QIST Workshop @ Kyoto, June 12 (2019)

Symmetry, geometry, and topology in measurement-based quantum computation

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Motivation for HEP (Part I)

bulk-boundary correspondence via entanglement

HEP (e.g., AdS/CFT):

geometry/gravity at bulk

= correlations/complexity

at boundary

QI (e.g., MBQC):

lattice geometry/symmetry at bulk

= correlations/space-time complexity/

computational universality at boundary

Measurement-based quantum computation (MBQC) is a systematic way to connect entanglement to computational complexity.

- analogy with Page-Wooters's timeless construction of time (Part II)
- complexity = entanglement + classical communication (Part III)

2D cluster states and graph states

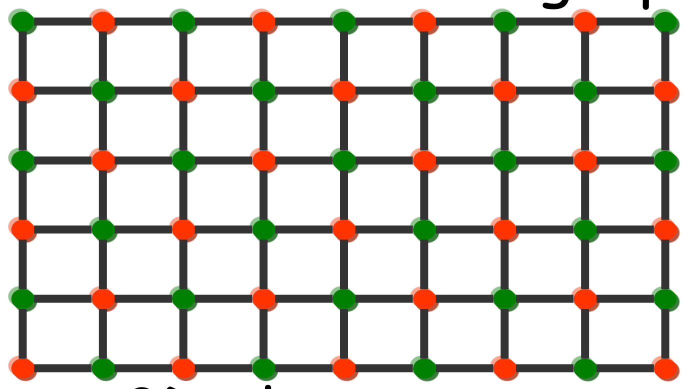
1. For a graph G,   [review: Hein et al., quant-ph/0602096]
   vertices = qubits, edges = Ising-type interaction pattern.
   degree is the number of edges from a vertex.

$$|G\rangle = \prod_{(a,b)\in edges} CZ^{(a,b)} |+\rangle^m, \quad CZ = \mathrm{diag}(1,1,1,-1), \; |+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

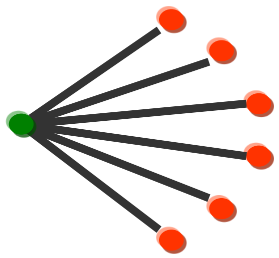
2. joint eigenstate of N commuting correlation operators for N qubits.

$$K_a|G\rangle = |G\rangle, \quad K_a = \sigma_x^{(a)} \bigotimes_{b\in N_a} \sigma_z^{(b)}$$

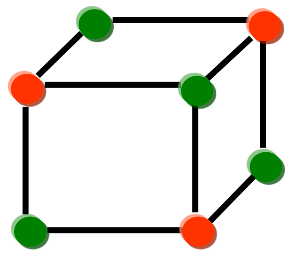
3. stabilizer states = graph state, up to local unitaries (not local Clifford).



2D cluster state



GHZ state



7-qubit Steane codeword

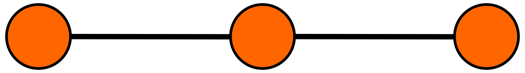
Simple example: 1D cluster state

[Briegel & Raussendorf, PRL 2001]

$$CZ = \mathrm{diag}(1,1,1,\text{-}1) = |0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes \sigma_z$$

$$|+\rangle|+\rangle = \tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|+\rangle$$
$$\rightarrow |0\rangle|+\rangle + |1\rangle|-\rangle \equiv |0\rangle_z|0\rangle_x + |1\rangle_z|1\rangle_x$$

Bell state
(max. entangled state)



two CZs are commuting !

$$|+\rangle|+\rangle|+\rangle = |+\rangle\left(|0\rangle + |1\rangle\right)|+\rangle$$
$$\rightarrow |+\rangle|0\rangle|+\rangle + |-\rangle|1\rangle|-\rangle \equiv |0\rangle_x|0\rangle_z|0\rangle_x + |1\rangle_x|1\rangle_z|1\rangle_x$$

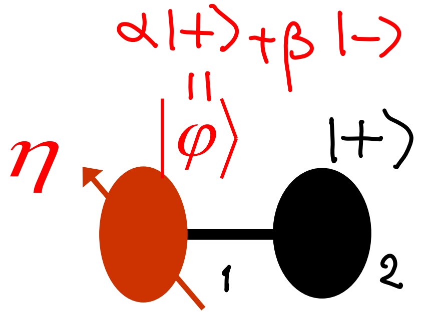
GHZ state

$$|+\rangle|+\rangle|+\rangle|+\rangle = \left(|0\rangle + |1\rangle\right)|+\rangle|+\rangle|+\rangle$$
$$\rightarrow \left(|0\rangle_z + |1\rangle_z \sigma_z\right)\left(|0\rangle_x|0\rangle_z|0\rangle_x + |1\rangle_x|1\rangle_z|1\rangle_x\right)$$
$$= |0\rangle_z|0\rangle_x|0\rangle_z|0\rangle_x + |0\rangle_z|1\rangle_x|1\rangle_z|1\rangle_x + |1\rangle_z|0\rangle_x|0\rangle_z|0\rangle_x - |1\rangle_z|1\rangle_x|1\rangle_z|1\rangle_x$$

cluster state
(not GHZ)

Key idea of MBQC

Steering quantum information by quantum correlation (entanglement)



$CZ\,|\varphi\rangle_1|+\rangle_2$

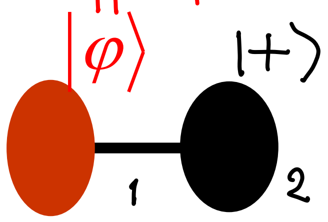
$$|\varphi\rangle_1 \mapsto X^{s_1} H R_z(\eta) |\varphi\rangle_2$$

local measurement by angle η

measurement outcome (with equal probability 1/2) $s_1 \in \{0,1\}$

single-qubit gate on the cluster state

$\alpha|+\rangle + \beta|-\rangle$ $= |\varphi\rangle$



$CZ|\varphi\rangle_1|+\rangle_2$

$$|\varphi\rangle_1 \mapsto X^{s_1} H R_z(\eta)|\varphi\rangle_2$$

$|\pm,\eta\rangle = |0\rangle \pm e^{i\eta}|1\rangle$

eigenstates of

$\sigma(\eta) \equiv \cos\eta\,\sigma_x + \sin\eta\,\sigma_y$

"xy-plane"

$$CZ|\varphi\rangle_1|+\rangle_2 = \left(\alpha|+\rangle_1 + \beta|-\rangle_1\right)|0\rangle_2 + \left(\alpha|-\rangle_1 + \beta|+\rangle_1\right)|1\rangle_2$$

$$= |+\rangle_1\left(H|\varphi\rangle\right) + |-\rangle_1\left(XH|\varphi\rangle\right)$$

$$= |+,\eta\rangle_1\left(H\underbrace{\begin{bmatrix}1 & \\ & e^{-i\eta}\end{bmatrix}}_{R_z(\eta)}|\varphi\rangle\right)_2 + |-,\eta\rangle_1\left(XH\begin{bmatrix}1 & \\ & e^{-i\eta}\end{bmatrix}|\varphi\rangle\right)_2$$

$R_z(\eta)$ logical rotation around z axis

$$|Cl\rangle = \sum_{\alpha_1,\ldots,\alpha_N=\pm} \mathrm{tr}\left[M[\alpha_N]\cdots M[\alpha_1]\right]|\alpha_1\ldots\alpha_N\rangle$$

$$M[+] = H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\quad M[-] = XH$$

Matrix Product State

Motivation for HEP (Part II)

relative state using a hidden U(1) degree of freedom η in cluster state

Page-Wootters's timeless construction of time (1983): precursor of Feynman clock

Wheeler-DeWitt equation: wavefunction of universe is static.

$$\mathscr{H} |\Psi\rangle^{AB} = 0$$

Consider a joint system by clock A and system B,

$$\mathscr{H} = h^A \otimes \mathbf{1}^B - \mathbf{1}^A \otimes h^B \qquad |t\rangle^A = e^{ih^A t} |o\rangle^A$$

Schroedinger equation of B relative to clock A

$$i\frac{\partial}{\partial t}\langle t|\Psi\rangle = \langle t| h^A \otimes \mathbf{1} |\Psi\rangle = \langle t| \mathscr{H} + \mathbf{1} \otimes h^B |\Psi\rangle = h^B \langle t|\Psi\rangle.$$

cluster-state example: $h^A = \sigma_z, \quad h^B = \sigma_x$

Universality in cluster-state MBQC model

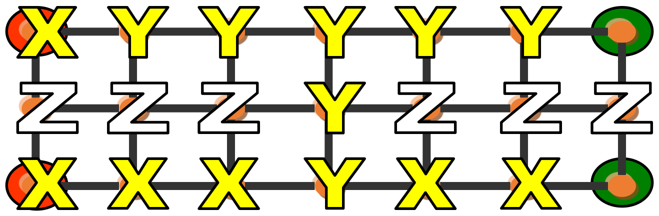
Raussendorf, Briegel, PRL 86, 5188 (2001)

Universality of 2D cluster states

the family of 2D cluster states with various sizes is a universal resource in measurement-based quantum computation (MBQC).

proof:
the universal set of gates {CNOT, SU(2)} is simulatable and composable as the circuit model.
Note that there are no exponential overheads in numbers of qubits, local quantum operations, and classical communication (LOCC).

CNOT



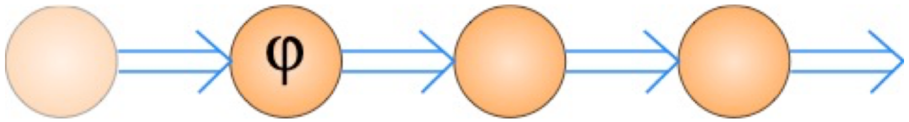
$|\psi_{in}\rangle$ $\qquad |\psi_{out}\rangle \overset{LU}{=} \mathrm{CNOT}|\psi_{in}\rangle$

SU(2) with Euler angles

Motivation for HEP (Part III)

cluster model ~ Page-Wootters mechanism + classical communication

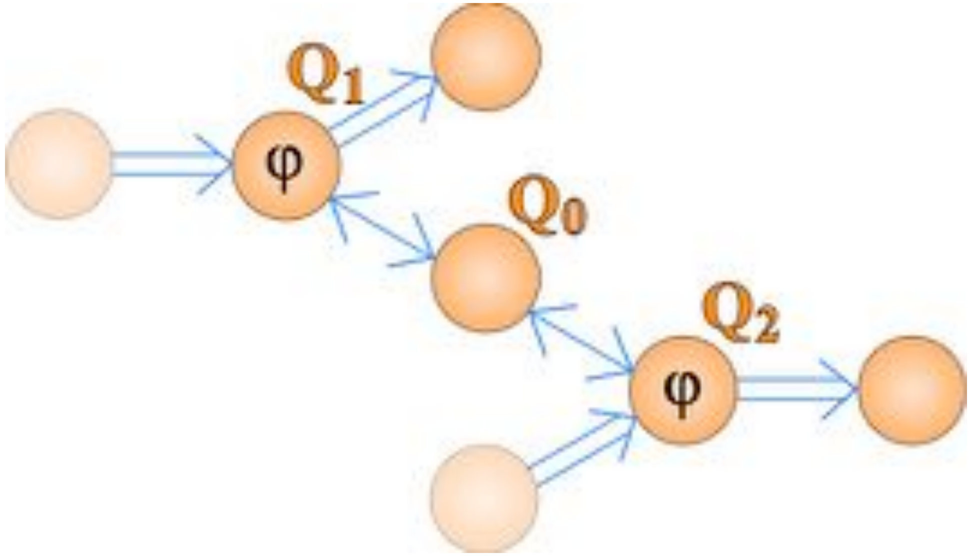
Miyake, arXiv:1111.2855 (2011)



Note Holevo theorem: 1 cbit out of 1 qubit

- simulate "time" direction

  1 ebit + 2 parallel cbits (teleportation)



- simulate "space" direction

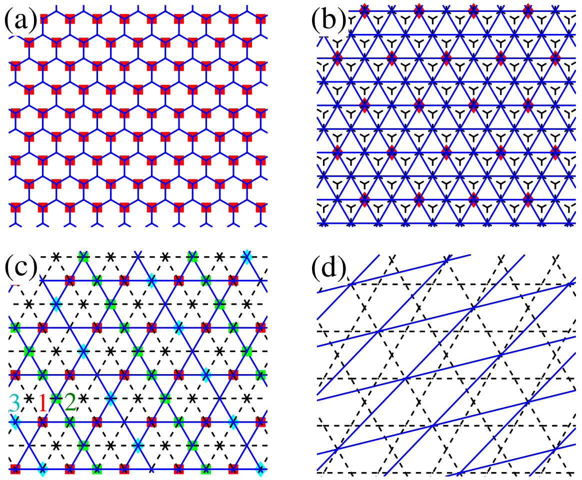
  1 ebit + 2 opposite cbits (synchronization of local clocks)

Measurement-based quantum computation (MBQC) allows to count space and time complexity in terms of entanglement and classical information

(Incomplete) zoo of universal entanglement for MBQC

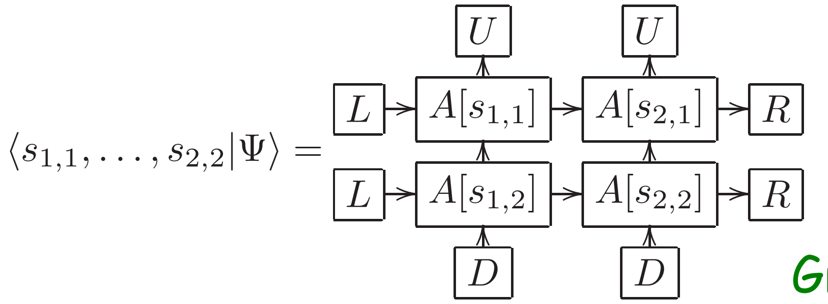
What kinds of entangled states are computationally universal under LOCC?

other graph states, connection to graph theory



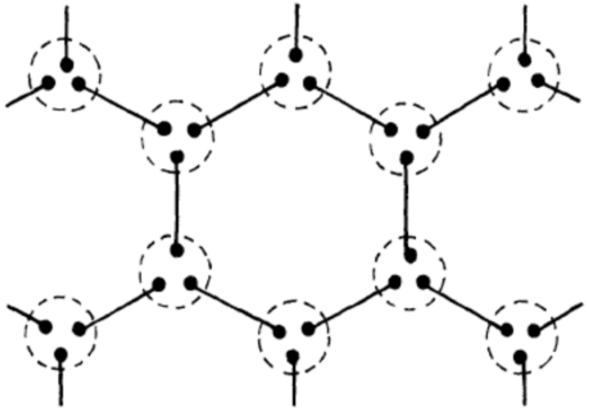
Van den Nest et al., 2006

tensor network states



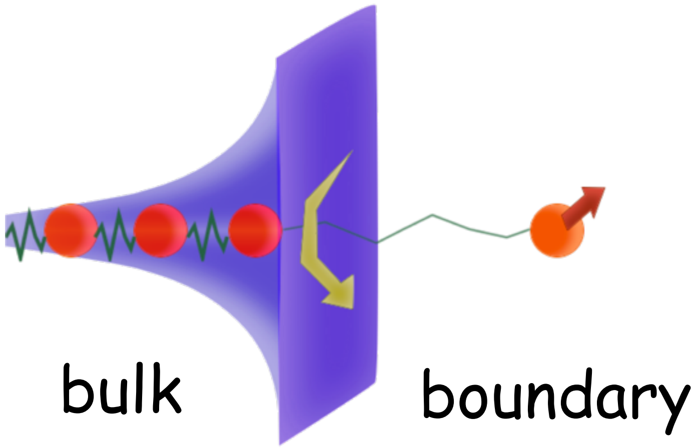
Gross, Eisert, 2007

AKLT states by two-body Hamiltonian (with exp. decaying two-point correlation)



Wei, Affleck, Raussendorf, 2011; Miyake 2011

symmetry-protected topologically ordered (SPTO) phase



Miyake 2010; Else et al., 2012; Miller, Miyake, 2015; Stephen et al., 2017; Raussendorf et al., 2019 ← newly for 2D

Motivations for HEP

bulk-boundary correspondence via entanglement

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What kinds of entangled states are computationally universal under LOCC?

2D SPTO phase of cluster states by sublattice symmetry

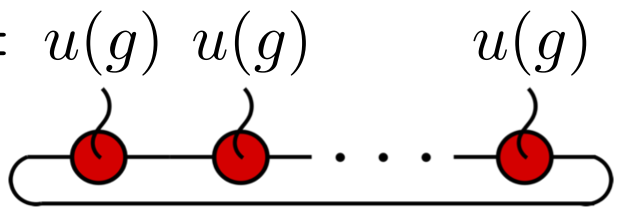
working with

Austin Daniel
Rafael Alexander

Symmetry-protected topological order

- Symmetry
  - Family of states with a common symmetry
  - Generally consider finite abelian groups G such as copies of $\mathbb{Z}_2$.
- Global symmetries act in an "on-site" manner.

$S(g) =$



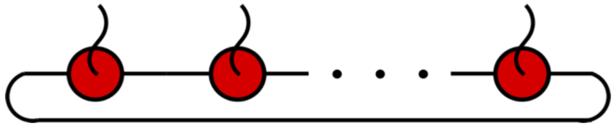
$u(g)$ = Onsite representation

$S(g)$ = Global representation

- X-type symmetries of cluster states define SPTO.
  - Want to do MBQC in a way that is compatible with symmetry.
  - Mainly restrict to X- measurements. (non-network MBQC)

Symmetry-protected topological order

- Topological
  - For periodic boundary conditions the ground state is unique

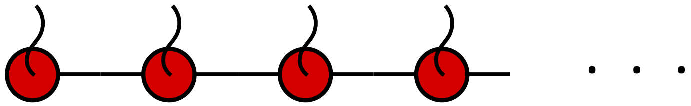


$$H = -\sum_{j=1}^{N} Z_{j-1} X_j Z_{j+1}$$

*Local Hamiltonian terms give complete set of commuting observables.

invariant under global X's ($\mathbb{Z}_2$), or even-site X's and odd-site X's ($\mathbb{Z}_2 \times \mathbb{Z}_2$)

  - Degeneracy of ground states occurs for open boundary (fractionalized edge states)



$$H = -\sum_{j=2}^{N-1} Z_{j-1} X_j Z_{j+1}$$

$$X^L = X_1 Z_2 \simeq \otimes_j X_{2j+1}$$
$$Z^L = Z_1 \simeq \otimes_j X_{2j}$$

bulk-boundary correspondence in MBQC:
universality at boundary in terms of bulk entanglement

Symmetry-protected topological order

- Topological considerations of symmetry
  - For 1D SPTO the symmetry is represented projectively at the boundary

Linear Representation $\rightarrow$ $u(g)u(h) = u(gh)$

Projective Representation $\rightarrow$ $V(g)V(h) = \omega(g,h)V(gh)$

- where $\omega(g,h) \in U(1)$

$\mathbb{Z}_2 \times \mathbb{Z}_2$

Linear Rep. $\longrightarrow$ $xy = z$

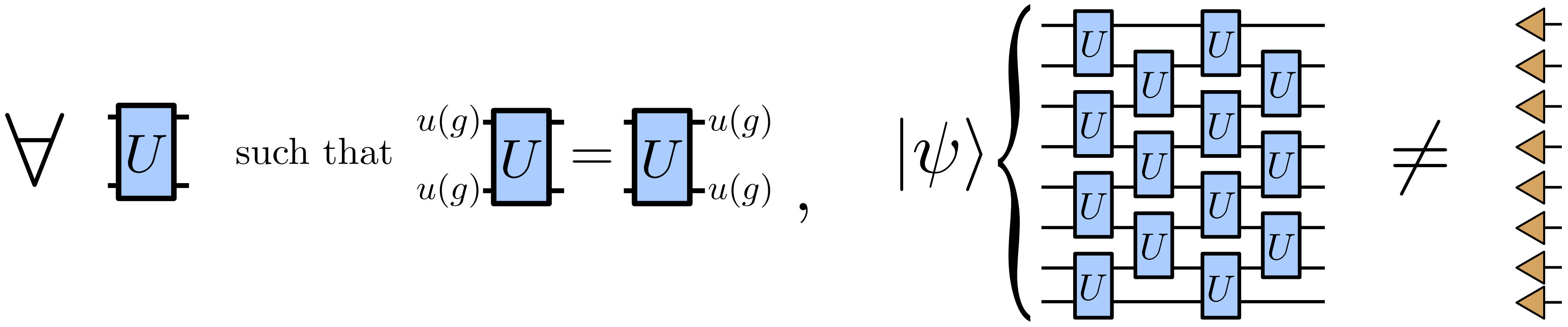
Projective Rep. $\longrightarrow$ $XY = iZ$

- The only non-trivial projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is equivalent to the Pauli matrices.
- For each copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$ we get a qubit degree of freedom at the edge!

Phase of symmetry-protected topological order

- Symmetry-protected
  - No symmetry respecting perturbation can lift the degeneracy.
  - Ground states in same phase related by a symmetric local unitary (SLU).

<u>constant-depth local quantum circuit</u>



- No intrinsic topological order!

X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 82, 1555138 (2010).

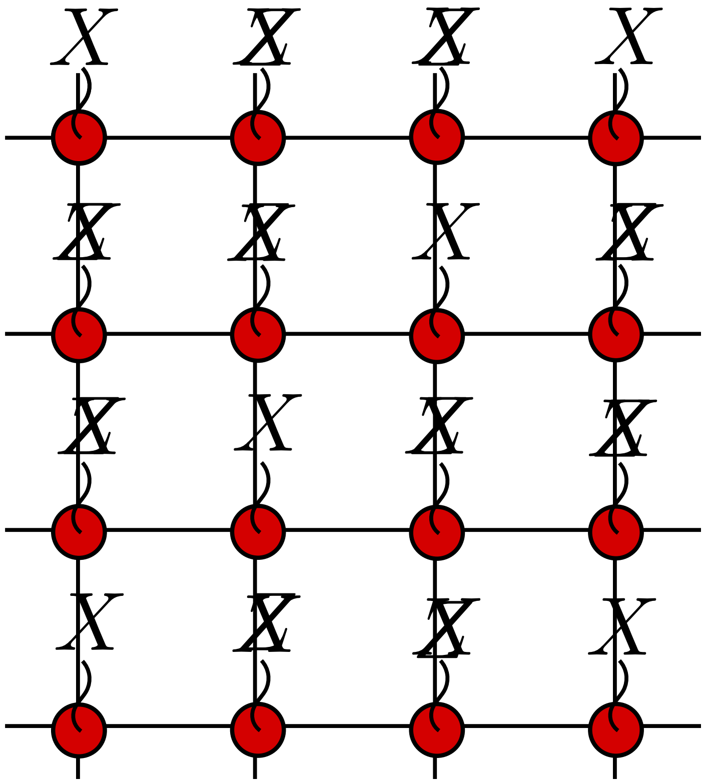
Symmetries of 2D cluster states

- Cluster states have many fancy symmetries
- Biggest symmetry group is full stabilizer group.

$$\mathcal{S} = \langle \{ X_v \bigotimes_{l \in \mathcal{N}(v)} Z_l \, | \forall v \} \rangle$$

- Smallest is global $\mathbb{Z}_2$ symmetry.
  - Apply all stabilizers!
  - One symmetry generator

*Note for odd degree lattice the action is a global Y.

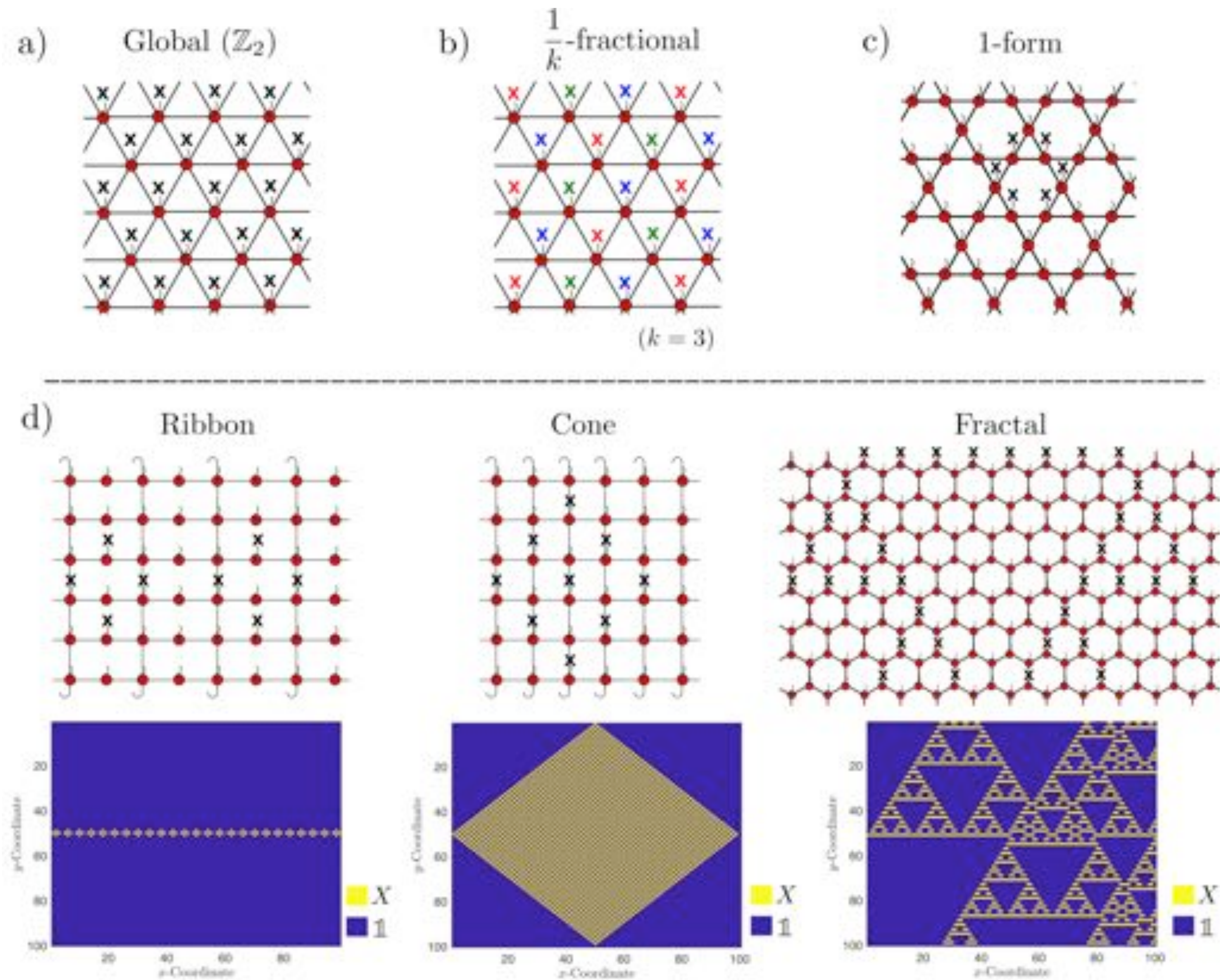


lattice (subsystem) symmetry in 2D lattices

Consider all SPTO states which <u>commute</u> with these subsystem symmetry.

All are connected to the cluster state on a corresponding lattice by SLU circuits.

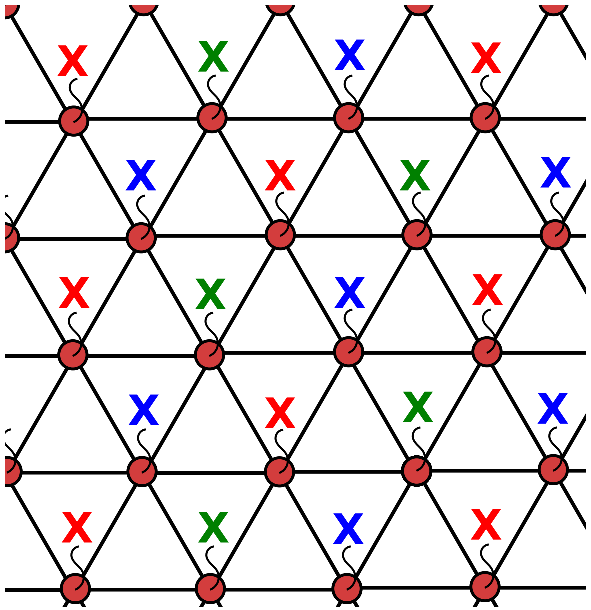
Note: most of following explanations can be done on the cluster state



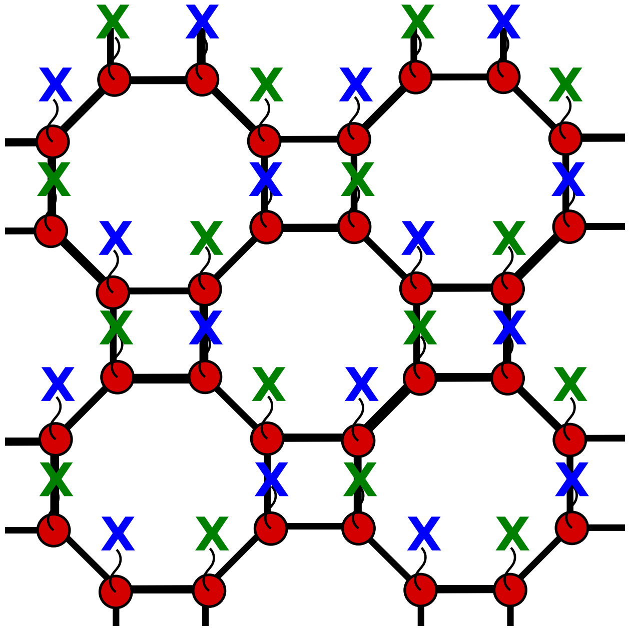
Symmetries of cluster states

- 1/k-Fractional symmetry.
  - On a k-colorable graph, apply stabilizers on each vertex of a given color.
  - Well studied symmetry. Common for defining symmetric phases of matter.

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$



$\mathbb{Z}_2 \times \mathbb{Z}_2$



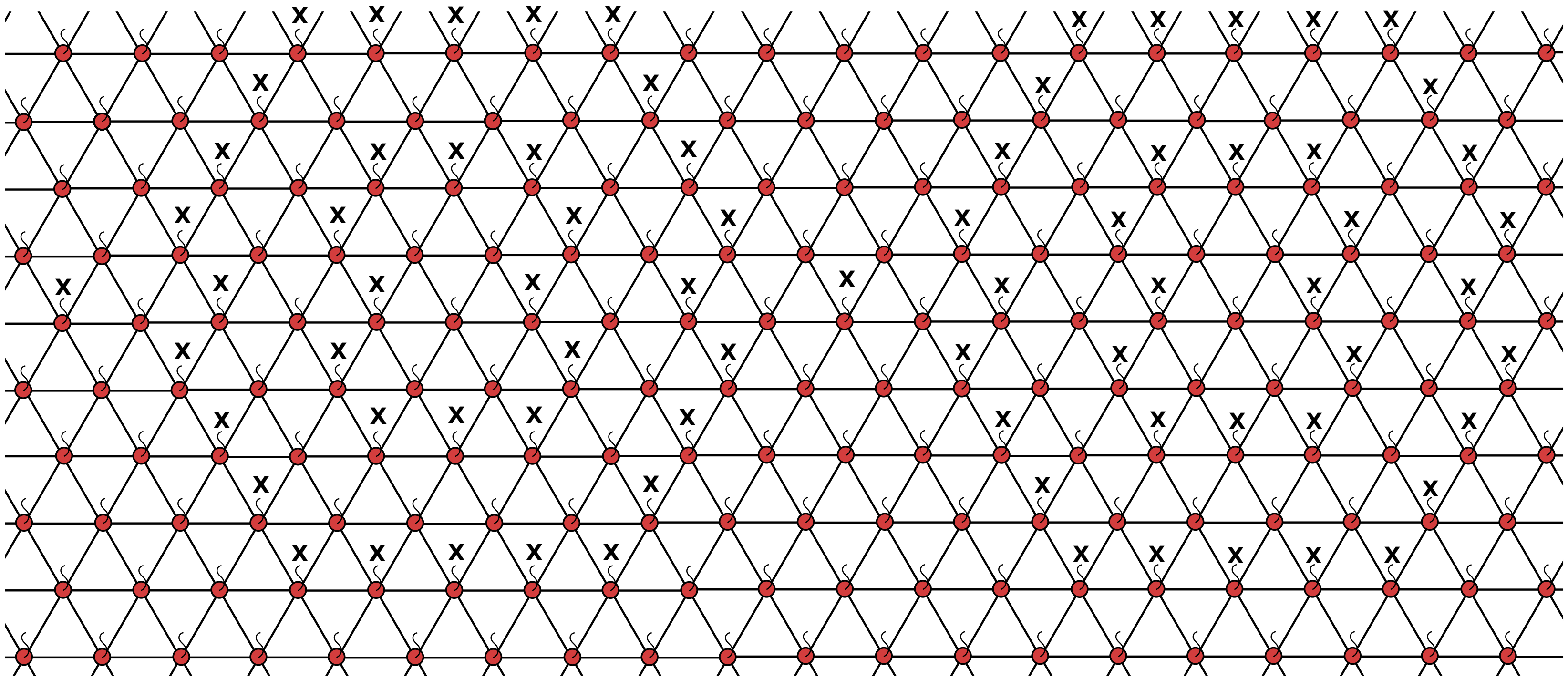
k-colorable

$\Downarrow$

$\mathbb{Z}_2^k$

Symmetries of cluster states

- Subsystem symmetry
  - Apply stabilizer on some site. Try to add as few more to cancel all Z 's.
  - Periodic structure for periodic boundaries. This is the main topic of this work.

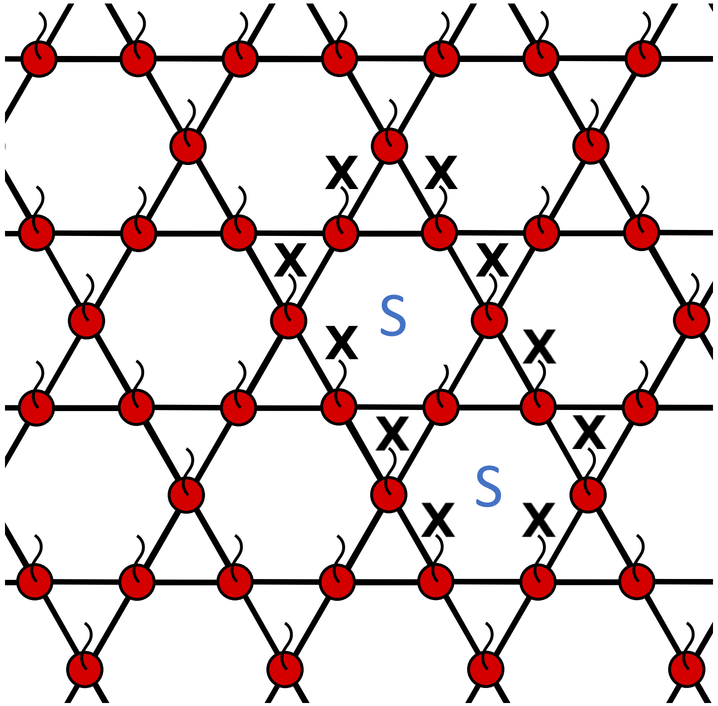


$\mathbb{Z}_2^{2n}$

*Symmetry group is subextensively large.

Symmetries of cluster states

- 1-form symmetry.
  - Closed loops of X - operators.
  - Deformable = Product of two loops is a bigger loop



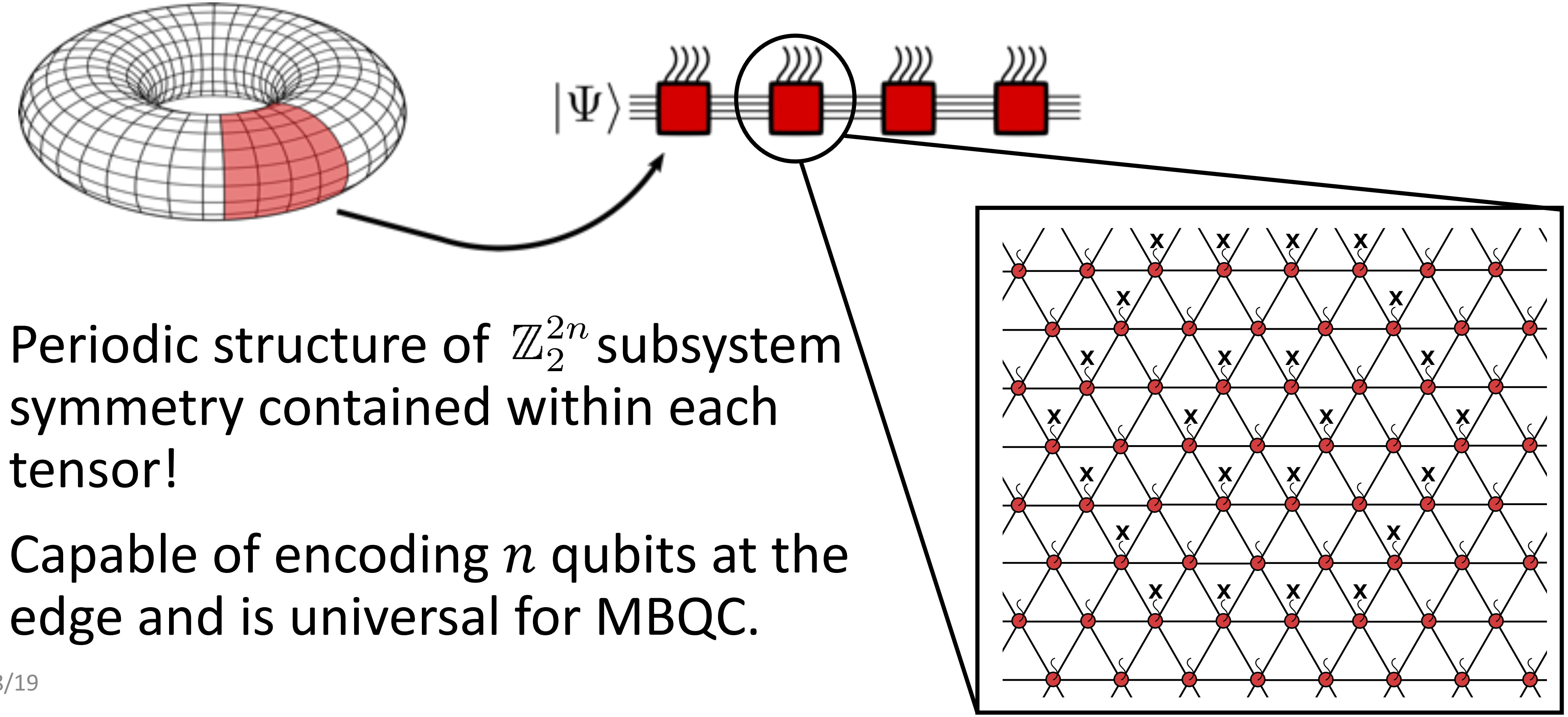
$$\mathbb{Z}_2^{\mathcal{O}(Nn)}$$

*Symmetry group is extensively large.

From 2D to quasi-1D

Raussendorf, Okay, Wang, Stephen, Nautrup, PRL 122, 090501 (2019)

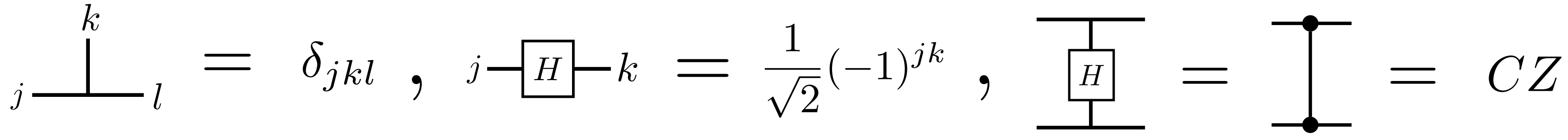
- Embed a 2D lattice cluster state on a torus and group together an $n \times n$ block of sites.



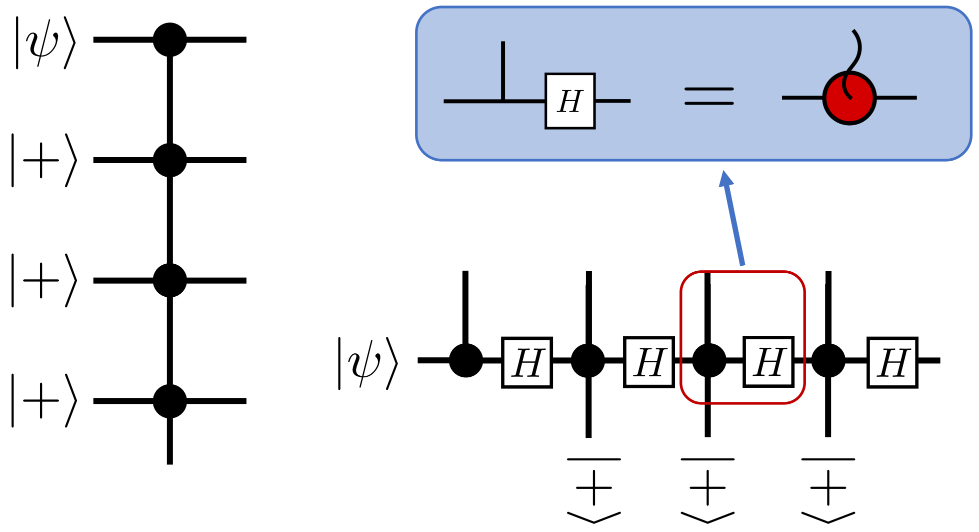
- Periodic structure of $\mathbb{Z}_2^{2n}$ subsystem symmetry contained within each tensor!
- Capable of encoding n qubits at the edge and is universal for MBQC.

Tensor Networks and MBQC

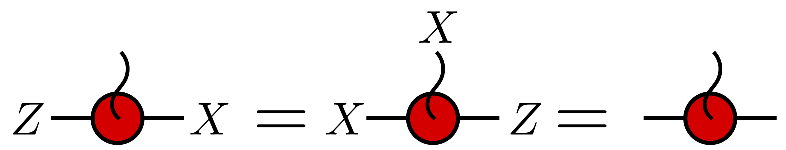
- The matrix product state description of the 1D cluster state is useful.



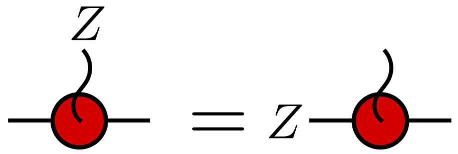
Construction of MPS



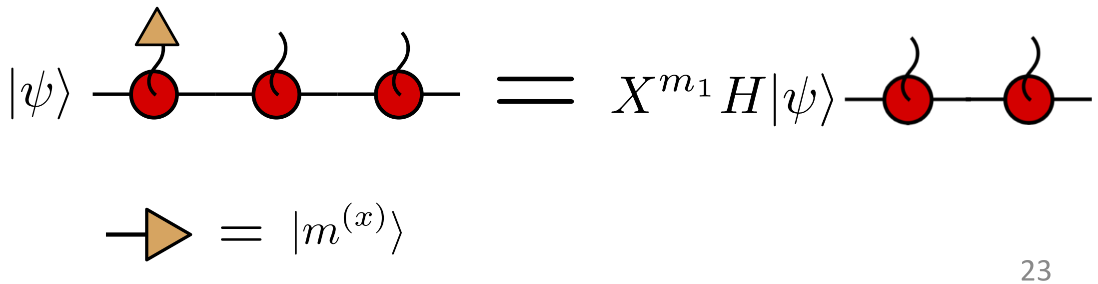
Symmetries:



* TN equivalent of stabilizers

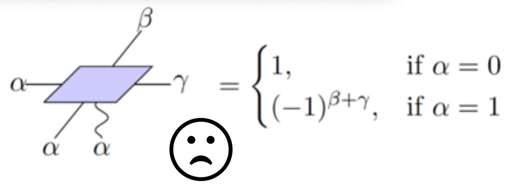


X - measurement implements H

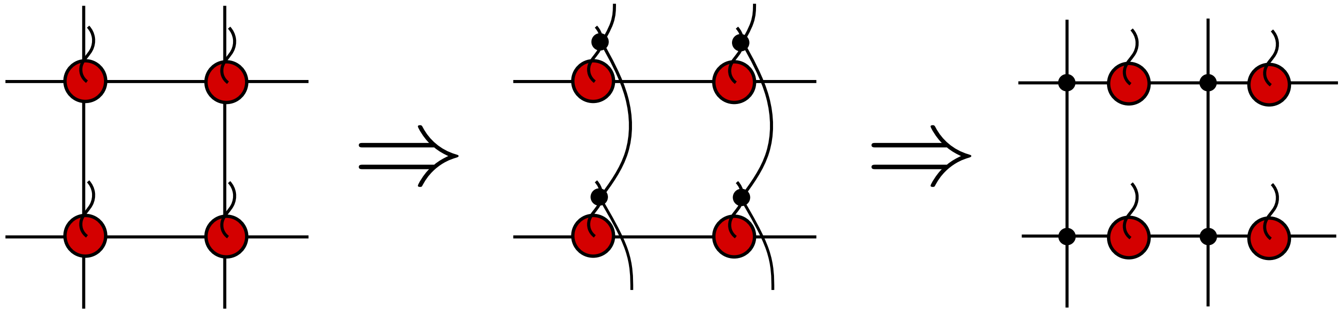


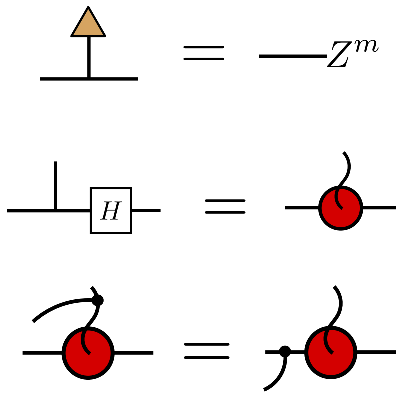
Convenient tensor networks for 2D cluster states

As opposed to…

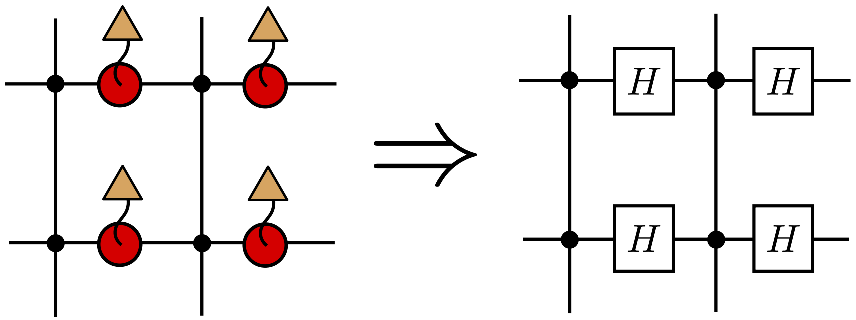


- Think of 2D cluster state as coupled 1D cluster states.





- X - basis measurements turn the TN into a Clifford circuit.



Quantum Cellular Automata

- <u>QCA</u> = Set of Hilbert spaces + translation invariant local update rules
  - i.e. A quantum circuit applied iteratively in time.
  - Evolution specified by transfer matrix. $T|\psi(t)\rangle = |\psi(t+1)\rangle$
- Clifford QCA are efficiently simulable! (Gottesmann Knill)

$$P(\xi) = \bigotimes_{j=1}^{N} X_j^{\xi_j^{(x)}} Z_j^{\xi_j^{(z)}} \Rightarrow \quad P(\xi) \equiv \xi \in \mathbb{F}_2^{2n}$$

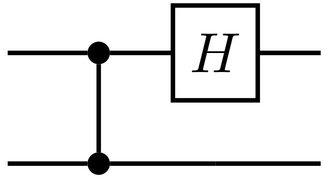
$$T \in \mathrm{Aut}(\mathbb{F}_2^{2n}) \ \text{ such that } \ T\Sigma T^{\top} = \Sigma$$

$$\Sigma = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

i.e. T is a binary matrix

*Binary representation is important for simulation

$$\begin{matrix} X_1 \\ X_2 \\ Z_1 \\ Z_2 \end{matrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} =$$



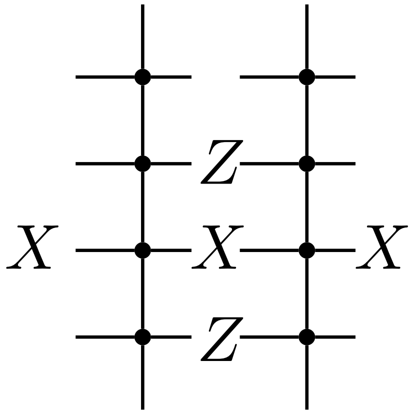
Stephen, D. T., et. Al. R. (2018). *ArXiv:1806.08780*

Quantum Cellular Automata

- Clifford QCA can be classified into 3 types. Guetschow et al., JMP 51, 015203 (2010)

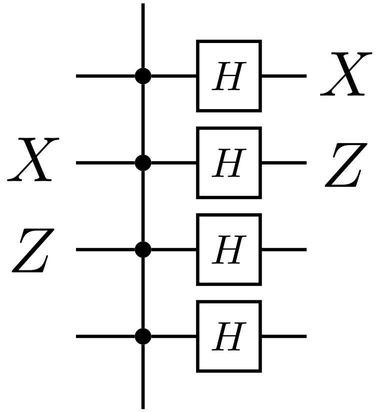
Periodic

- Period is constant and independent of system size



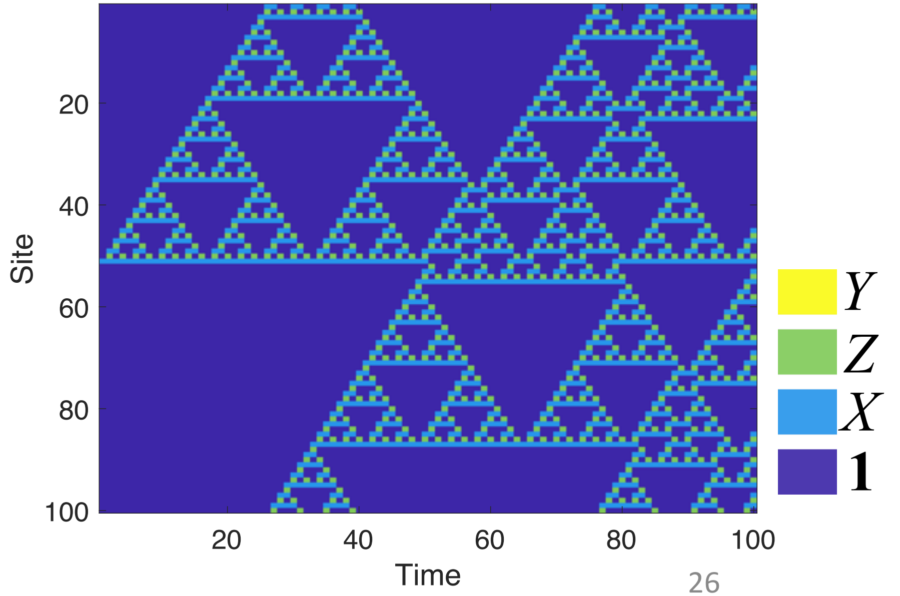
Glider

- Supports gliders (eigenoperators up to translation)
- Period linear in system size



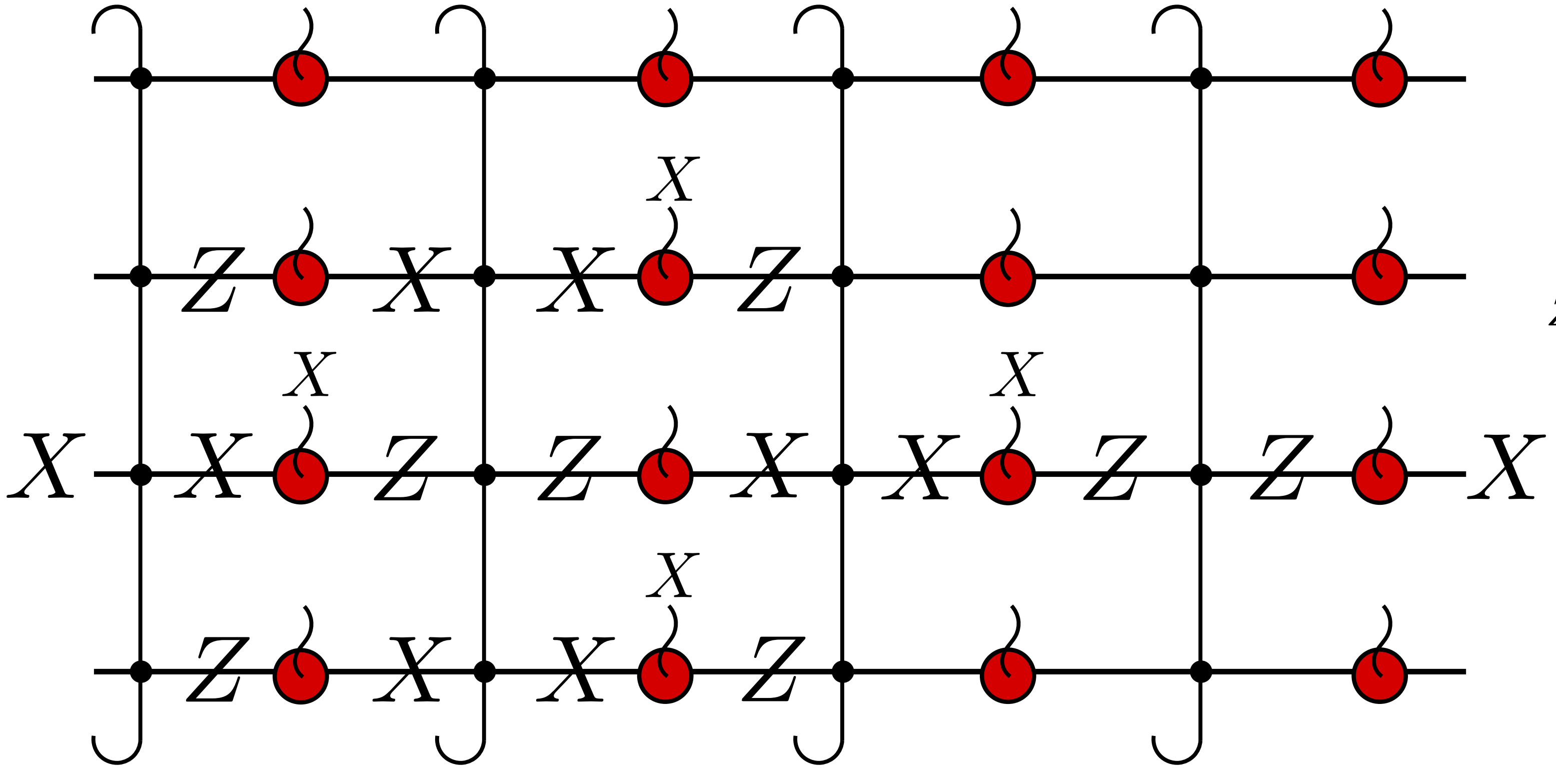
Fractal

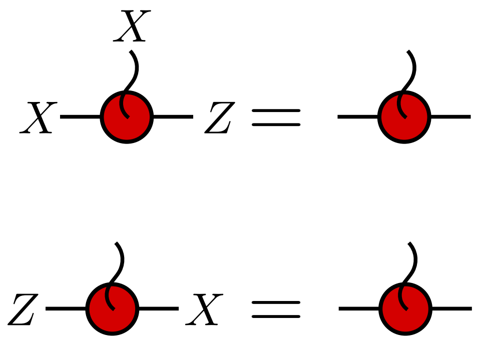
- Operator support is fractal
- Period varies wildly



QCA and subsystem symmetries

- There is a 1-1 correspondence between QCA evolution and subsystem symmetries of cluster phases.

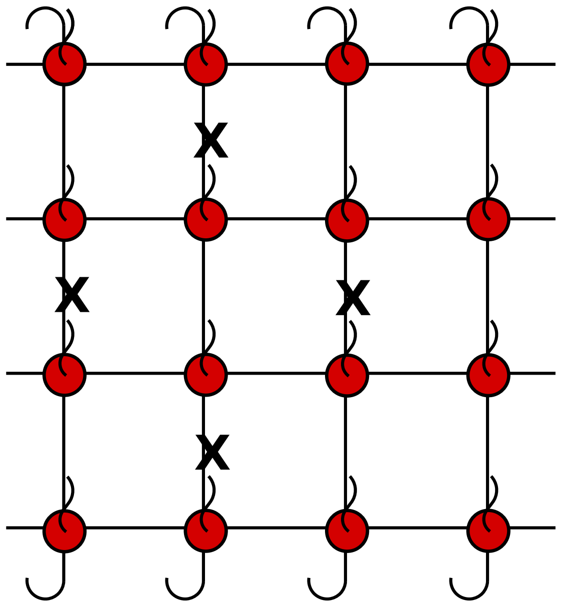




QCA and subsystem symmetries

- There is a 1-1 correspondence between QCA evolution and subsystem symmetries of cluster phases.

- 2n generators of Pauli group = 2n real-space symmetry generators.
- Subsystem symmetries define a SPT phase, cluster phase, universal for MBQC.
- We study cluster phases of Archimedean lattices.

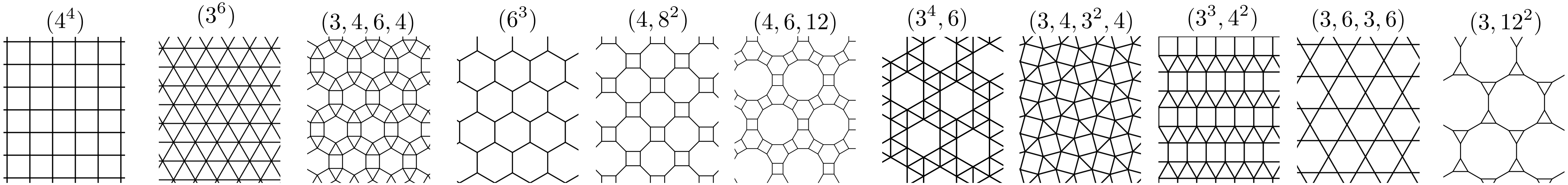


2D cluster SPTO phase by subsystem symmetry

Daniel, Alexander, Miyake, in preparation

All ground states (which are not necessarily stabilizer states) in 2D cluster phase on a 2D Archimedean lattice with Ribbon/Cone/Fractal subsystem symmetry are universal for MBQC.

- Archimedean lattices are vertex translative (each vertex locally looks the same).



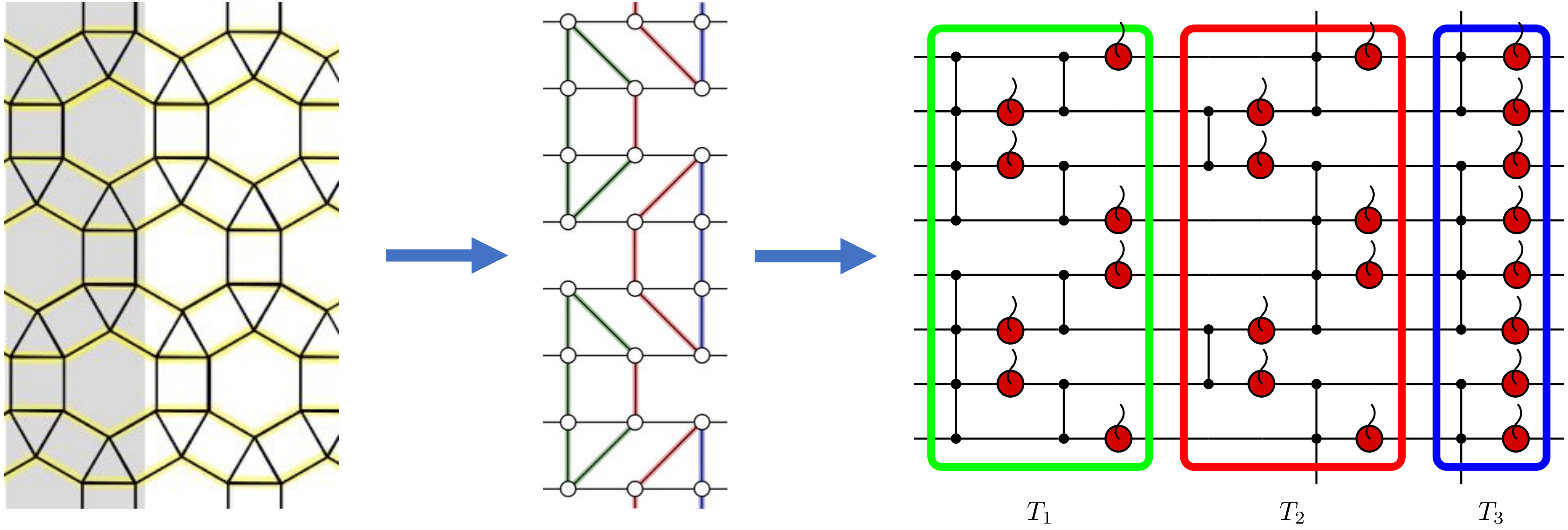
Real space symmetry	Real space symmetry group	Virtual space QCA structure	Computational phase	Lattices
Ribbon	$\mathbb{Z}_2^{2n}$	Periodic	Yes	Rectangular*
Cone	$\mathbb{Z}_2^{2n}$	Glider	Yes	(4^4)*, (3^6), $(3,4,6,4)$
Fractal	$\mathbb{Z}_2^{2n}$	Fractal	Yes	(6^3)*, $(4,8^2)$, $(4,6,12)$, $(3^4,6)$, $(3,4,3^2,4)$, $(3^3,4^2)$
1 - Form	$\mathbb{Z}_2^{\mathcal{O}(nN)}$	No	No	$(3,6,3,6)$, $(3,12^2)$
$\frac{1}{k}$Fractional	$\mathbb{Z}_2^k$	-	-	All

*Previously known

Raussendorf et al., PRL 122, 090501 (2019)
Devakul, Williamson PRA 98, 022332 (2018)
Stephen et al., 1806.08780

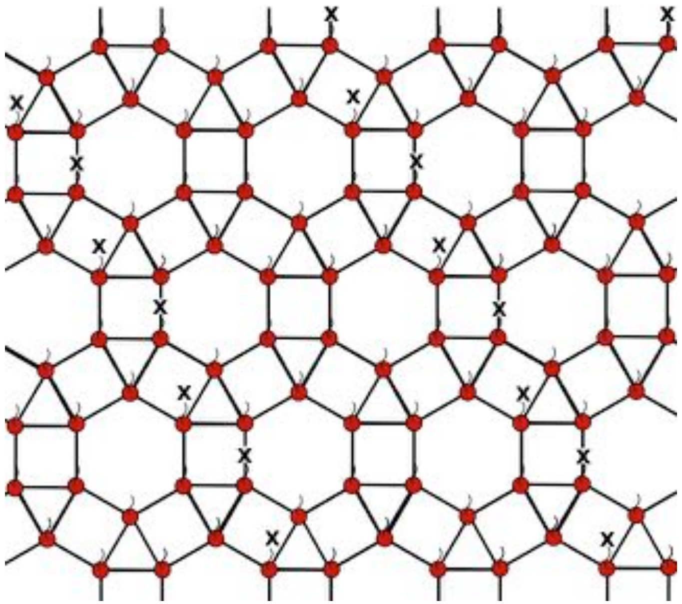
Lattices supporting glider QCA

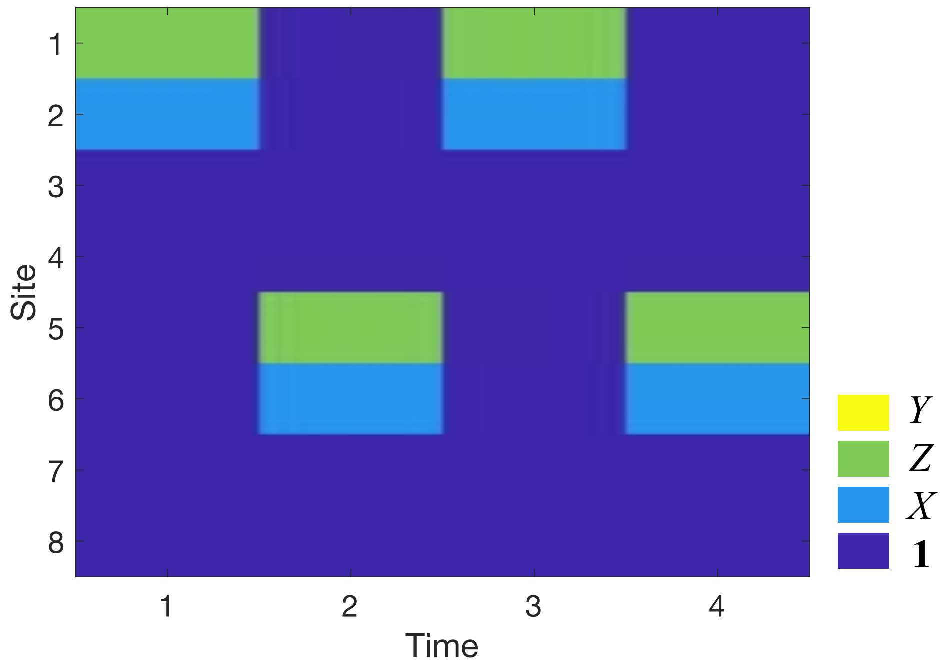
- Consider the $(3,4,6,4)$ lattice. We first construct tensor network description.



Lattices supporting glider QCA

- Gliders are operators whose support is translated by the QCA.





Lattices supporting glider QCA

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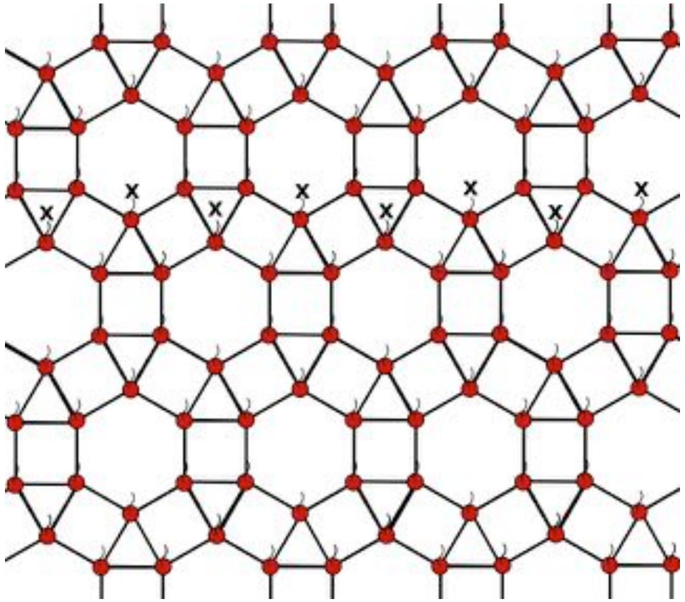

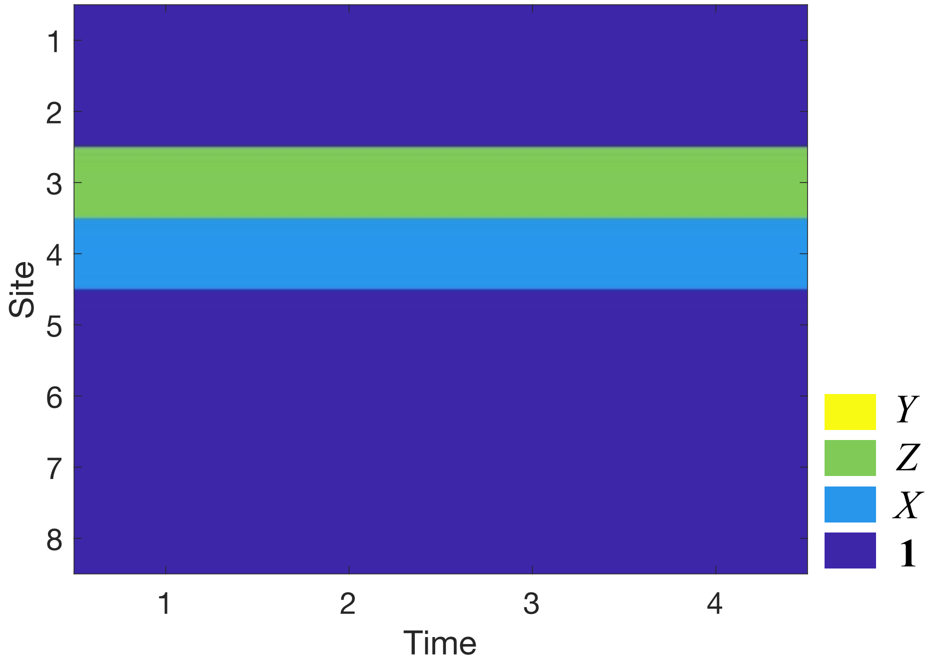

Lattices supporting glider QCA

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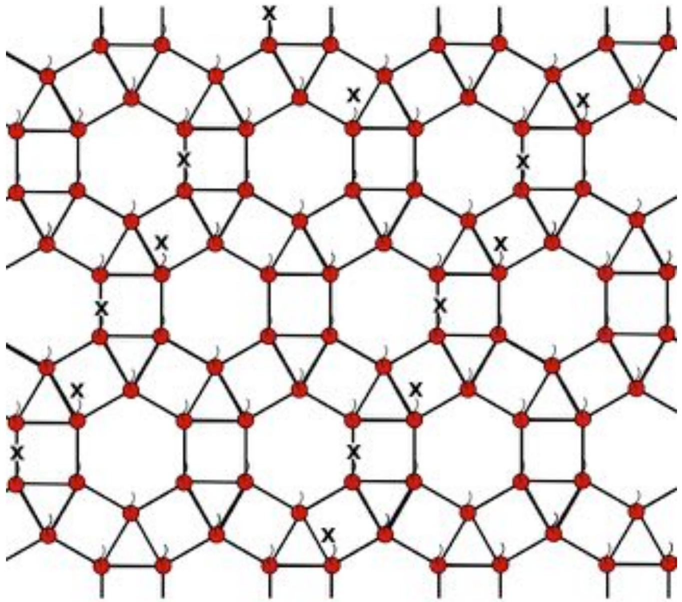
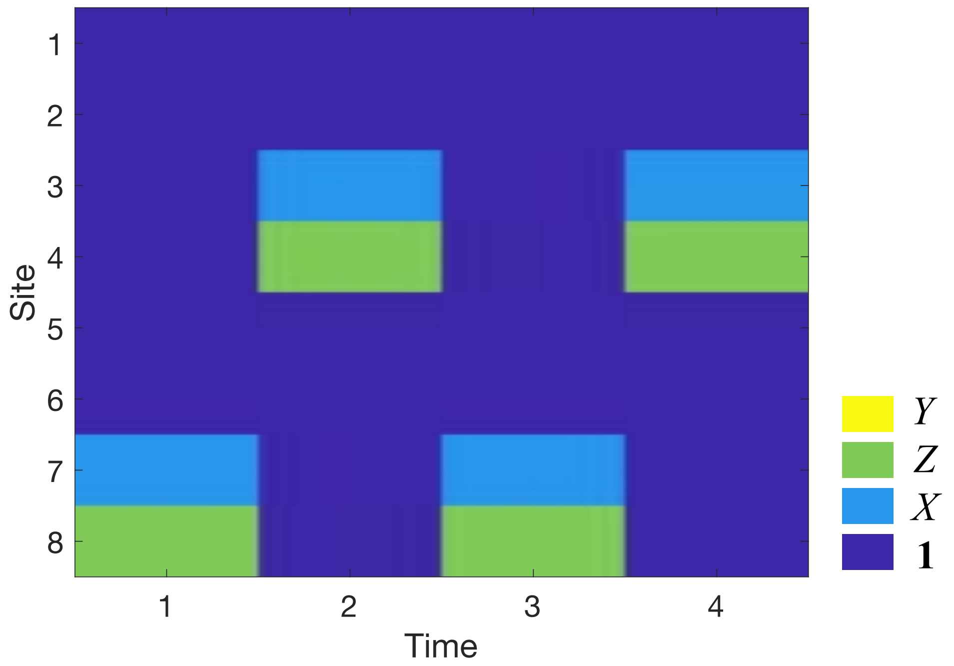
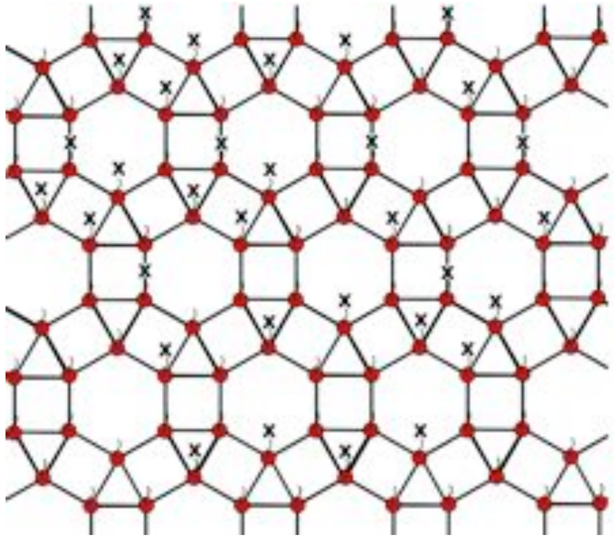


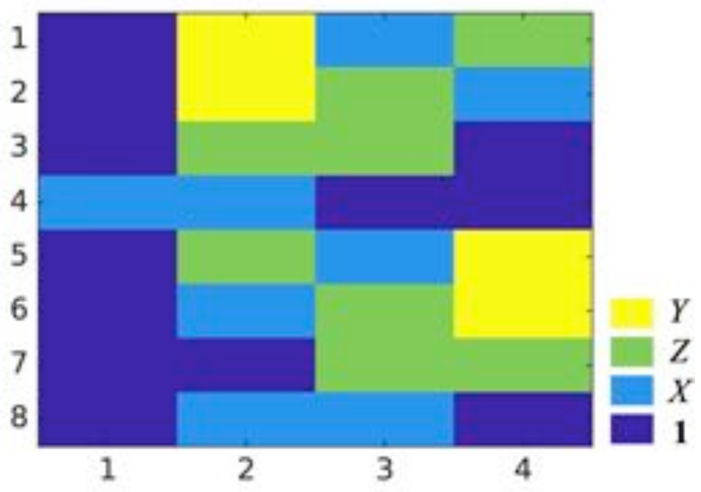


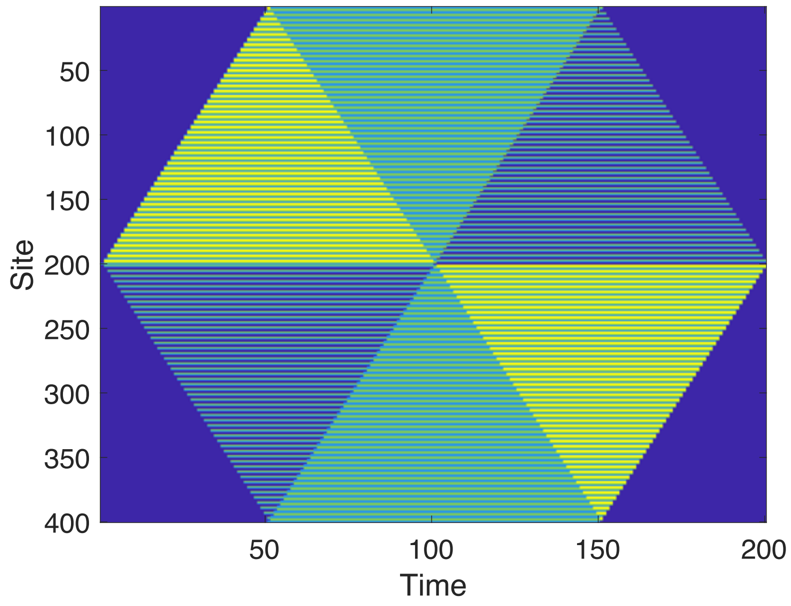
Note: line-like symmetry does not provide glider QCA in general.

Lattices supporting glider QCA

- Defining symmetry for the SPTO (cluster phase) is the cone symmetry.



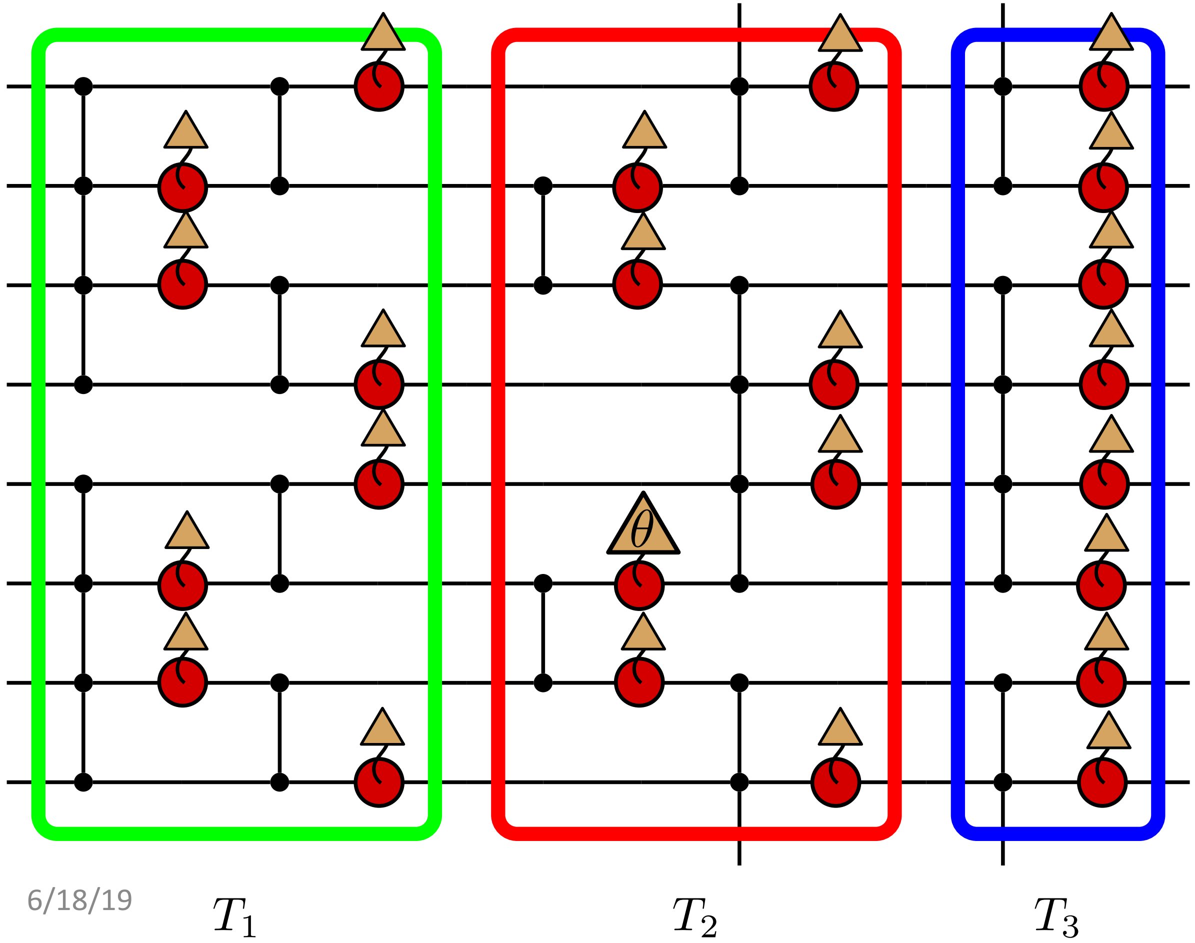


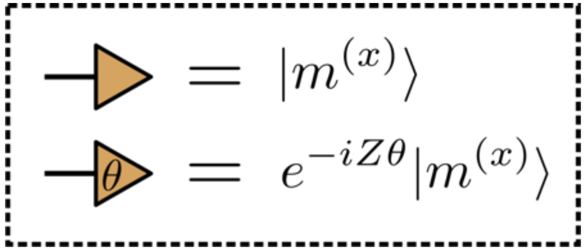


Note: line-like symmetry, subgroup of cone symmetry, does not provide glider QCA.

Lattices supporting glider QCA

- Universal gates achieved via measurement in (X,Y)-plane.





Lattices supporting glider QCA

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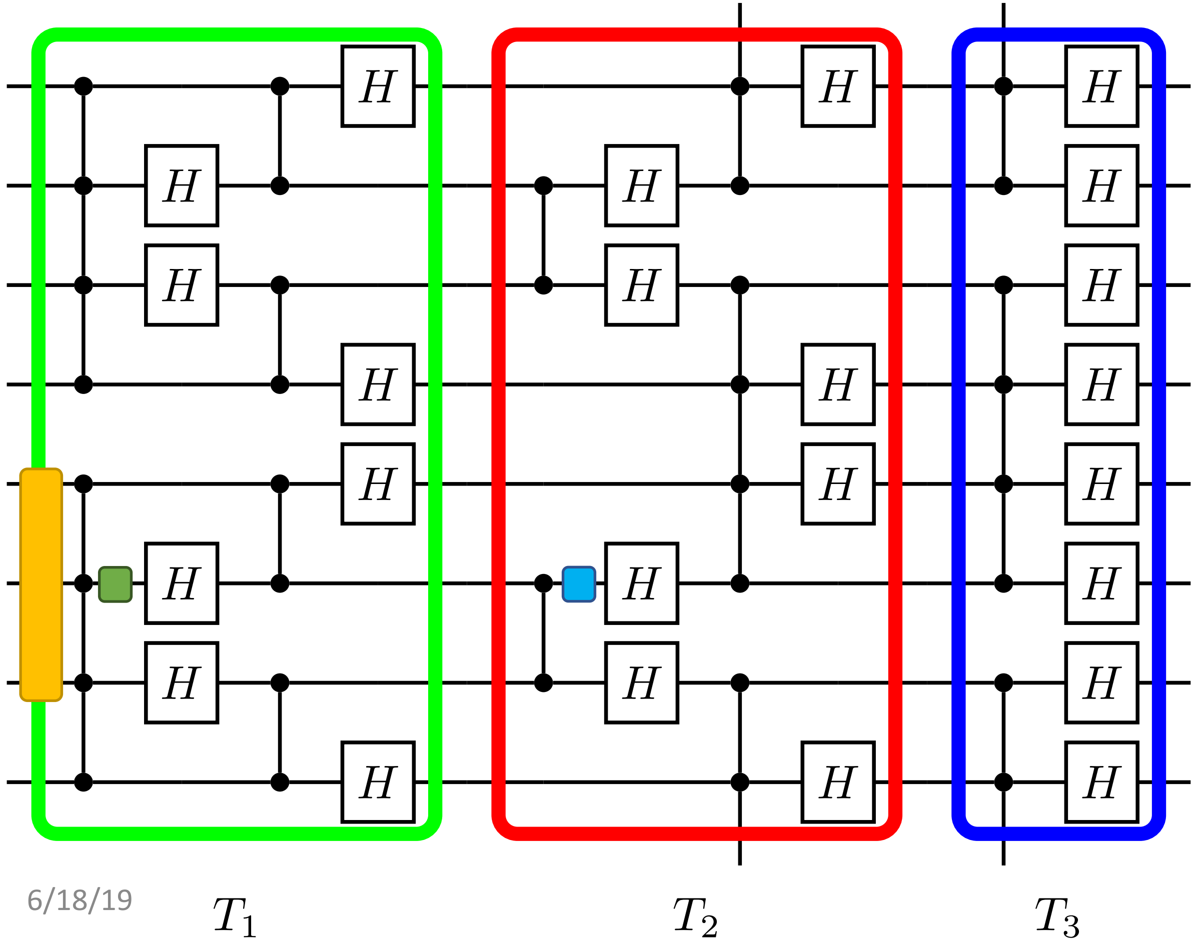
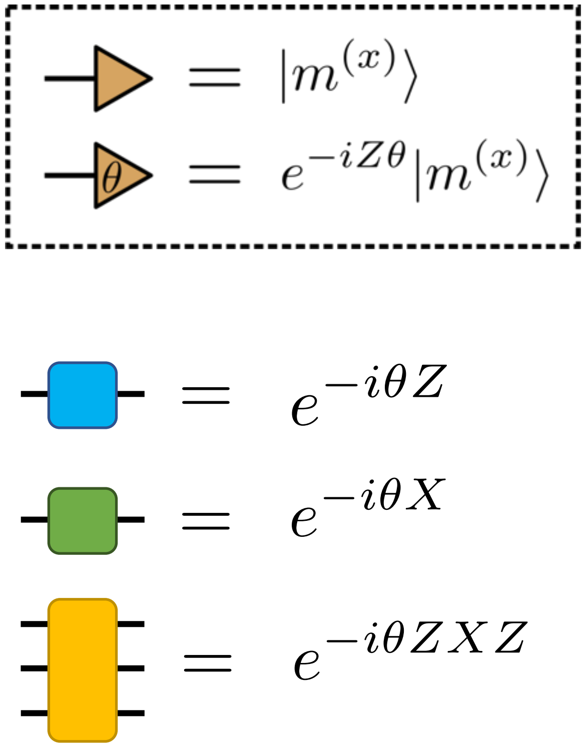




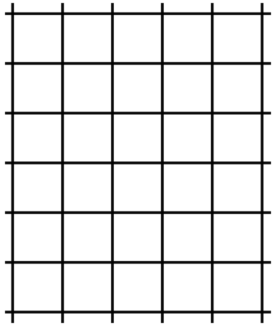
Lattices supporting glider QCA

- By analogous argument we find the following gate set.

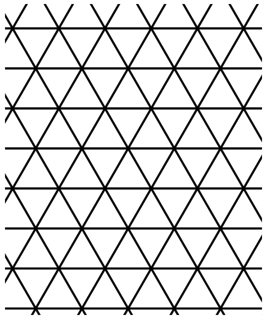
Gates	Measurement
$e^{-i\theta Z_l}$	$(1, l)$
$e^{-i\theta X_l}$	(n, l)
$e^{-i\theta Z_{4l} X_{4l+1} Z_{4l+2}}$	$(2, 4l+1)$

- Other lattices supporting glider QCA and cone symmetries.

(4^4)

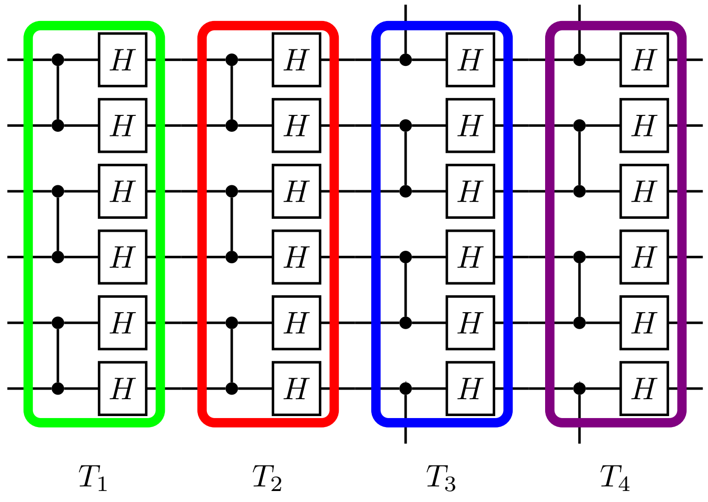


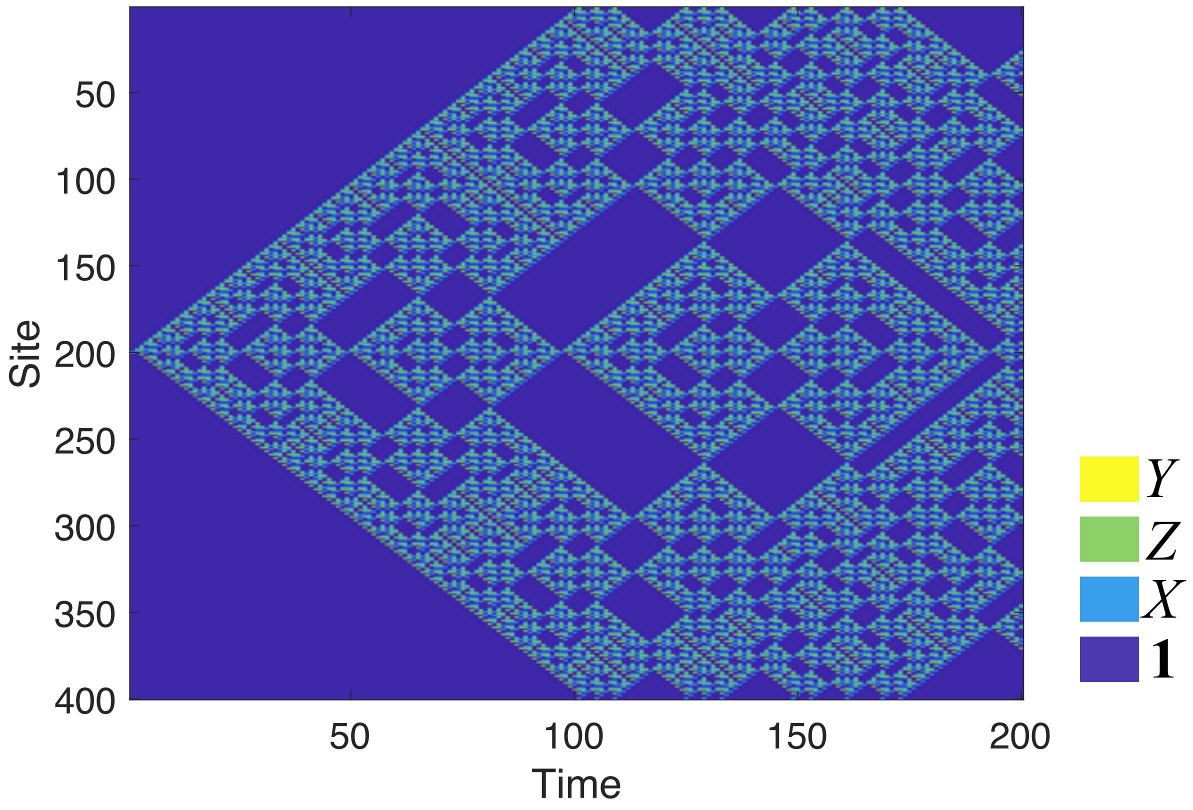
(3^6)



Lattices supporting fractal QCA

- Fractal QCA are characterized by operators supported on a fractal subset of space-time points.

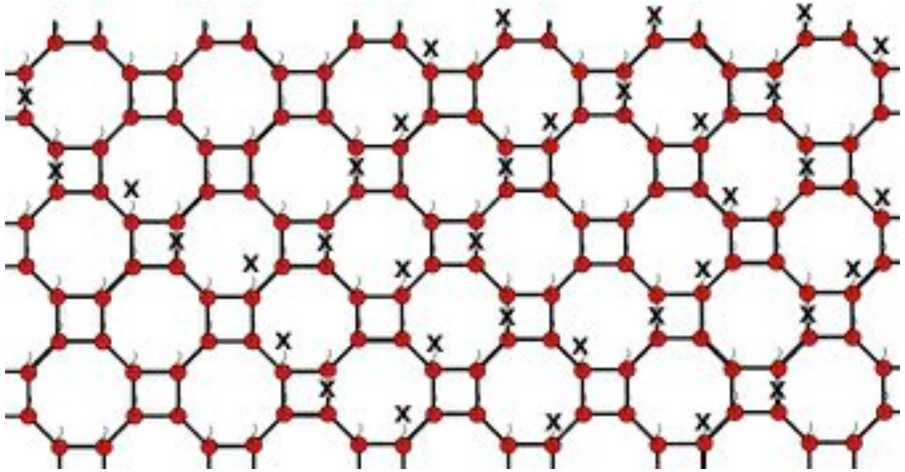


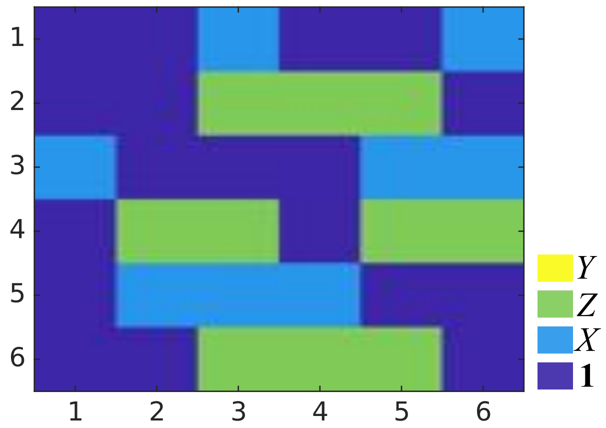


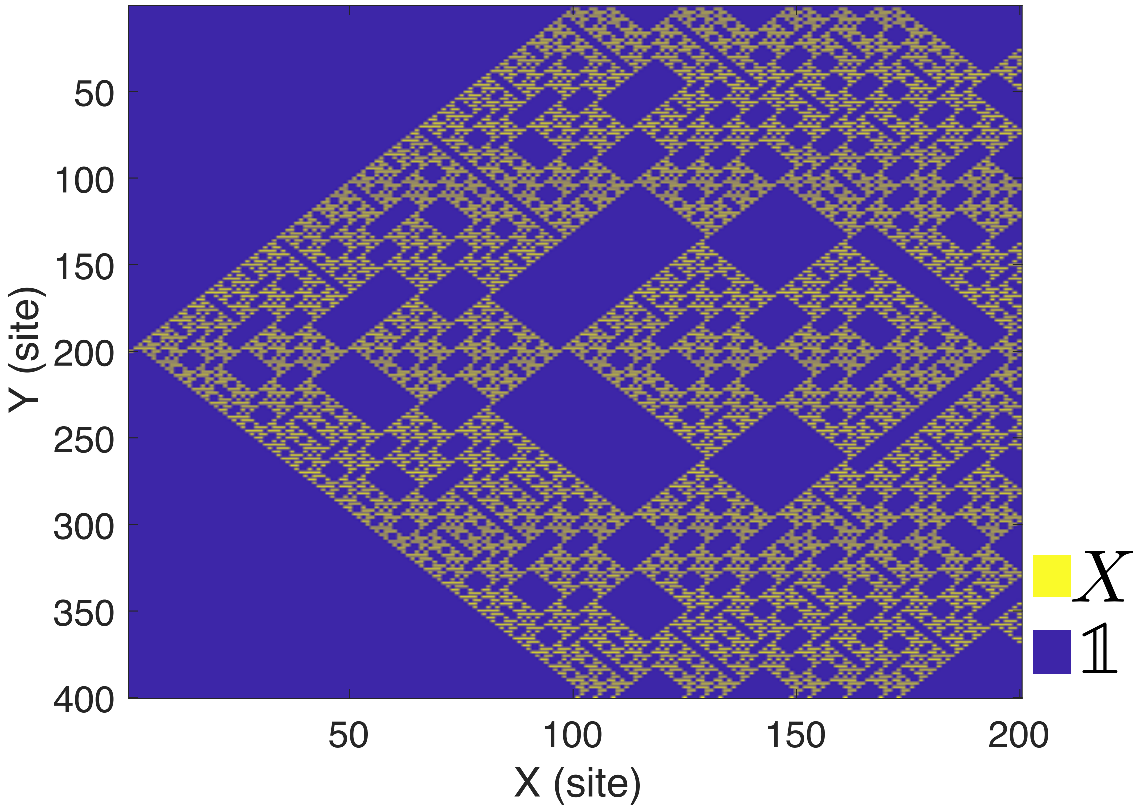
- QCA period varies spuriously with system size.

Lattices supporting fractal QCA

- The defining symmetry of the cluster phase is the fractal symmetry.







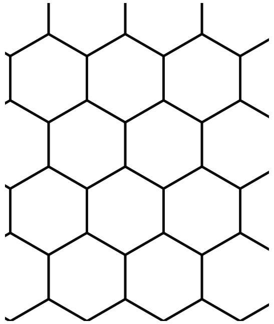
Lattices supporting fractal QCA

- For the $(4, 8^2)$ lattice the following gate set can be derived.

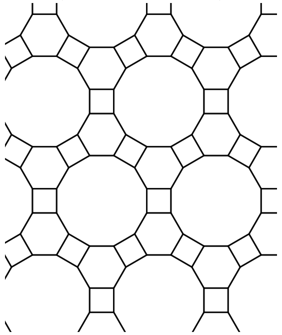
Gates	Measurement
$e^{-i\theta Z_l}$	$(1, l)$
$e^{-i\theta X_l}$	(τ, l)
$e^{-i\theta Z_{2l-1} X_{2l}}$	$(2, 2l-1)$
$e^{-i\theta Z_{2l} X_{2l+1}}$	$(\tau - 1, 2l)$

- Other lattices supporting fractal QCA and fractal symmetries.

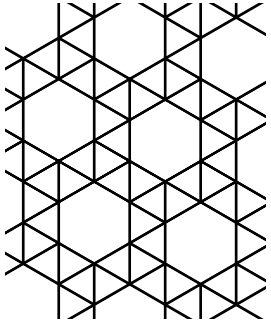
(6^3)



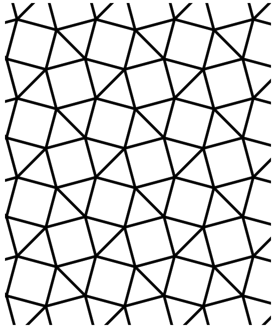
$(4, 6, 12)$



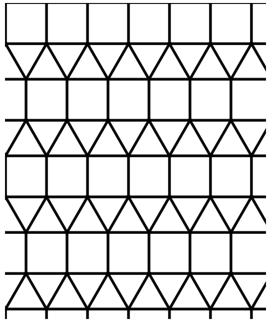
$(3^4, 6)$



$(3, 4, 3^2, 4)$

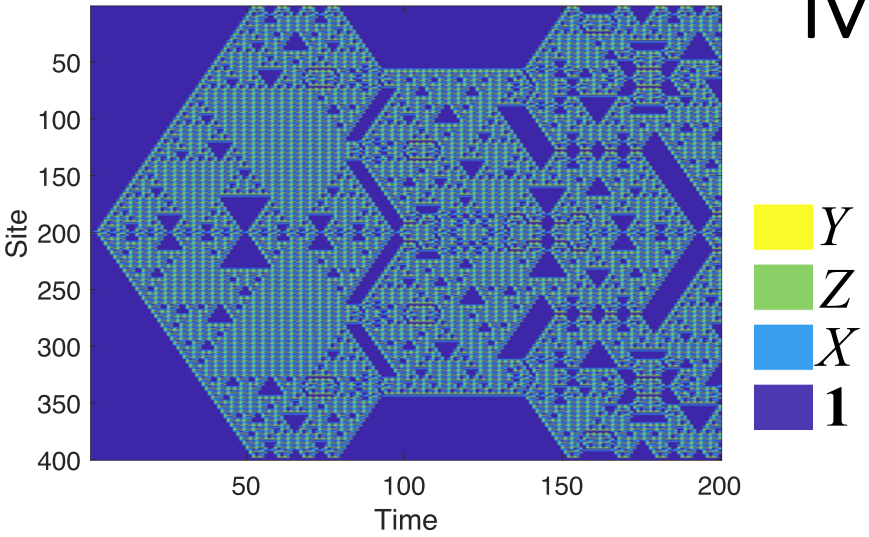


$(3^3, 4^2)$

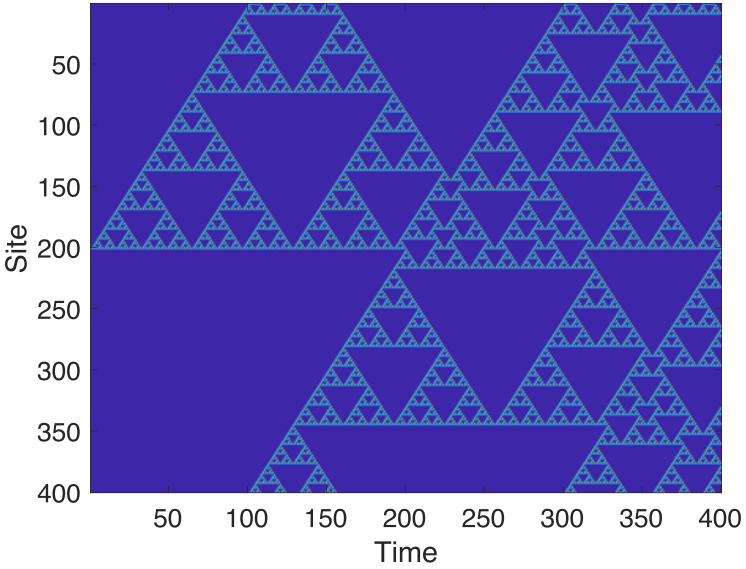


More fractal QCAs

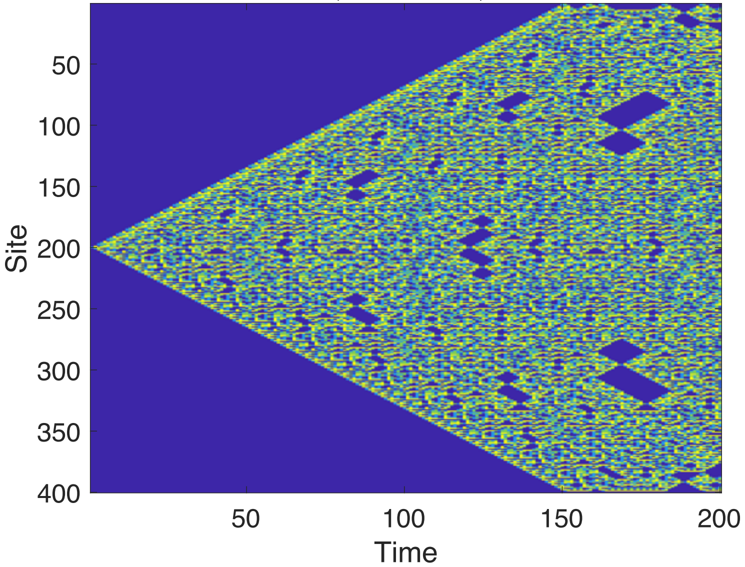
$(4, 6, 12)$



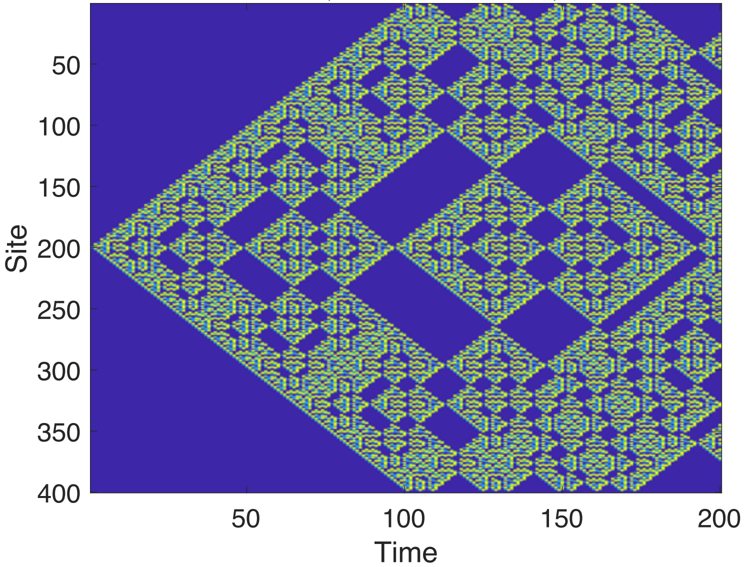
(6^3)



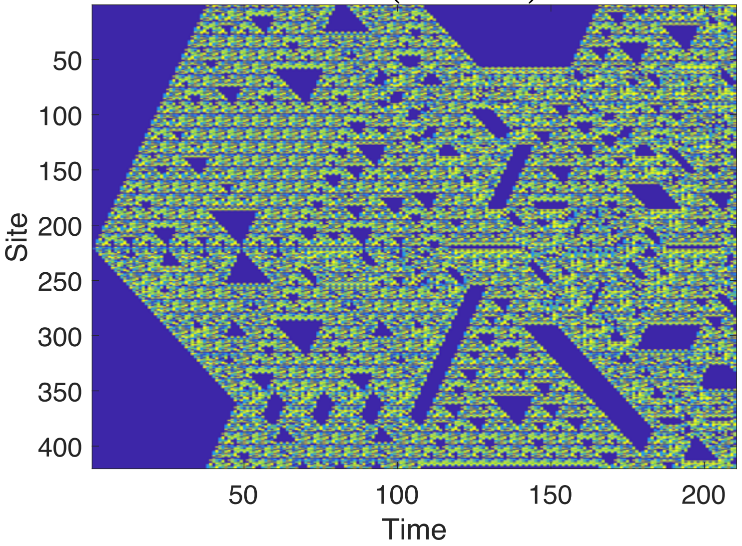
$(3^3, 4^2)$



$(3, 4, 3^2, 4)$

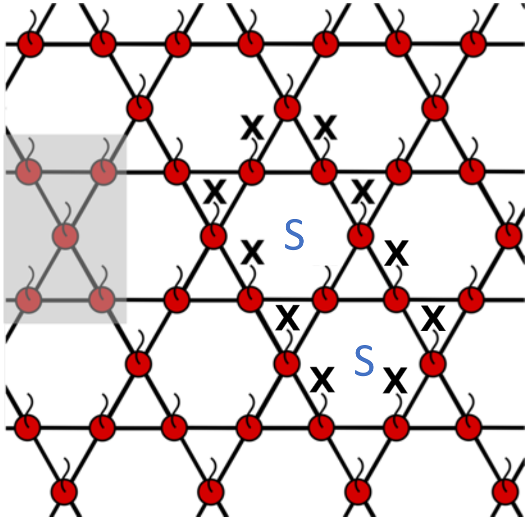


$(3^4, 6)$

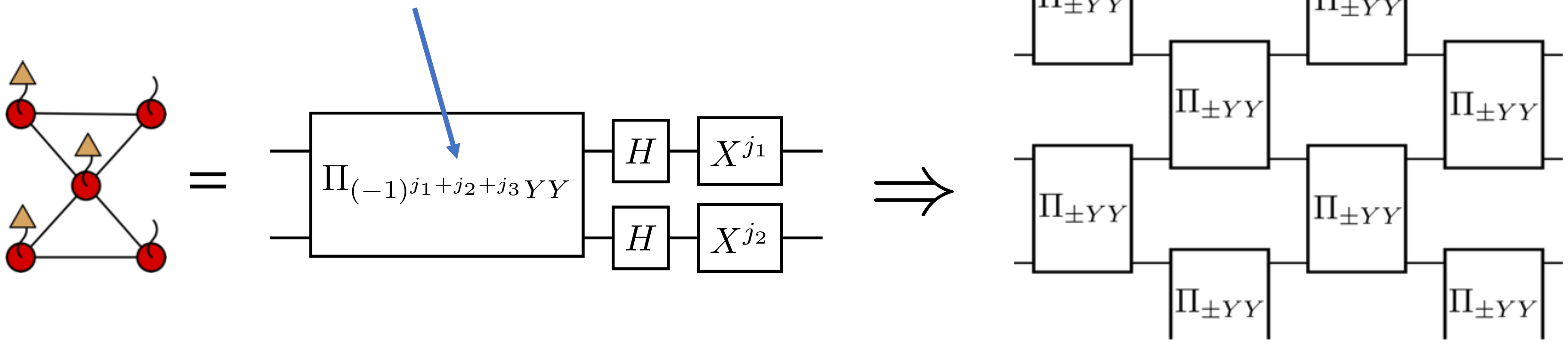


Lattices with no QCA structure

- Lattices with 1-form symmetries prevent unital QCA.
  - 1-form symmetries are closed loops of operators
  - Symmetry operators are deformable as multiplying two together gives a larger loop.
  - Pauli-X measurements at the edge implement YY parity measurements.



Measurement outcome tells us YY-parity of two qubits.



*This lattice can teleport a single qubit encoded in a repetition code.

Summary and Outlook for HEP

Summary:

- bulk-boundary correspondence via entanglement
- Measurement-based quantum computation (MBQC) allows to count space and time complexity in terms of entanglement and classical information

All ground states (which are not necessarily stabilizer states) in 2D cluster phase on a 2D Archimedean lattice with Ribbon/Cone/Fractal subsystem symmetry are universal for MBQC.

Outlook:

- universality -> scrambling in 3 classes of Clifford QCA
- 1-form symmetry, gauge theories, spurious topological entropy
- QCA for quantum field theory
- subsystem symmetries for non-Eucleadian lattices