d CFTs [e.g. Casini, Huerta]
  - Holographic cosmology [eg: Anninos, Denef, Harlow; Hertog, Hawking]



- General results: N. Bobev, P. Bueno, Y. Vreys JHEP 1707 (2017) 093; S. Fischetti, T. Wiseman, JHEP 1712 (2017) 133

$$F_{\mathbb{S}^3_\varepsilon} = F_{\mathbb{S}^3_0} - \frac{\pi^4}{6} C_T \varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

- Round sphere is local extremum; leading correction controlled by central charge; **cubic correction difficult**
  - Einstein gravity not helpful →

$$F^{\mathrm{Ein}}_{\mathbb{S}^3_\varepsilon} = F_{\mathbb{S}^3_0} - \frac{\pi^4}{6} C_T \varepsilon^2$$

Euclidean Taub-NUT/Bolt-AdS solutions

$$ds^2 = V(r)(d\tau + 2n\cos\theta d\phi)^2 + \frac{dr^2}{V(r)} + (r^2 - n^2)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

"NUT" parameter

- Fixed points of Killing vector ∂_τ (where $V(r_+) = 0$) of two types:
  - NUT type: 0-dimensional, i.e. $r_+ = n$
  - Bolt: 2-dimensional, i.e. $r_+ > n$

Boundary Geometry

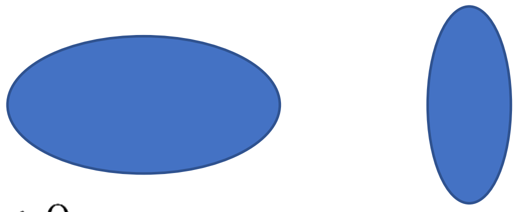
- The boundary geometry is a squashed 3-sphere

$$\frac{{}^{(3)}ds^2}{r^2} = \frac{4n^2}{f_\infty L^2}(d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2 + \mathcal{O}(r^{-2})$$

$$\frac{\tau}{2n}$$

- Useful to introduce squashing parameter

$$\varepsilon := \frac{4n^2}{f_\infty L^2} - 1$$



$\varepsilon < 0$ $\varepsilon > 0$

- The case $\varepsilon = 0$ corresponds to round 3-sphere

Higher-curvature gravity

$$\int d^{d+1}x\sqrt{-g}\,[-2\Lambda + R] \Rightarrow \int d^{d+1}x\sqrt{-g}\,\underbrace{\left[-2\Lambda + R + f_2\left(\mathcal{R}^2\right) + f_3\left(\mathcal{R}^3\right) + \cdots\right]}_{\mathcal{L}\left(g^{\alpha\beta},\, R_{\mu\nu\sigma\rho}\right)}$$

$R_{\mu\nu}R^{\mu\nu}$ $\quad R^2 \quad$ $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$

- Renormalization of quantum fields in curved space; renormalizable quantum gravity

  N. Birrell, P. Davies, (1982) "Quantum Fields in Curved Space"
  K. Stelle, Phys. Rev. D **16**, 953 (1977)

- Low energy effective actions in string theory

  For example, D. Gross, E. Witten, Nucl. Phys. B277 (1986)

- **Toy models for black hole thermodynamics and holography**

  M. Brigante, H. Liu, R. C. Myers, S. Shenker, S. Yaida, Phys. Rev. D77 (2008) 126006
  R. Myers, M. Paulos, A. Sinha JHEP 1008 (2010) 035

Higher-curvature toy models & AdS/CFT

- "Bottom up" approach; impose consistency conditions, control calculations
- Introduce additional parameters (couplings); allows to distinguish various charges
- Has been used to identify 'universal relationships' that hold (or fail to hold) beyond holography

	$\langle T_{\mu\nu}T_{\alpha\beta}\rangle$	$\langle T_{\mu\nu}T_{\alpha\beta}T_{\gamma\delta}\rangle$	$s = C_S T^{d-1}$	$s_{\rm EE} \sim \frac{\partial A}{\delta^{d-2}} - 2\pi a^*$
	C_T	$C_T \cdot t_4$	C_S	a^*
Einstein	$\frac{\Gamma(d+2)}{8(d-1)\Gamma(d/2)\pi^{(d+2)/2}}\frac{\tilde{L}^{d-1}}{G}$	0	$\frac{\Gamma(d+1)\pi^{(2d-1)/2}2^{d-3}}{\Gamma(\frac{d+1}{2})\Gamma(d/2)d^d}\frac{\tilde{L}^{d-1}}{G}$	$\frac{\pi^{(d-2)/2}}{8\Gamma(d/2)}\frac{\tilde{L}^{d-1}}{G}$
QTG $(d=4)$	$(1-3\mu f_\infty^2)C_T^{\rm E}$	$3780\mu f_\infty^2 C_T^{\rm E}$	$f_\infty^3 C_S^{\rm E}$	$\left(1+9\mu f_\infty^2\right)a^{*\rm E}$

$$1 - f_\infty + \mu f_\infty^3 = 0$$

R. Myers, B. Robinson JHEP 1008 (2010) 067
R. Myers, M. Paulos, A. Sinha JHEP 1008 (2010) 035

Higher-curvature toy models & AdS/CFT

- Has been used to identify 'universal relationships' that hold (or fail to hold) beyond holography
  - Higher-derivative corrections to η/s [Brigante, Liu, Myers, Shenker, Yaia, Edelstein, Camanho, Buchel, Paulos, Sinha]
  - Holographic c-theorems [Myers, Sinha]
  - Universal terms in holographic entanglement entropy [Bueno, Myers, Witczak-Krempa, Mezei]

- Problem: In general hard to construct explicit solutions (exceptions: Lovelock; quasi-topological gravity $d \geq 4$)

	C_T	$C_T \cdot t_4$	C_S	a^*
Einstein	$\frac{\Gamma(d+2)}{8(d-1)\Gamma(d/2)\pi^{(d+2)/2}} \frac{\tilde{L}^{d-1}}{G}$	0	$\frac{\Gamma(d+1)\pi^{(2d-1)/2}2^{d-3}}{\Gamma(\frac{d+1}{2})\Gamma(d/2)d^d} \frac{\tilde{L}^{d-1}}{G}$	$\frac{\pi^{(d-2)/2}}{8\Gamma(d/2)} \frac{\tilde{L}^{d-1}}{G}$
QTG ($d=4$)	$(1-3\mu f_\infty^2)C_T^{\mathrm{E}}$	$3780\mu f_\infty^2 C_T^{\mathrm{E}}$	$f_\infty^3 C_S^{\mathrm{E}}$	$\left(1+9\mu f_\infty^2\right) a^{*\mathrm{E}}$

R. Myers, B. Robinson JHEP 1008 (2010) 067
R. Myers, M. Paulos, A. Sinha JHEP 1008 (2010) 035

Our toy model: Einsteinian Cubic Gravity

$$I_E = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left[\frac{6}{L^2} + R - \frac{\mu L^4}{8}\mathcal{P}\right]$$

$$\mathcal{P} = 12 R_{a\ b}^{\ c\ d} R_{c\ d}^{\ e\ f} R_{e\ f}^{\ a\ b} + R_{ab}^{cd} R_{cd}^{ef} R_{ef}^{ab} - 12 R_{abcd} R^{ac} R^{bd} + 8 R_a^b R_b^c R_c^a$$

P. Bueno, P. Cano PRD 2016

- Member of a class of theories known as "generalized quasi-topological gravities"; amenable to explicit calculations

RAH, D. Kubiznak, R. Mann PRD 95, (2017)
P. Bueno, P. Cano CQG 34 175008 2017

	C_T	$C_T \cdot t_4$	C_S	a^*
Einstein	$\frac{\Gamma(d+2)}{8(d-1)\Gamma(d/2)\pi^{(d+2)/2}}\frac{\tilde{L}^{d-1}}{G}$	0	$\frac{\Gamma(d+1)\pi^{(2d-1)/2}2^{d-3}}{\Gamma(\frac{d+1}{2})\Gamma(d/2)d^d}\frac{\tilde{L}^{d-1}}{G}$	$\frac{\pi^{(d-2)/2}}{8\Gamma(d/2)}\frac{\tilde{L}^{d-1}}{G}$
ECG ($d=3$)	$(1-3\mu f_\infty^2)C_T^{\mathrm{E}}$	-1260$\mu f_\infty^2 C_T^{\mathrm{E}}$	$\left(1-\frac{27}{4}\mu\right) f_\infty^2 C_S^{\mathrm{E}}$	$\left(1+3\mu f_\infty^2\right) a^{*\mathrm{E}}$

P. Bueno, P. Cano, A. Ruiperez JHEP 03 (2018) 150

Some properties of the GQT theories

- Second-order linearized equations for gravitational perturbations on AdS (perturbatively ghost-free; unitary CFT)
- Second-order (integrated) field equations for static spherically symmetric metrics
- Vacuum static spherically symmetric black holes characterized by mass alone (no "higher derivative hair")
- Black hole thermodynamics can be studied non-perturbatively in the higher-order couplings

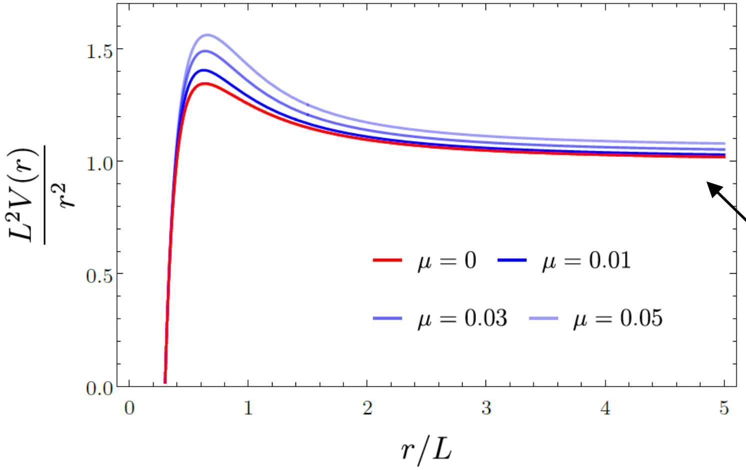
RAH, D. Kubiznak, R. Mann PRD 95, (2017)
P. Bueno, P. Cano CQG 34 175008 2017
J Ahmed, **RAH**, R. Mann, M. Mir 1705 (2017) 134
X. Feng, H. Huang, Z. Mai, H. Lu, PRD 96 (2017)
RAH JHEP 1709 (2017) 082

$$I_{\mathrm{E}} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left[\frac{6}{L^2} + R - \frac{\mu L^4}{8}\mathcal{P}\right]$$

Solving the equations

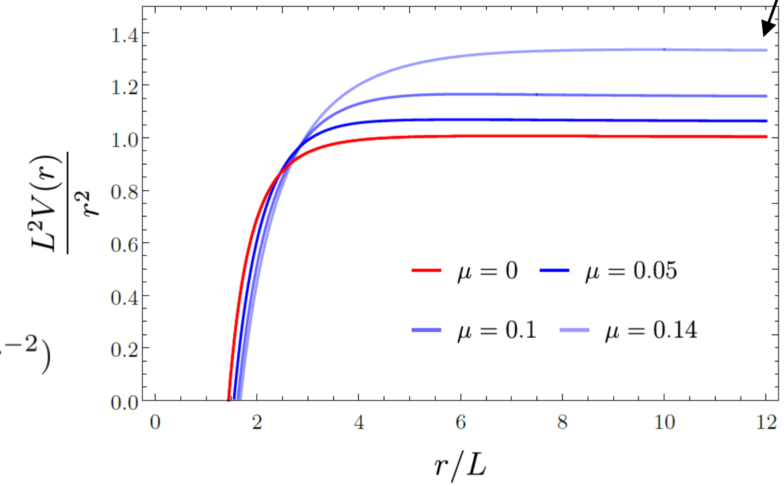
- Relatively complicated equations; full solution requires numerics
- Like black hole case, essential features can be obtained without approximations/numerics
- Thermodynamics obtained fully analytically; solutions characterized by mass alone $V(r) = f_\infty \frac{r^2}{L^2} + 1 - 5f_\infty \frac{n^2}{L^2} - \frac{2GM}{r(1-3f_\infty^2\mu)} + \mathcal{O}(r^{-2})$

Numerical NUT solution



$V(r) \sim f_\infty \frac{r^2}{L^2}$

Numerical Bolt solution



P. Bueno, P. Cano, **RAH**, R. Mann JHEP 1810 (2018) 095

$$I_{\mathrm{E}} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left[\frac{6}{L^2} + R - \frac{\mu L^4}{8}\mathcal{P}\right]$$

Evaluating the on-shell Euclidean action

- Euclidean on-shell action of asymptotically AdS solution can be computed using

$$I_E = -\int_{\mathcal{M}} d^D x\sqrt{g}\mathcal{L}(g^{ef}, R_{abcd}) - \frac{2a^*}{\Omega_{(D-2)}\tilde{L}^{D-2}}\int_{\partial\mathcal{M}}\sqrt{h}\left[K + \text{counterterms}\right],$$

Usual Einstein gravity boundary & counterterms!

R. Emparan, C. V. Johnson and R. C. Myers, PRD (1999)

Provided that linearized perturbations on vacuum obey second-order EOM

$$a^* = -\frac{\pi^{(D-1)/2}\tilde{L}^D}{(D-1)\Gamma\left[\frac{D-1}{2}\right]}\mathcal{L}|_{\mathrm{AdS}}\,.$$

R. Myers, A. Sinha JHEP 1101 (2011) 125

P. Bueno, P. Cano, A. Ruiperez JHEP 03 (2018) 150

Euclidean action for NUTs: general result

- The same procedure can be carried out at higher-order in curvature; in higher dimensions $[(d+1)$ spacetime dimensions$]$

- A pattern emerges...

Lagrangian of theory evaluated on
auxiliary AdS $(d+1)$ geometry

$$R_{\alpha\beta}{}^{\mu\nu} = \frac{-f_\infty}{(1+\varepsilon)L^2}\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta\right)$$

$$F_{\mathbb{S}^d_\varepsilon} := -\log Z \approx I_E = -\frac{\pi^{\frac{(d+2)}{2}}}{\Gamma\left[\frac{d+2}{2}\right]} \frac{\mathcal{L}\left[f_\infty/(1+\varepsilon)\right] L^{d+1}}{\left[f_\infty/(1+\varepsilon)\right]^{\frac{(d+1)}{2}}}$$

Governs asymptotic (large r)
behavior of metric function

$$V(r) \sim f_\infty \frac{r^2}{L^2}$$

$$\varepsilon := \frac{(d+1)n^2}{f_\infty L^2} - 1$$

Squashing parameter
$\varepsilon = 0$ for round sphere

P. Bueno, P. Cano, **RAH**, R. Mann PRL 122 (2019)

Euclidean action for NUTs: general result

- Verified that this expression reproduces free energy of known Taub-NUT solutions in the literature
  - Einstein gravity A. Chamblin, R. Emparan, C. Johnson, R. Myers, PRD 59 (1999) B. Clarkson, L. Fatibene, and R. Mann, Nucl. Phys. B652, 348 (2003)
  - Lovelock gravity e.g. A. Khodam-Mohammadi and M. Monshizadeh, PRD 79, (2009)
  - Generalized quasi-topological theories P. Bueno, P. Cano, **RAH**, R. Mann JHEP 1810 (2018) 095

$$\boxed{F_{\mathbb{S}^d_\varepsilon} = -\frac{\pi^{\frac{(d+2)}{2}}}{\Gamma\left[\frac{d+2}{2}\right]}\frac{\mathcal{L}\left[f_\infty/(1+\varepsilon)\right]L^{d+1}}{\left[f_\infty/(1+\varepsilon)\right]^{\frac{(d+1)}{2}}}}$$

$$F_{\mathbb{S}^d_\varepsilon} := -\log Z$$

Universality of squashed sphere partition functions

$$F_{\mathbb{S}^d_\varepsilon} = -\frac{\pi^{\frac{(d+2)}{2}}}{\Gamma\left[\frac{d+2}{2}\right]} \frac{\mathcal{L}\left[f_\infty/(1+\varepsilon)\right] L^{d+1}}{\left[f_\infty/(1+\varepsilon)\right]^{\frac{(d+1)}{2}}}$$

- Take a particular theory, expand in the limit of small squashing, write in terms of boundary quantities

$$I_{\mathrm{E}} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left[\frac{6}{L^2} + R - \frac{\mu L^4}{8}\mathcal{P}\right]$$

$$C_T^{\mathrm{ECG}} = \frac{(1-3\mu f_\infty^2)3L^2}{\pi^3 f_\infty G}\,, \quad C_T^{\mathrm{ECG}} t_4^{\mathrm{ECG}} = -\frac{3780\mu f_\infty L^2}{\pi^3 G}$$

Central charge

One of three constants characterizing stress tensor three-point function

$$\boxed{F_{\mathbb{S}^3_\varepsilon} = F_{\mathbb{S}^3_0} - \frac{\pi^2 C_T}{96}\varepsilon^2\left[1 - \frac{t_4}{630}\varepsilon + \mathcal{O}(\varepsilon^2)\right]}$$

P. Bueno, P. Cano, **RAH**, R. Mann PRL 122 (2019)

Universality of squashed sphere partition functions

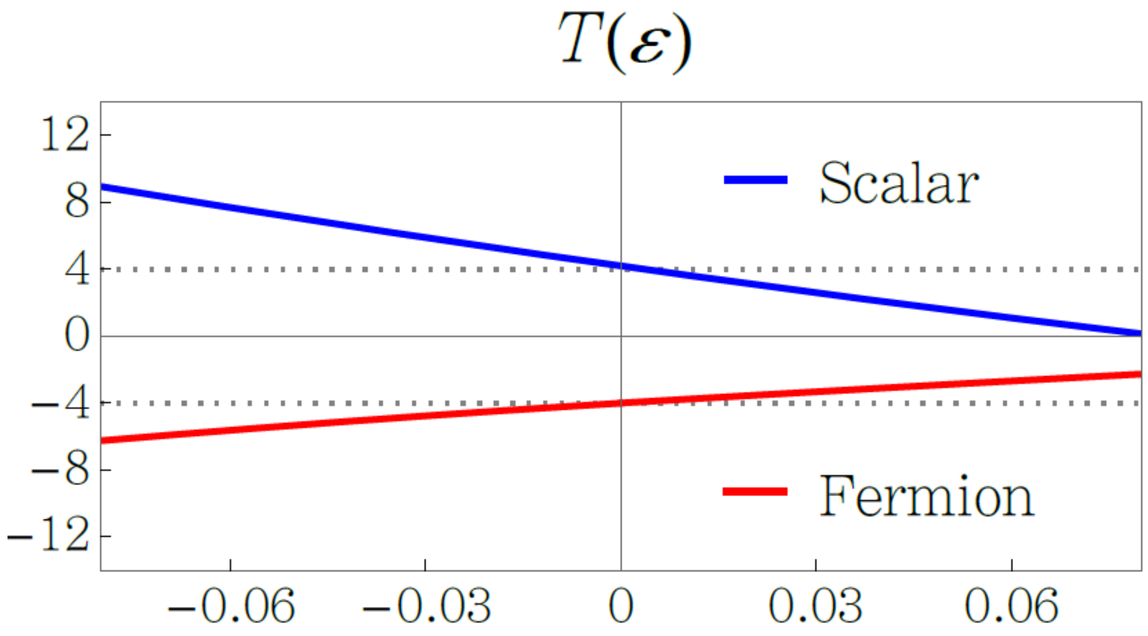
- It was shown in general (without resorting to holography) that the quadratic correction to the partition function is governed by the central charge N. Bobev, P. Bueno, Y. Vreys JHEP 1707 (2017) 093
- What's new? The cubic deformation is controlled by the stress tensor charge t_4

$$F_{\mathbb{S}^3_\varepsilon} = F_{\mathbb{S}^3_0} - \frac{\pi^2 C_T}{96}\varepsilon^2\left[1 - \frac{t_4}{630}\varepsilon + \mathcal{O}(\varepsilon^2)\right]$$

- Does it hold beyond holography?

Testing against free-fields

$$T(\varepsilon) \equiv \frac{630}{\varepsilon}\left[1 + \frac{6(F_{\mathbb{S}^3_\varepsilon} - F_{\mathbb{S}^3_0})}{\pi^4 C_T \varepsilon^2}\right]$$



- Function designed to pick out the term proportional to ε^3
- Numerical computations for free fields agree with holographic prediction

Methods of: N. Bobev, P. Bueno, Y. Vreys JHEP 1707 (2017) 093

$$t_4^{\mathrm{s}} = +4$$

$$t_4^{\mathrm{f}} = -4$$

$$F_{\mathbb{S}^3_\varepsilon} = F_{\mathbb{S}^3_0} - \frac{\pi^2 C_T}{96}\varepsilon^2\left[1 - \frac{t_4}{630}\varepsilon + \mathcal{O}(\varepsilon^2)\right]$$

Conclusions & Takeaway

- Identified a class of higher curvature theories that serve as relatively nice toy models

- Constructed the first explicit examples of (curvature corrected) Taub-NUT/Bolt solutions beyond Lovelock theory; explicit expression for free energies

$$F_{\mathbb{S}^d_\varepsilon} = -\frac{\pi^{\frac{(d+2)}{2}}}{\Gamma\left[\frac{d+2}{2}\right]} \frac{\mathcal{L}\left[f_\infty/(1+\varepsilon)\right] L^{d+1}}{\left[f_\infty/(1+\varepsilon)\right]^{\frac{(d+1)}{2}}}$$

- 'Universal' behavior for the partition function in $d = 3$

$$F_{\mathbb{S}^3_\varepsilon} = F_{\mathbb{S}^3_0} - \frac{\pi^2 C_T}{96}\varepsilon^2 \left[1 - \frac{t_4}{630}\varepsilon + \mathcal{O}(\varepsilon^2)\right]$$