Shape Derivatives of Entanglement Entropy and the Light Ray OPE

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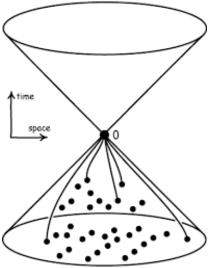
Based upon recent work:
Leichenauer, AL and Shahbazi-Moghaddam, arXiv:1802.02584.
And forthcoming work: Balakrishnan, Chandrasekaran, Faulkner, AL and Shahbazi-Moghaddam, arXiv:1906.xx.

Energy in Quantum Field Theories

Lots of recent progress has been made by connecting constraints from causality, quantum information theory and chaos to energy conditions.



Averaged Null Energy Condition

Non-local bound on stress energy

$$\hat{\mathcal{E}}_+(y) = \int_{-\infty}^{\infty} dv \, \langle T_{vv}(u = 0, v, y) \rangle \geq 0$$

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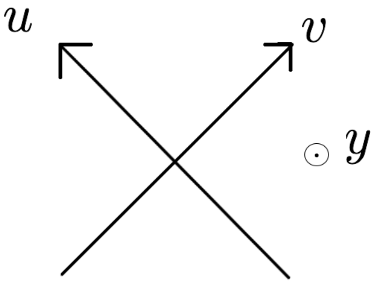

$u = x - t$ and $v = x + t$

Proved using a multitude of techniques. See [Faulkner et al. (2016)], [Hartman et al. (2016)]

A Local Quantum Energy Condition

Can we say anything about local energy density?

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The Quantum Null Energy Condition (QNEC)

$$\boxed{\langle T_{vv}(y_0)\rangle \geq \frac{1}{2\pi}\frac{d}{d\lambda}\left(\left.\frac{\delta S(\mathcal{R}(\lambda))}{\delta V(y_0)}\right|_{V(y;\lambda)}\right)}$$

$$S(\mathcal{R}(\lambda)) = -Tr[\rho_{\mathcal{R}}\log\rho_{\mathcal{R}}],\ \ \rho_{\mathcal{R}} = Tr_{\bar{\mathcal{R}}}\left|\psi\right\rangle\left\langle\psi\right|$$

- Proof uses causality as well as methods from quantum information, quantum chaos...
- Connects energy and entanglement

Conjectured: [Bousso et al. (2015)]. Proofs: [Bousso et al. (2015)], [Koeller and Leichenauer (2015)], [Balakrishnan, Faulkner, Khandker and Wang (2017)]

A Local Quantum Energy Condition

$$\boxed{\langle T_{vv}(y)\rangle \geq \frac{\hbar}{2\pi}\frac{d}{d\lambda}\left(\left.\frac{\delta S(\mathcal{R}(\lambda))}{\delta V(y)}\right|_{V(y;\lambda)}\right)}$$

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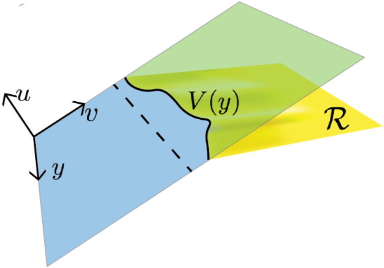
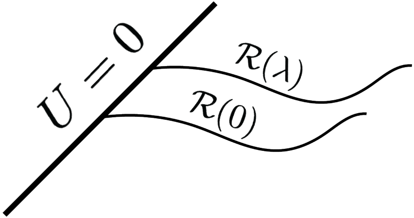
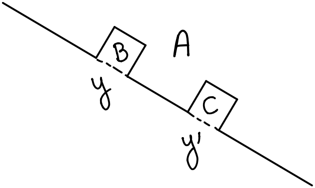




$$V(y;\lambda) = \dot{V}(y)\lambda$$

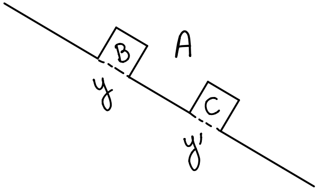
Second Variations of the Entanglement Entropy



$$\frac{d}{d\lambda}\left(\frac{\delta S}{\delta V(y)}\bigg|_{V(y;\lambda)}\right)$$
$$= \int d^{d-2}y' \frac{\delta^2 S}{\delta V(y)\delta V(y')}\frac{d}{d\lambda}V(y';\lambda)$$

Let's look at second variations of the entanglement entropy with respect to the entangling surface position.

Second Variations of the Entanglement Entropy



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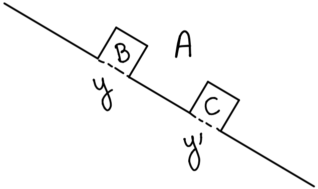
Let's look at second variations of the entanglement entropy with respect to the entangling surface position.

$$\frac{\delta^2 S}{\delta X^\mu(y)\delta X^\nu(y')}k^\mu k^\nu = S''_{vv}(y)\delta^{d-2}(y-y') + \text{off-diagonal}$$

- S'' stands for the "diagonal" variation of the entropy.

Second Variations of the Entanglement Entropy



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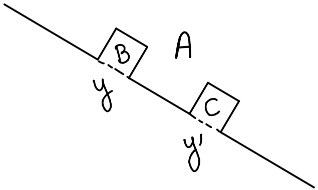
- S'' stands for the "diagonal" variation of the entropy.
- Strong sub-additivity - $S(A) - S(AB) \leq S(AC) - S(ABC)$ - implies that the off-diagonal second variations are non-positive.

The Diagonal QNEC



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By making $\frac{d}{d\lambda}V(y;\lambda) = \theta(\epsilon - |y - y_0|)$ and taking $\epsilon \to 0$ can make the QNEC become

$$T_{vv}(y_0) \geq \frac{1}{2\pi}\frac{d}{d\lambda}\left(\frac{\delta S}{\delta V(y_0)}\right) \implies T_{vv} \geq \frac{1}{2\pi}S''_{vv}$$

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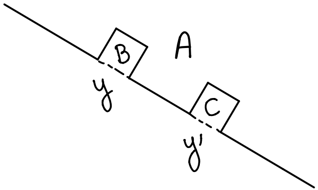

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This is the “diagonal QNEC”

Saturation of the Diagonal QNEC

$$\boxed{S''_{vv} = 2\pi \langle T_{vv} \rangle}$$

- New strong evidence that this equality holds for all QFTs with an *interacting* UV fixed point.
- N.B. *interactions are key*; Not saturated in free fields! [Bousso et al. (2015)]

$\Longrightarrow$ The full (integrated) QNEC is a result of strong sub-additivity and *is not always saturated*.

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$\implies$ The full (integrated) QNEC is a result of strong sub-additivity and *is not always saturated*.

$$\text{SSA} \implies \langle T_{vv}(y_0) \rangle \geq \frac{1}{2\pi} \frac{d}{d\lambda} \left(\frac{\delta S}{\delta V(y_0)} \right)$$

Entanglement Entropy in Near Vacuum States

Goal: Explicitly compute $\frac{\delta^2 S}{\delta V(y) \delta V(y')}$ in a special class of states.

$$|\psi\rangle = |\Omega\rangle + i\lambda O_{\mathcal{R}} |\Omega\rangle + \mathcal{O}(\lambda^2)$$

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$$\Downarrow$$

$\rho_{\mathcal{R}} = \sigma_{\mathcal{R}} + \lambda\delta\rho + \mathcal{O}(\lambda)^2,\ \delta\rho \sim \sigma_{\mathcal{R}} O_{\mathcal{R}}$ *for a flat* $V(y) = 0$ *profile*

$$\frac{\delta^2 S}{\delta V(y)\delta V(y')} = \frac{\delta^2 \Delta S}{\delta V(y)\delta V(y')}$$

Entanglement Entropy in Near Vacuum States

For such states, just expand $\Delta S(\rho) = -Tr[\rho \log \rho] + Tr[\sigma \log \sigma]$ using BCH...

$$\Delta S(\rho) \sim -\langle \log \sigma_{\mathcal{R}} \rangle_{\psi} + \langle \log \sigma_{\mathcal{R}} \rangle_{vac}$$
$$+ \lambda^2 \int_{-\infty}^{\infty} ds \frac{\langle O_{\mathcal{R}} e^{isK^{\text{vac}}} O_{\mathcal{R}} \rangle_{vac}}{\sinh^2((s - i\epsilon)/2)} + \mathcal{O}(\lambda^3)$$

[Faulkner, 1412.5648]

where

$$K^{\text{vac}} = -\log \sigma_{\mathcal{R}} + \log \sigma_{\bar{\mathcal{R}}}$$

for V(y) = 0

Entanglement Entropy in Near Vacuum States

What about for $V \neq 0$? Need this case for entropy variations...

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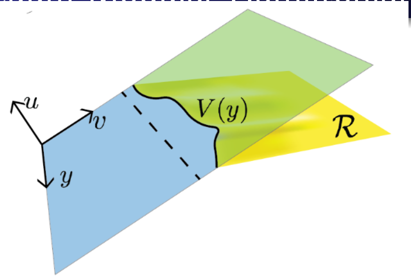
$$\Delta S[V(y)] \sim -\langle \log \sigma_{\mathcal{R}}[V(y)] \rangle_{\psi} + \langle \log \sigma_{\mathcal{R}}[V(y)] \rangle_{vac} + \int_{-\infty}^{\infty} ds \frac{\langle O_{\mathcal{R}} e^{isK^{\text{vac}}[V(y)]} O_{\mathcal{R}} \rangle}{\sinh^2((s - i\epsilon)/2)} + \mathcal{O}(\lambda^3)$$

where now

$$K^{\text{vac}}[V(y)] = -\log \sigma_{\mathcal{R}}[V(y)] + \log \sigma_{\bar{\mathcal{R}}}[V(y)]$$

Detour into Vacuum Modular Hamiltonians

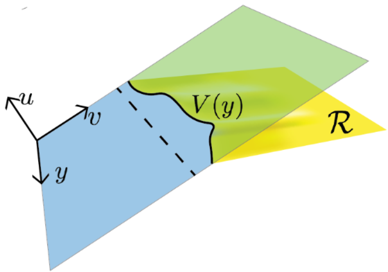


Simple form for vacuum "modular Hamiltonian" $-\log \sigma_{\mathcal{R}}[V(y)]$

$$- \langle \log \sigma_{\mathcal{R}} \rangle_{\psi} [V(y)] + \langle \log \sigma_{\mathcal{R}} \rangle_{vac} [V(y)] := \langle \Delta H^{\text{vac}}_{\mathcal{R}} \rangle_{\psi}$$
$$= 2\pi \int d^{d-2}y \int_{V(y)}^{\infty} dv (v - V(y)) \langle T_{vv}(u=0, v, y) \rangle_{\psi}$$

$$\frac{\delta^2 \langle \Delta H^{vac}_{\mathcal{R}} \rangle}{\delta V(y) \delta V(y')} = \langle T_{vv}(y) \rangle \delta(y - y')$$

Detour into Vacuum Modular Hamiltonians



$$-\log\sigma_{\mathcal{R}} + \log\sigma_{\bar{\mathcal{R}}} = K_{vac}[V(y)]$$
$$= 2\pi \int d^{d-2}y \int_{-\infty}^{\infty} dv(v - V(y)) \left\langle T_{vv}(u=0,v,y)\right\rangle_{\psi}$$

Again using BCH, two derivatives of the vacuum modular flow give:

$$\frac{\delta^2}{\delta V(y)\delta V(y')} e^{isK_{vac}[V(y)]} = (e^s - 1)^2 \hat{\mathcal{E}}_+(y)\hat{\mathcal{E}}_+(y') e^{isK_{vac}[V(y)]}$$

Returning to our formula...

$$\Delta S[V(y)] \sim \langle \Delta H_{\mathcal{R}}^{vac} \rangle + \lambda^2 \int_{-\infty}^{\infty} ds \frac{\langle O_{\mathcal{R}} e^{isK^{\mathrm{vac}}[V(y)]} O_{\mathcal{R}} \rangle}{\sinh^2((s - i\epsilon)/2)} + \mathcal{O}(\lambda^3)$$

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Take two derivatives

$$\frac{\delta^2 \Delta S}{\delta V(y) \delta V(y')} \sim \langle T_{vv} \rangle_{\psi} \delta(y - y')$$
$$+ \int_{-\infty}^{\infty} ds e^{s} \langle O_{\mathcal{R}} \hat{\mathcal{E}}_{+}(y) \hat{\mathcal{E}}_{+}(y') e^{isK^{\mathrm{vac}}[V(y)]} O_{\mathcal{R}} \rangle + \mathcal{O}(\lambda^3)$$

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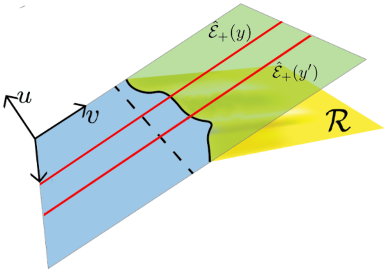
Are there any delta functions in $y - y'$ in the last term??



$$\int_{-\infty}^{\infty} ds\ e^{s} \langle O_{\mathcal{R}} \hat{\mathcal{E}}_+(y) \hat{\mathcal{E}}_+(y') e^{isK^{\text{vac}}[V]} O_{\mathcal{R}} \rangle$$

Take $y \to y'$: OPE?

OPE of Averaged Null Energy Operators

[Hofman & Maldacena, 2008] considered this OPE. Recent work [Kologlu, Kravchuk, Simmons-Duffin, & Zhiboedov 2019].

$$\hat{\mathcal{E}}_+(y)\hat{\mathcal{E}}_+(y') \sim \sum_i \frac{c_i \mathbb{O}^i_{++}(y')}{|y-y'|^{2(d-2)-\tau_i(J=3)}} + (y\text{-descendants})$$

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- $\mathbb{O}^i_{++}$ should be thought of as the integral of a non-local spin-3 operator (e.g. $\int dv \int_0^\infty ds \frac{\partial\phi(v)\partial\phi(v+s)}{(s+i\epsilon)^2}$)

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- For free theories, this OPE contains a delta function!
- For non-free theories (i.e. with a "twist gap"), delta function $\rightarrow$ integrable power law divergence

No time for...

- Argument for general states (displacement operator OPE on the twist defect)
- Argument from algebraic QFT