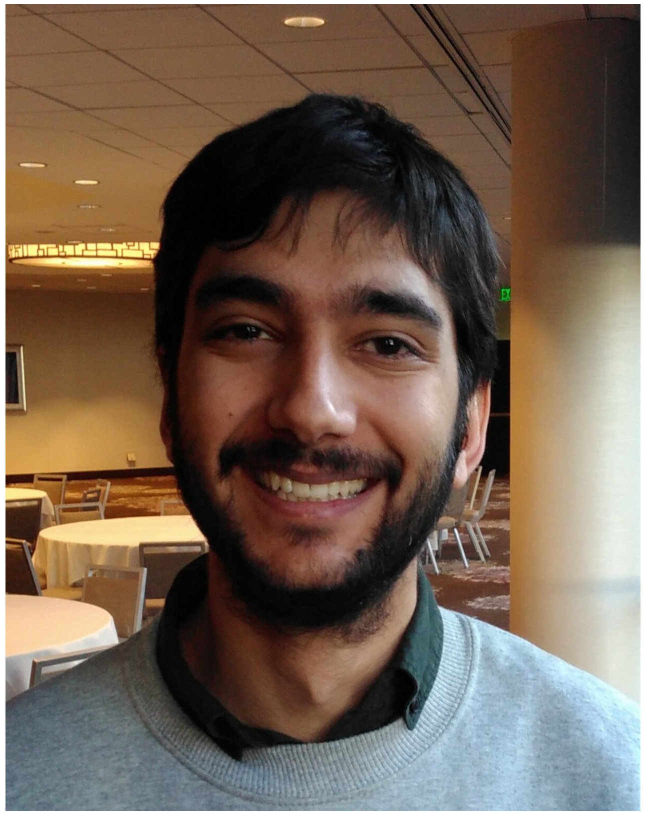
X-Batteries

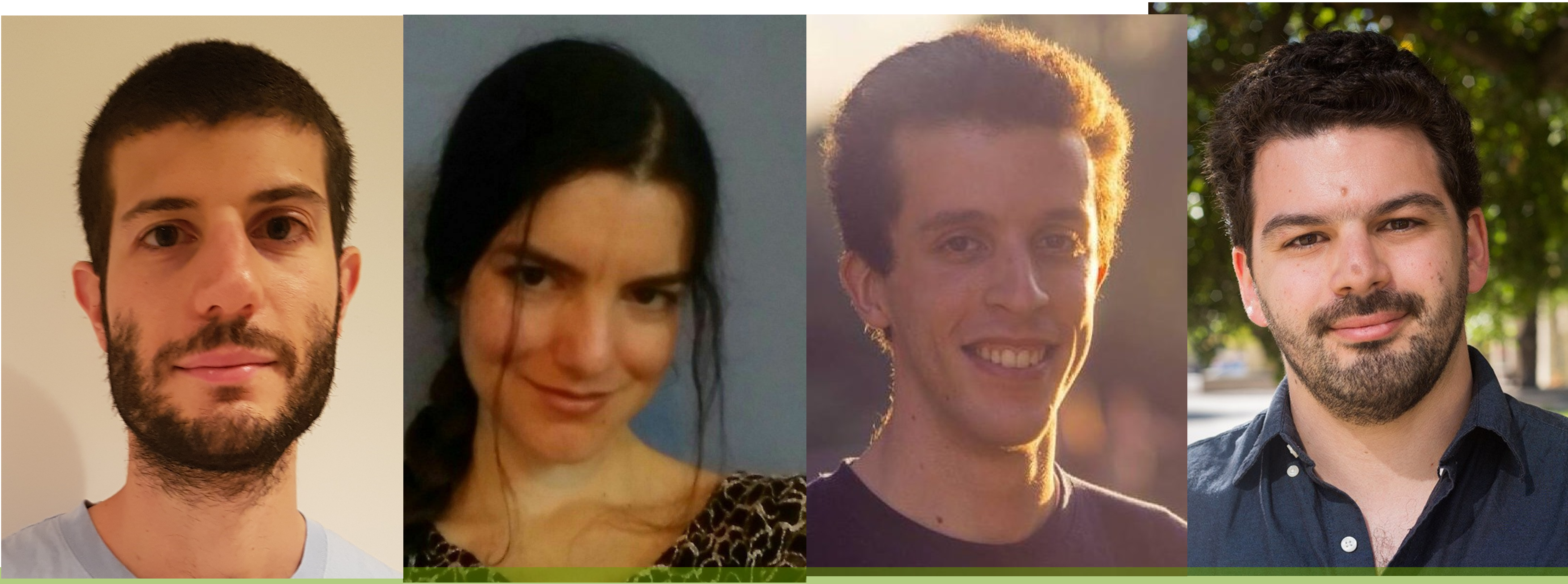
Majorization and Fluctuations
Phys. Rev. X 6, 041017 (2016)
Fluctuation Theorems for Entanglement arXiv:1709.06139



Alvaro Alhambra, Lluis Masanes, Jonathan Oppenheim, Chris Perry

X-Batteries

1st law of quantum resource theories



Carlo Sparaciari et. al. arXiv:1806.04937

UCL

Information is physical

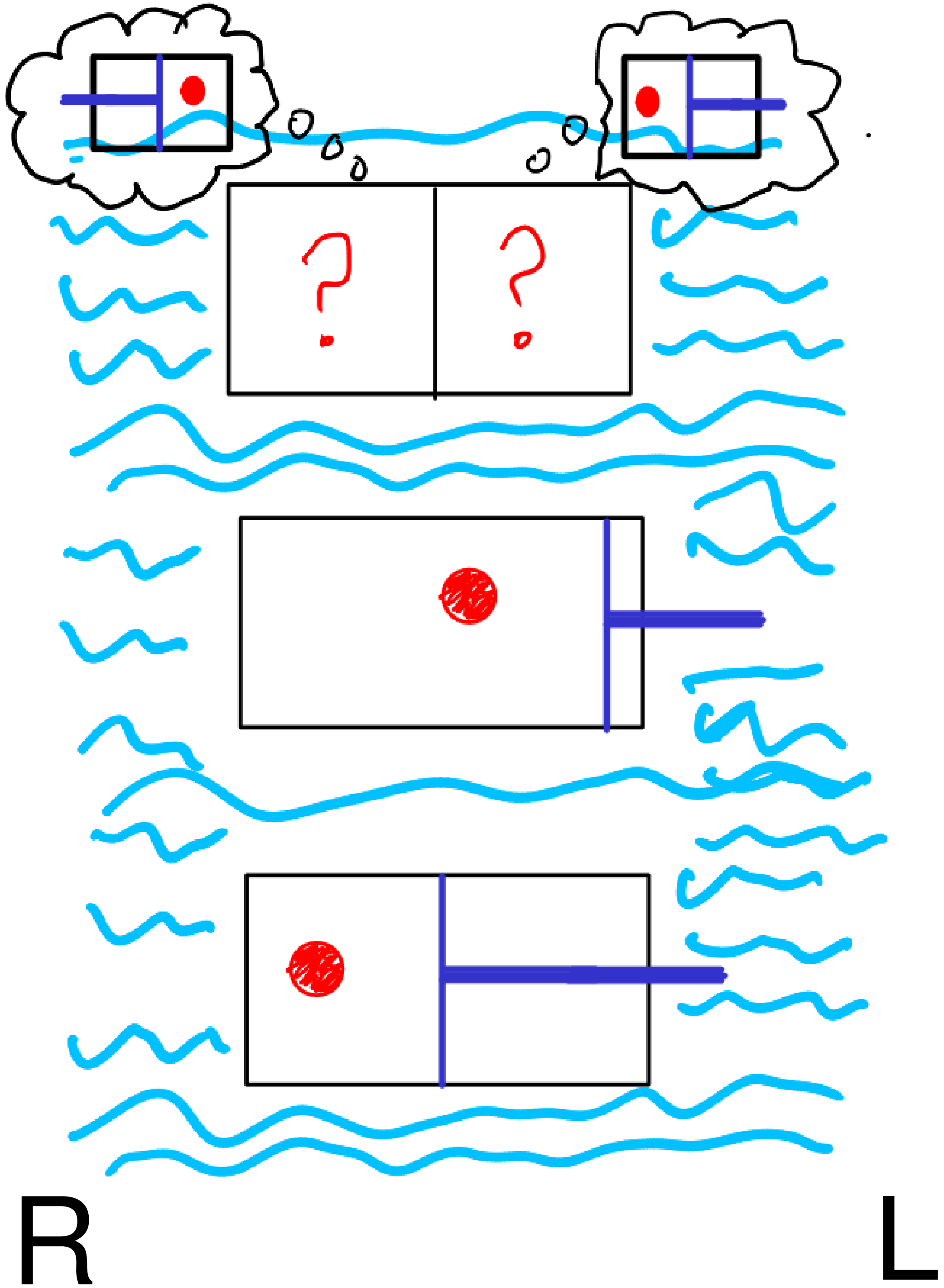
Maxwell

Szilard

Landauer

Bennett

$W=kT\log2$



What do we mean by W=kTlog2?

(consider the limit of perfect erasure)

A)We can achieve W=kTlog2 on average, but there will be fluctuations around this value.

B)We can achieve perfect erasure.

C)Using slightly more work on average than kTlog2, enables you to sometimes gain work by erasing the bit.

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D)None of these statements are true.

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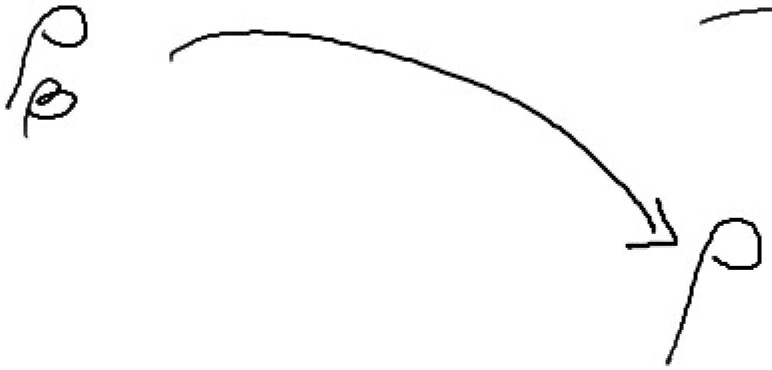
D) None of these statements are true.

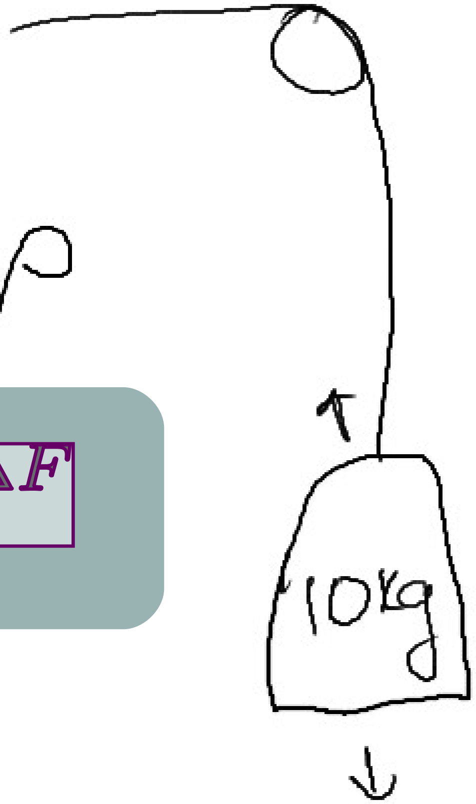
E) This quiz is undecidable.

Corrections to second law



$$\langle e^{-\beta w} \rangle = e^{-\beta \Delta F}$$

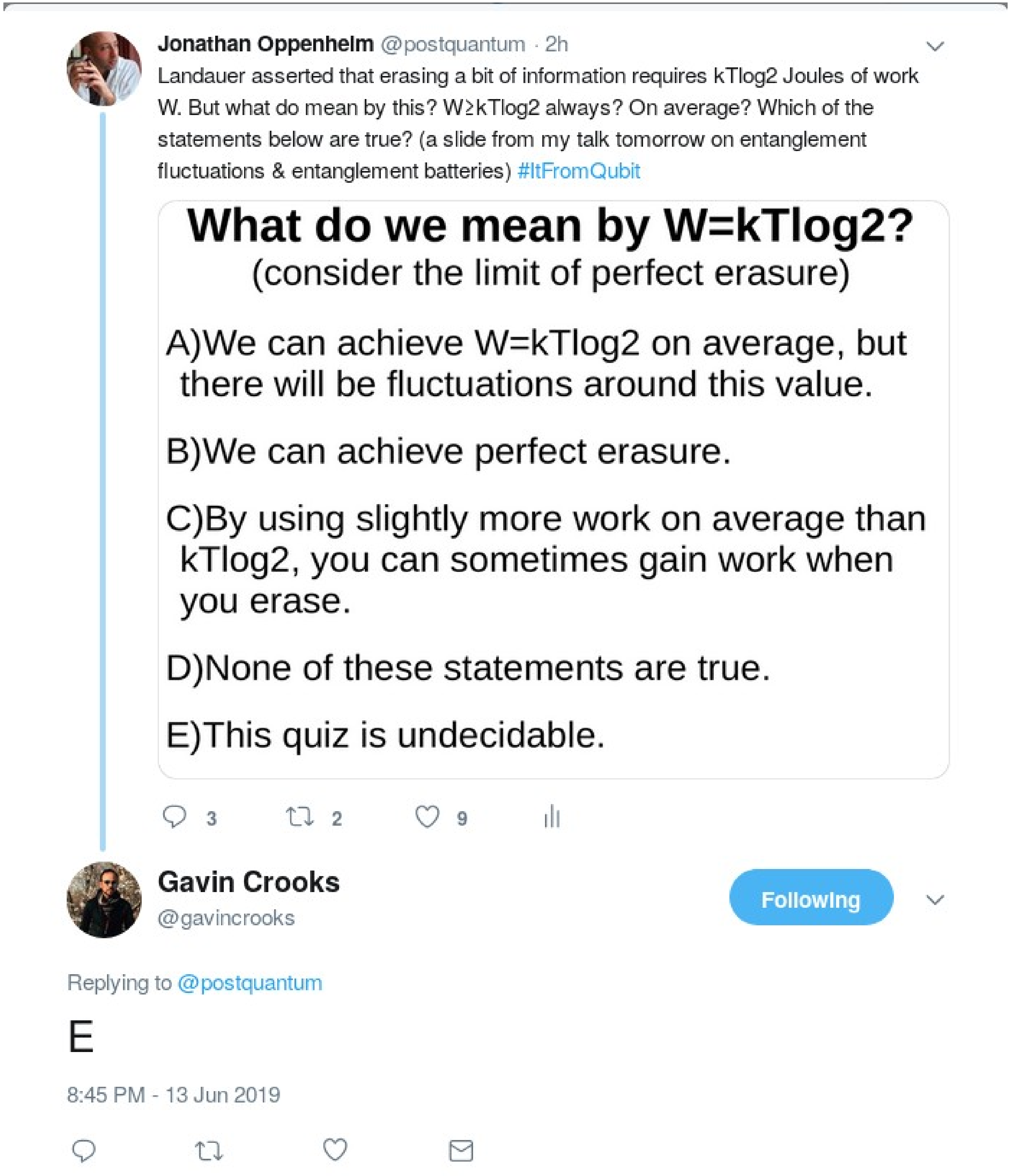
Jarzynski, Crooks



Standard 2nd law

$$W \leq \Delta F$$

$$\sum_{k=1}^{N} \frac{\beta^k}{k!} \langle (f_{s'} - f_s + w)^k \rangle \leq 0$$



Jonathan Oppenheim @postquantum · 2h

Landauer asserted that erasing a bit of information requires kTlog2 Joules of work W. But what do mean by this? W≥kTlog2 always? On average? Which of the statements below are true? (a slide from my talk tomorrow on entanglement fluctuations & entanglement batteries) #ItFromQubit

What do we mean by W=kTlog2?

(consider the limit of perfect erasure)

A)We can achieve W=kTlog2 on average, but there will be fluctuations around this value.

B)We can achieve perfect erasure.

C)By using slightly more work on average than kTlog2, you can sometimes gain work when you erase.

D)None of these statements are true.

E)This quiz is undecidable.

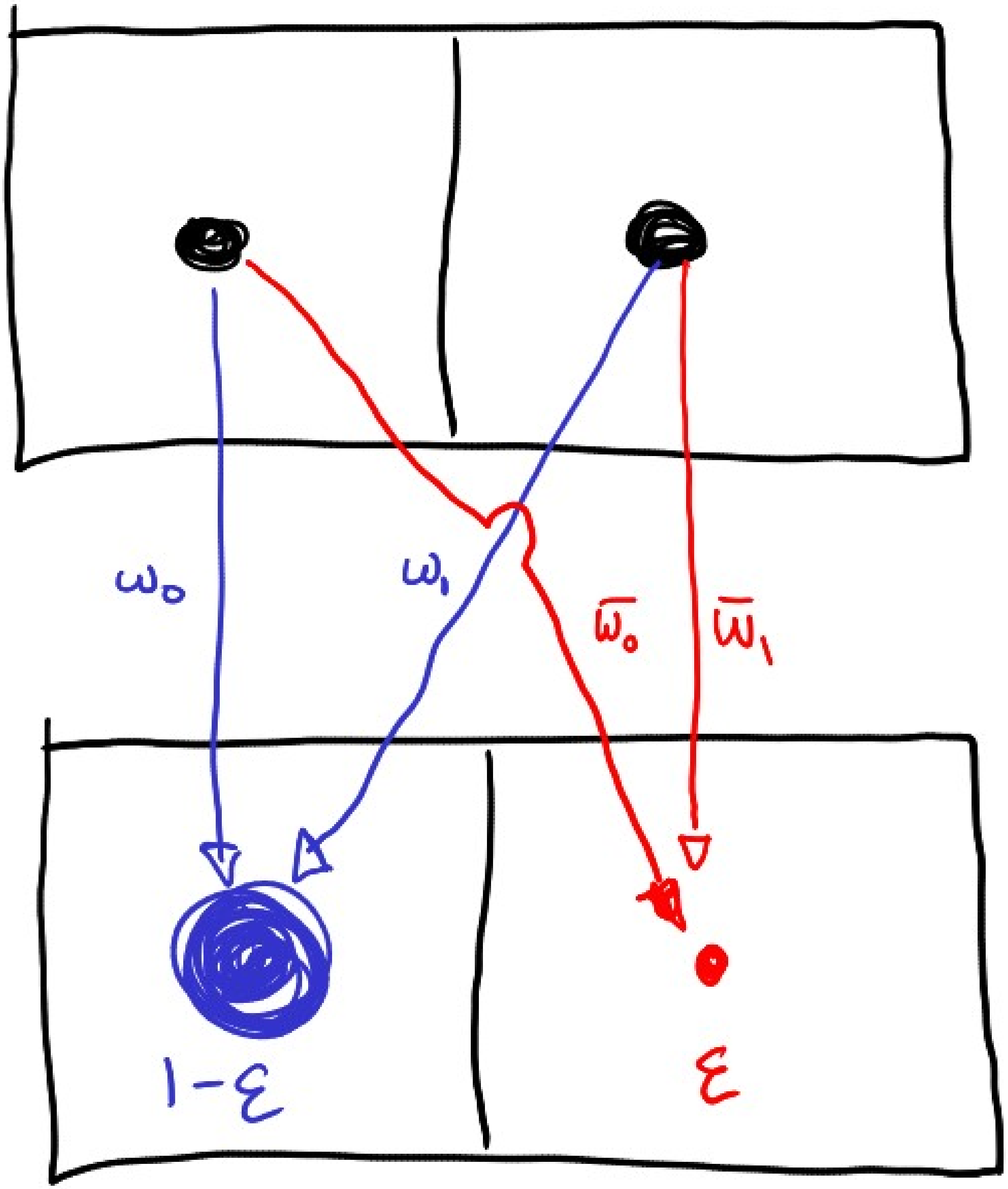
Gavin Crooks
@gavincrooks

Replying to @postquantum

E

8:45 PM - 13 Jun 2019

Fluctuating work in erasure

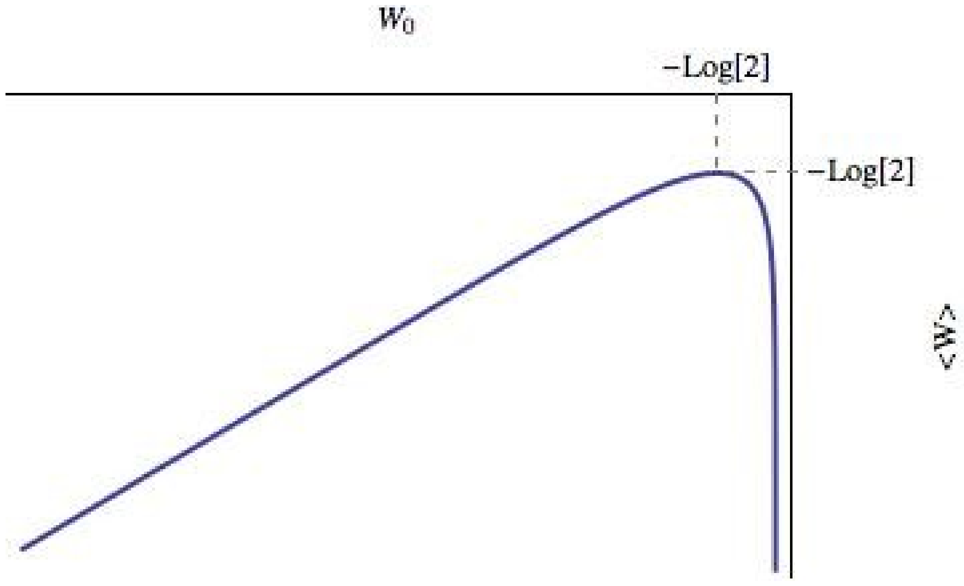


$$\sum_{s',w} P(s',w|s)=1$$

$$\sum_{s,w} P(s',w|s)\,e^{\beta(E_{s'}-E_s+w)}=1$$

$$e^{\beta w_o}+e^{\beta w_1}=1/(1-\epsilon)$$

$$e^{\beta \bar{w}_o}+e^{\beta \bar{w}_1}=1/\epsilon$$

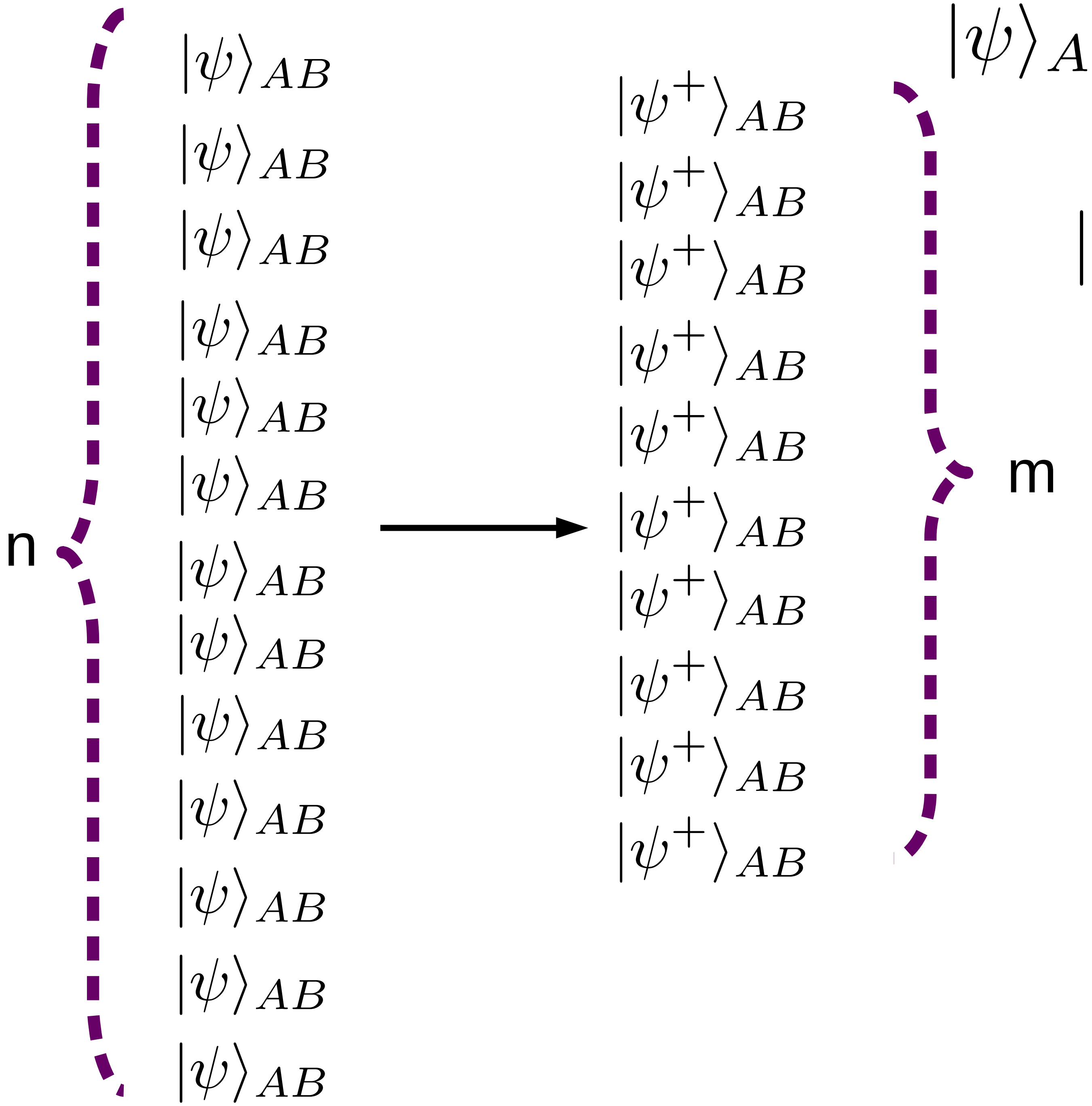


The 4 questions

1) What does W=kTlog2 mean? Average?
2) Entanglement dilution vs concentration?
3) Is a heat bath a bank?
4) What do we do about embezzlement?

Pure state entanglement theory

entanglement concentration



$$n\left\{\begin{array}{l}|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\\|\psi\rangle_{AB}\end{array}\right. \longrightarrow \left.\begin{array}{l}|\psi^+\rangle_{AB}\\|\psi^+\rangle_{AB}\\|\psi^+\rangle_{AB}\\|\psi^+\rangle_{AB}\\|\psi^+\rangle_{AB}\\|\psi^+\rangle_{AB}\\|\psi^+\rangle_{AB}\\|\psi^+\rangle_{AB}\\|\psi^+\rangle_{AB}\\|\psi^+\rangle_{AB}\end{array}\right\}m$$

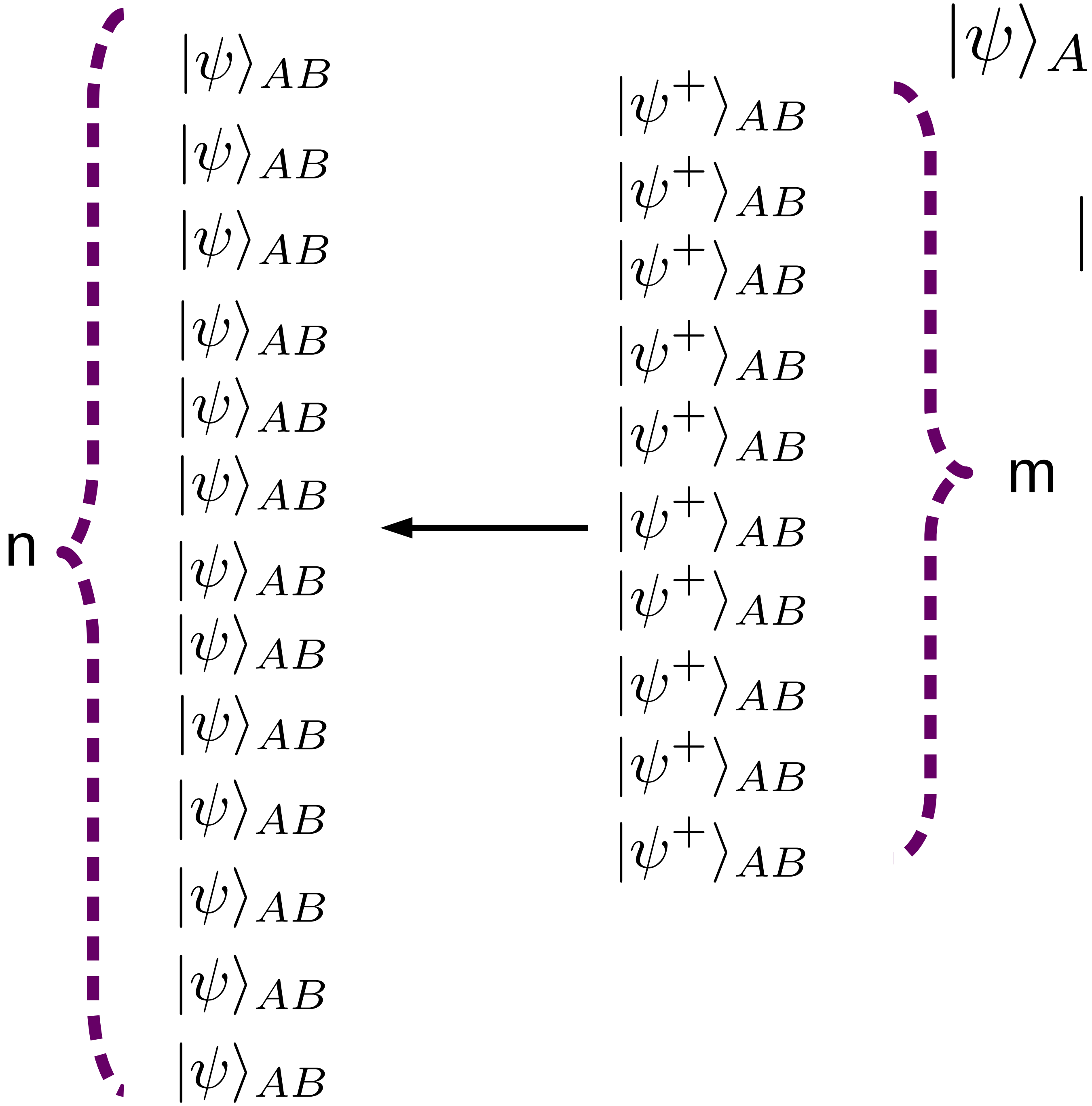
$$|\psi\rangle_{AB} = \sqrt{p}|00\rangle_{AB} + \sqrt{1-p}|11\rangle_{AB}$$

$$|\psi\rangle_{AB}^{\otimes n} = \sum_{k,\pi_k} p(k)|\pi_k\rangle_A|\pi_k\rangle_B$$

m fluctuates around $\langle m\rangle = nS(\rho_A)$

Pure state entanglement theory

entanglement dilution



n

$|\psi\rangle_{AB}$

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$|\psi^+\rangle_{AB}$

m

$$|\psi\rangle_{AB} = \sqrt{p}|00\rangle_{AB} + \sqrt{1-p}|11\rangle_{AB}$$

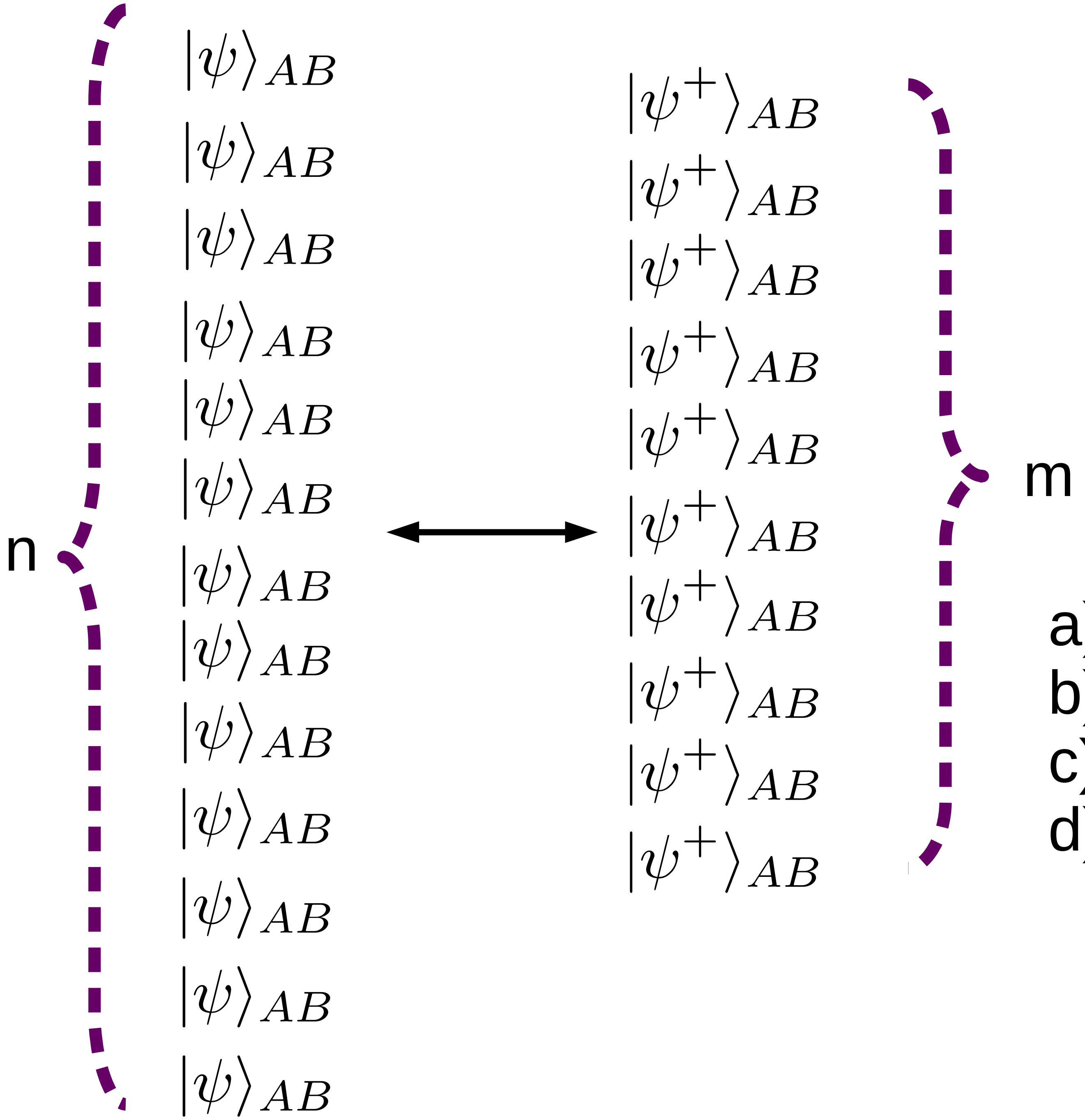
$$|\psi\rangle_{AB}^{\otimes n}$$

Alice compresses $\rho_B^{\otimes n}$ and teleports it to Bob using m ebits.

We require $m \geq nS(\rho_A) + O(\sqrt{n})$

Pure state entanglement theory

Entanglement cycle



Why does this dilution protocol require:

$$m \geq nS(\rho_A) + O(\sqrt{n})$$

But concentration has:

$$\langle m \rangle = nS(\rho_A)$$

a) Optimal dilution protocol?
b) Reversibility?
c) Can we characterize the fluctuations?
d) Do we require many copies?

The 4 questions

1) What does W=kTlog2 mean? Average?
2) Entanglement dilution vs concentration?
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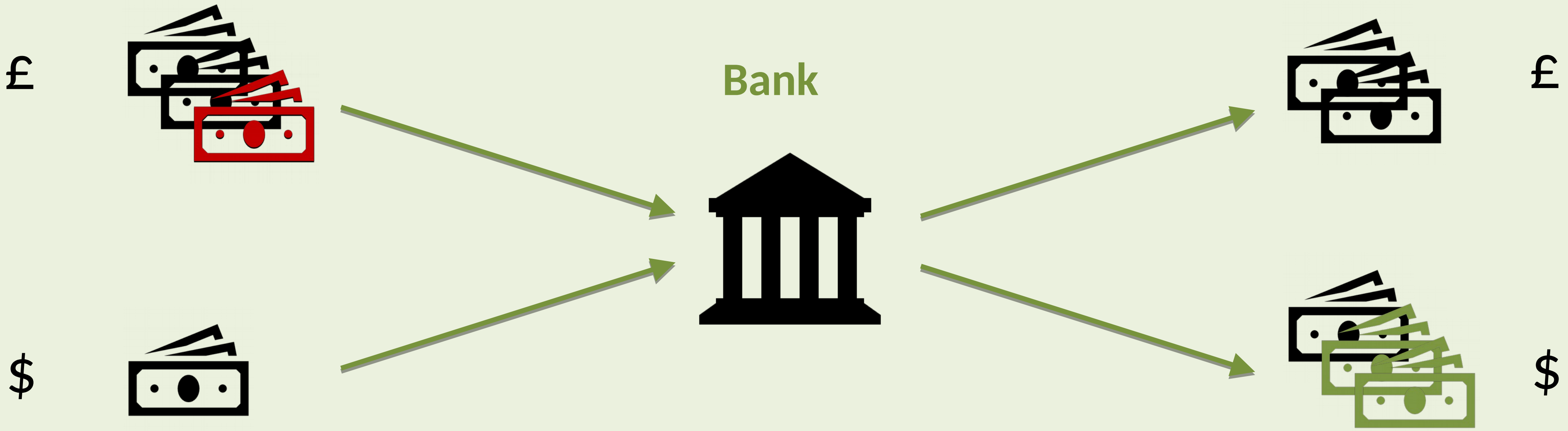
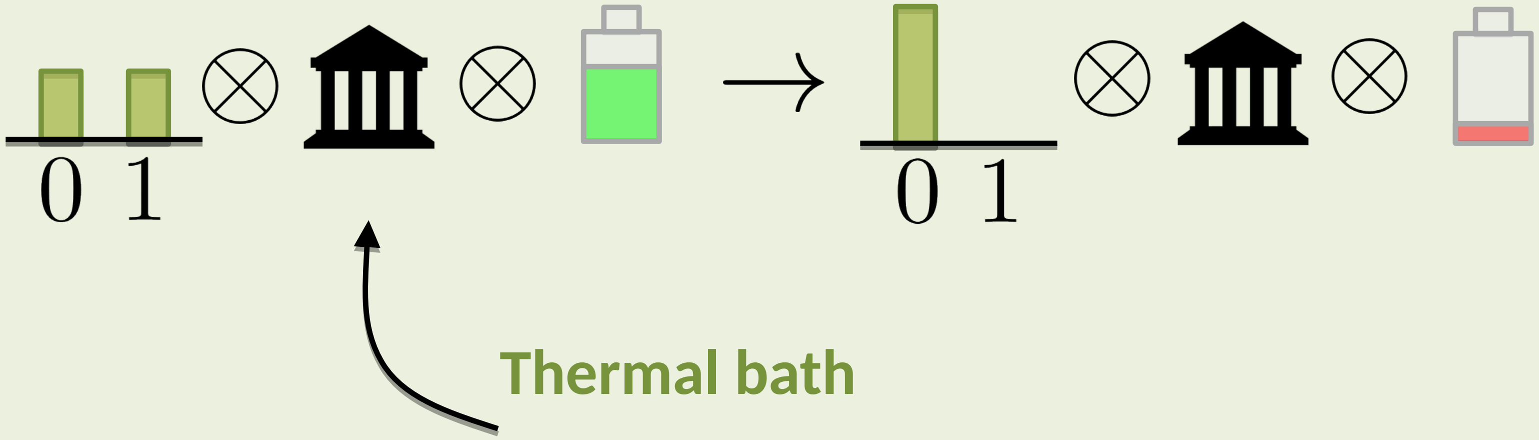
Banks and interconversion

Can we interconvert between resources?



- Cannot get dollars/pounds for free
- The bank fixes an exchange rate
- After the exchange, the rate does not change

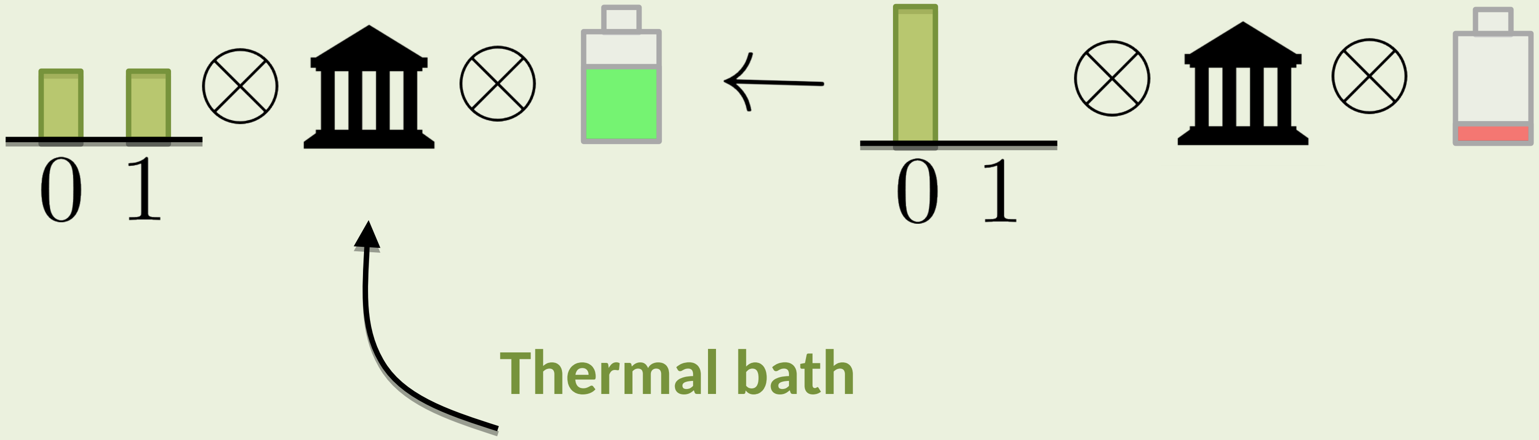
Interconversion and banks: Landauer's erasure



Thermal bath

- **Energy (work)** ΔE is added to the thermal bath
- **Purity (neg-entropy)** ΔJ is taken from the thermal bath
- Exchange rate depends on **temperature** $\Delta J = -\beta \Delta E$
- The thermal bath is left (almost) **unchanged**
- The thermal state converts work into purity with temperature as an exchange rate

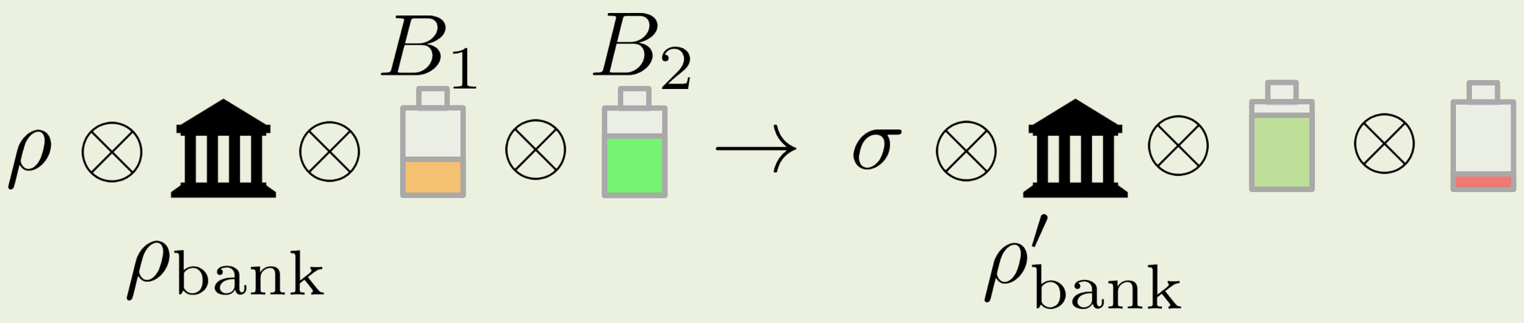
Interconversion and banks: Maxwell’s demon



- **Energy (work)** ΔE is taken from the thermal bath
- **Purity (neg-entropy)** ΔJ is injected into the thermal bath
- Exchange rate depends on **temperature** $\Delta J = -\beta \Delta E$
- The thermal bath is left (almost) **unchanged**
- The thermal state converts purity into work with temperature as an exchange rate

The First Law

- **Main system** : $\rho \to \sigma$
- **Bank** $\mathcal{F}_{\text{bank}}^{\bar{M}_1,\bar{M}_2}$: allows for exchange of resources
- **Battery** B_1: exchange first resource
- **Battery** B_2: exchange second resource



First Law :
$$M_{\text{bank}}^{\bar{M}_1,\bar{M}_2}(\rho) - M_{\text{bank}}^{\bar{M}_1,\bar{M}_2}(\sigma) = \alpha\,\Delta W_1 + \beta\,\Delta W_2$$

For thermodynamics, we get:
$$\Delta U = \Delta Q - \Delta W$$

The 4 questions

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Pure state entanglement theory

Single copy transformations

$$|\psi\rangle_{AB} \to |\phi\rangle_{AB}$$

iff $\sum_{i}^{k} q_i^{\downarrow} \geq \sum_{i}^{k} p_i^{\downarrow} \quad \forall k$

$$|\psi\rangle_{AB} = \sum \sqrt{p_i}|ii\rangle_{AB}$$

$$|\phi\rangle_{AB} = \sum \sqrt{q_i}|ii\rangle_{AB}$$

Asymptotic limit

$$|\psi\rangle_{AB}^{\otimes n} \otimes |\psi^{+}\rangle^{\otimes n\Delta S} \to |\phi\rangle_{AB}^{\otimes n}$$

$$\rho_{\psi} = tr_B|\psi\rangle\langle\psi|_{AB}$$

$$\rho_{\phi} = tr_B|\phi\rangle\langle\phi|_{AB}$$

$$\Delta S = S(\rho_{\phi}) - S(\rho_{\psi})$$

Embezzling state

van Dam, Hayden (2002)

Single copy transformations

$$|\psi\rangle_{AB} \rightarrow |\phi\rangle_{AB}$$

if $\sum_i^k \psi_i^\downarrow \geq \sum_i^k \phi_i^\downarrow \; \forall k$

Always

$$|\mu(n)\rangle := \frac{1}{\sqrt{C(n)}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} |j\rangle_A |j\rangle_B$$

$$|b\rangle := \sum_{j=1}^{N} \frac{1}{\sqrt{N}} |jw\rangle \qquad |b-w\rangle := \sum_{j=0}^{N-1} \frac{1}{\sqrt{N}} |jw\rangle$$

$$\langle b|b-w\rangle = 1 - \tfrac{2}{N}$$

Pure state entanglement theory with an entanglement battery

work is

$$|e_x\rangle_{AB} = \left(\frac{1}{\sqrt{2^x}}\left(|1\rangle|1\rangle + |2\rangle|2\rangle\right)\right)^{\otimes x} \otimes \left(|0\rangle|0\rangle\right)^{n-x}$$

$$|b\rangle_W = \sum_{x=1}^{n} \alpha_x |e_x\rangle$$

$$|\Psi\rangle := \sum_{i,x} \sqrt{p_{ix}}|ii\rangle \otimes |e_x\rangle \xrightarrow{\Lambda} |\Phi\rangle := \sum_{j,x'} \sqrt{q_{jx'}}|jj\rangle \otimes |e_{x'}\rangle$$

Pure state entanglement theory with an entanglement battery

Single copy transformations

$$|\psi\rangle_{AB} \rightarrow |\phi\rangle_{AB}$$

iff $\langle w\rangle \geq S(\rho_\psi) - S(\rho_\phi)$

Reversible on single copy level

$$|\psi\rangle_{AB} \leftarrow |\phi\rangle_{AB}$$

$-\langle w\rangle$

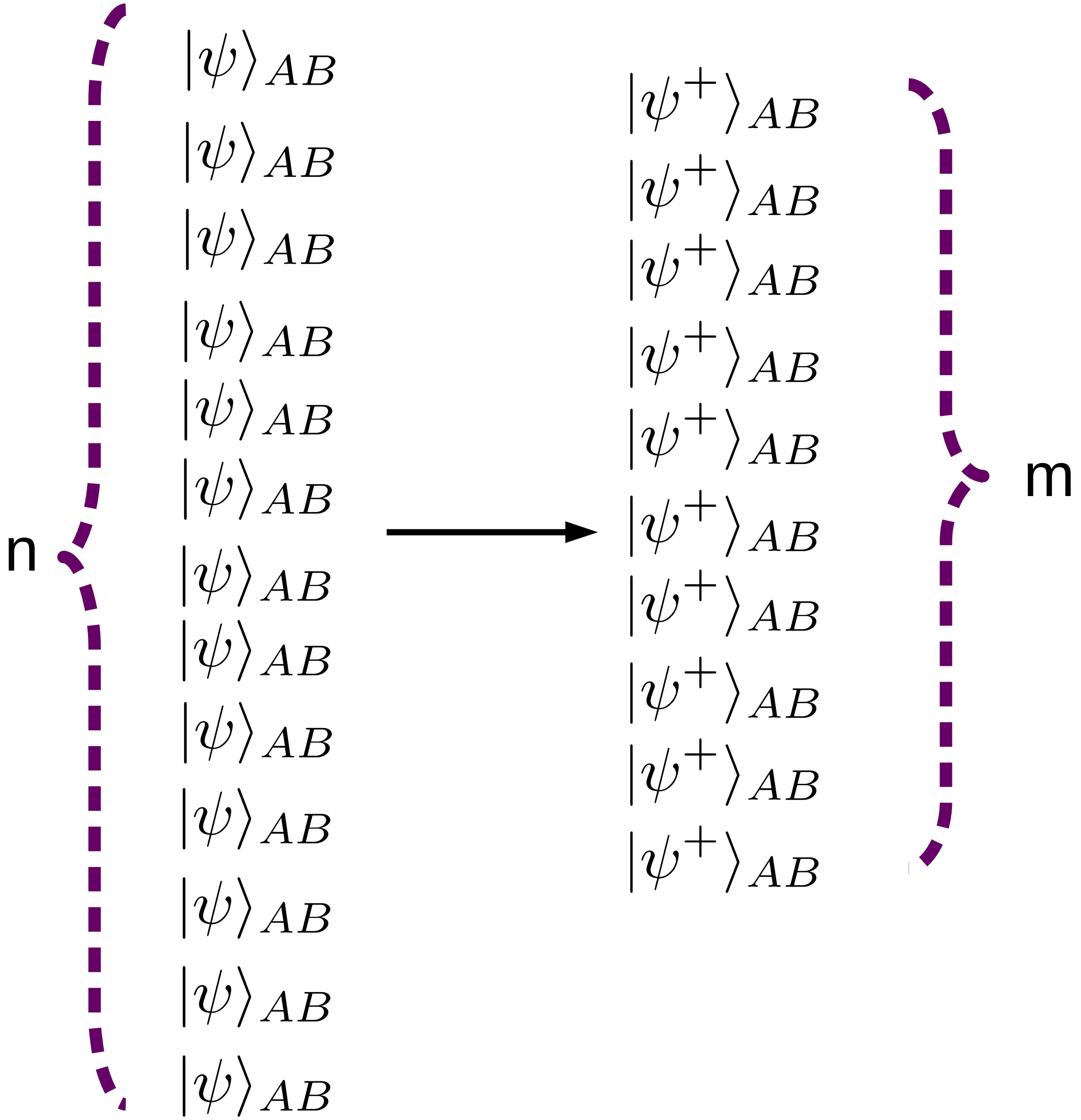
$$\left\langle 2^{w-\log q_j+\log p_i}\right\rangle = 1$$

$$\langle 2^w\rangle = \frac{d}{d'}$$

$$\langle e^{\beta w}\rangle = \frac{Z'}{Z}$$

Pure state entanglement theory

entanglement concentration



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w/ entanglement battery

$|\psi\rangle_{AB} \rightarrow |\psi^+\rangle_{AB}$
$|\psi\rangle_{AB} \rightarrow |00\rangle_{AB}$
$|\psi\rangle_{AB} \rightarrow |\psi^+\rangle_{AB}^{\otimes 2}$
$|\psi\rangle_{AB} \rightarrow |00\rangle_{AB}$
$|\psi\rangle_{AB} \rightarrow |\psi^+\rangle_{AB}$
$|\psi\rangle_{AB} \rightarrow |\psi^+\rangle_{AB}^{\otimes 3}$
$|\psi\rangle_{AB} \rightarrow |00\rangle_{AB}$
$|\psi\rangle_{AB} \rightarrow |00\rangle_{AB}$
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$|\psi\rangle_{AB} \rightarrow |00\rangle_{AB}$
$|\psi\rangle_{AB} \rightarrow |\psi^+\rangle_{AB}$

Pure state entanglement theory with an entanglement battery

$$|\psi\rangle_{AB} \otimes |b\rangle_W \to\approx |\phi\rangle_{AB} \otimes |b'\rangle_W$$

$$|b\rangle_W = \sum_{x=1}^{n} \alpha_x |e_x\rangle$$

$$\alpha_x = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(x-n/2)^2}{2\delta^2}}$$

$$\left\langle 2^{w-\log q_j+\log p_i} \right\rangle = 1$$

$$\langle 2^w \rangle = \frac{d}{d'}$$

$$P^{\mathrm{rev}}(-w) = P(w) 2^w \frac{d'}{d}$$

no work to fluctuating work

doubly stochastic maps

$$\sum_{s'} P(s'|s)=1$$

$$\sum_{s} P(s'|s)=1$$

majorisation

Gibbs-stochastic maps

$$\sum_{s'} P(s'|s)=1$$

$$\sum_{s} P(s'|s)\,e^{\beta(E_{s'}-E_s)}=1$$

thermo-majorisation

fluctuating work

$$\sum_{s'} P(s'|s)=1$$

$$\sum_{s,w} P(s',w|s)\,e^{\beta(E_{s'}-E_s+w)}=1$$

linear program

Why I like resource theories

Thermodynamics

- Many second laws
- Work fluctuations

$$\boxed{\langle e^{-\beta w}\rangle = e^{-\beta\Delta F}}$$

Majorisation and Fluctuations
Phys. Rev. X 6, 041017 (2016)

$$\sum_i^k p_i^{\downarrow} \geq \sum_i^k q_i^{\downarrow} \quad \forall k$$

Entanglement theory

- Majorisation criteria
- Entanglement fluctuations

Why I like resource theories

Thermodynamics

Something new about Landauer erasure

$$\boxed{\langle e^{-\beta w}\rangle = e^{-\beta \Delta F}}$$

$$D_\alpha(\rho||\rho_\beta) \geq D_\alpha(\sigma||\rho_\beta)$$

$$\langle 2^w \rangle = \frac{d}{d'}$$

$$\sum_i^k p_i^\downarrow \geq \sum_i^k q_i^\downarrow \quad \forall k$$

Majorisation and Fluctuations
Phys. Rev. X 6, 041017 (2016)

Entanglement theory

Something new about entanglement distillation

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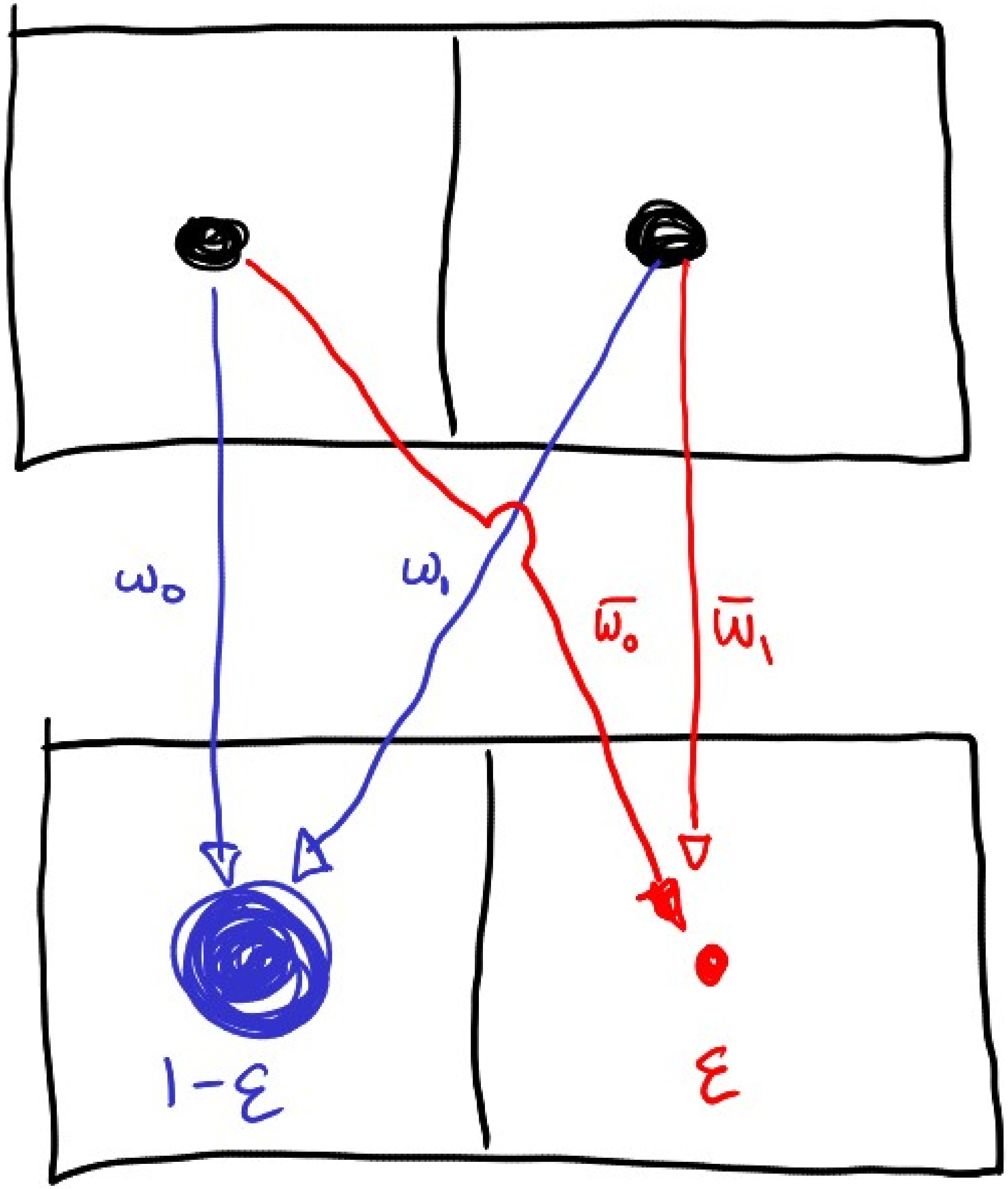
Fluctuating work in erasure



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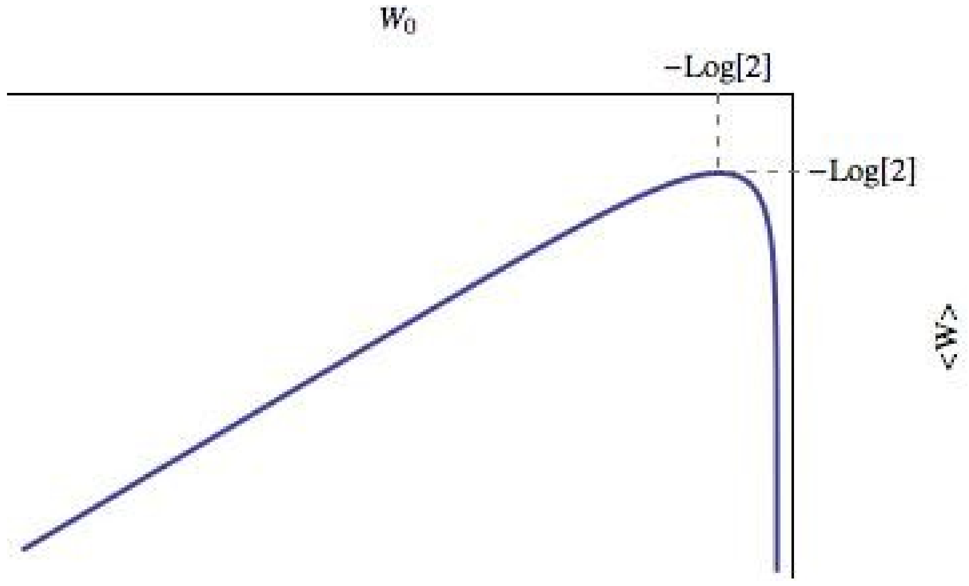
$$\sum_{s,w} P(s',w|s)\,e^{\beta(E_{s'}-E_s+w)}=1$$

$$e^{\beta w_o}+e^{\beta w_1}=1/(1-\epsilon)$$

$$e^{\beta \bar{w}_o}+e^{\beta \bar{w}_1}=1/\epsilon$$



Summary

- Majorisation → fluctuation relations
- Pure state entanglement criteria ↔ 2nd laws of thermo
- Entanglement fluctuation theorem ↔ Work fluctuation theorem
- Two kinds of batteries: 1st law of resource theories
- I want an entanglement battery!

Outlook and open questions

- Other theories with fluctuation relations?
  - e.g. Coherence (Morris & Adesso; 1802.059191802.05919)
- More connections between resource theories:
  - Relative entropy distance as unique measure (Horodecki, JO; quant-ph/0207177)
  - More 1st law examples? (Sparaciari et. al.)
  - Destruction of the resource (Groisman et. al. 2005)
  - Many second laws for black holes (Alice Bernamonti et. al. 2018)