Two-dimensional AKLT states as (1) ground states of gapped Hamiltonians and (2) resource for quantum computation

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support:

Nothing more enjoyable than finally meeting the person whose work is the basis of your works

the ‘T’ in AKLT

And to meet old friends and new friends you can share work with and/or collaborate with

Acknowledgment

Collaborators: Robert Raussendorf, Ian Affleck, Valentin Murg, Artur Garica-Saez, Ching-Yu Huang, Abhishodh Prakash, **Nikko Pomata**, Hendrik Poulsen Nautrup, David Stephen, Dong-Sheng Wang,…

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Many of you in this wonderful workshop!

Outline

I. Introduction

II. AKLT models and states for universal quantum computation (in MBQC framework)

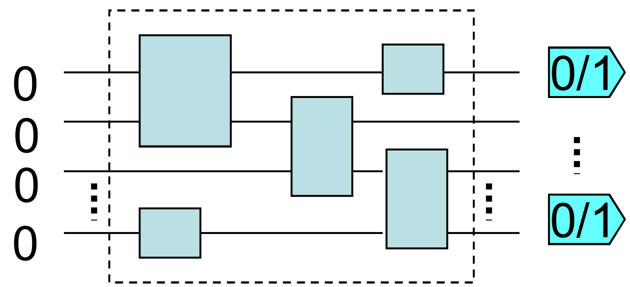
III. Nonzero gap for some 2D AKLT models

Ref: arXiv:1905.01275

IV. Summary

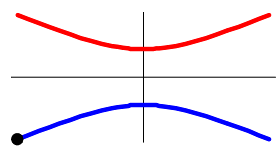
(Frameworks of) Quantum Computation

I. Circuit:



- ✓ Major scheme by most labs: IBM, Intel Rigetti, IonQ, Alibaba

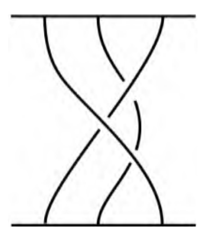
II. Adiabatic:



$$H(t) = \left(1 - \frac{t}{T}\right) H_{\text{initial}} + \frac{t}{T} H_{\text{final}}$$

- ✓ Approach by D-Wave

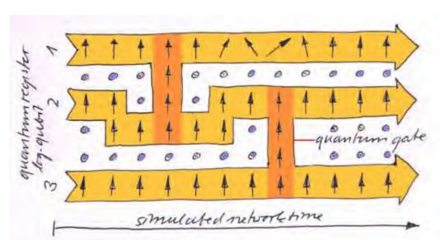
III. Topological:



quantum gates = braiding anyons

- ✓ Approach by Microsoft, Google uses a hybrid of III and I (circuit version of IV)

IV. Measurement -based:



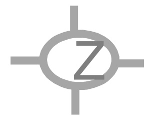
local measurement is the only operation needed

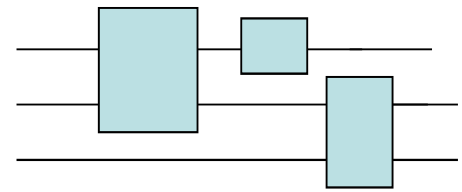
- ✓ Used in photonic systems, such as PsiQuantum

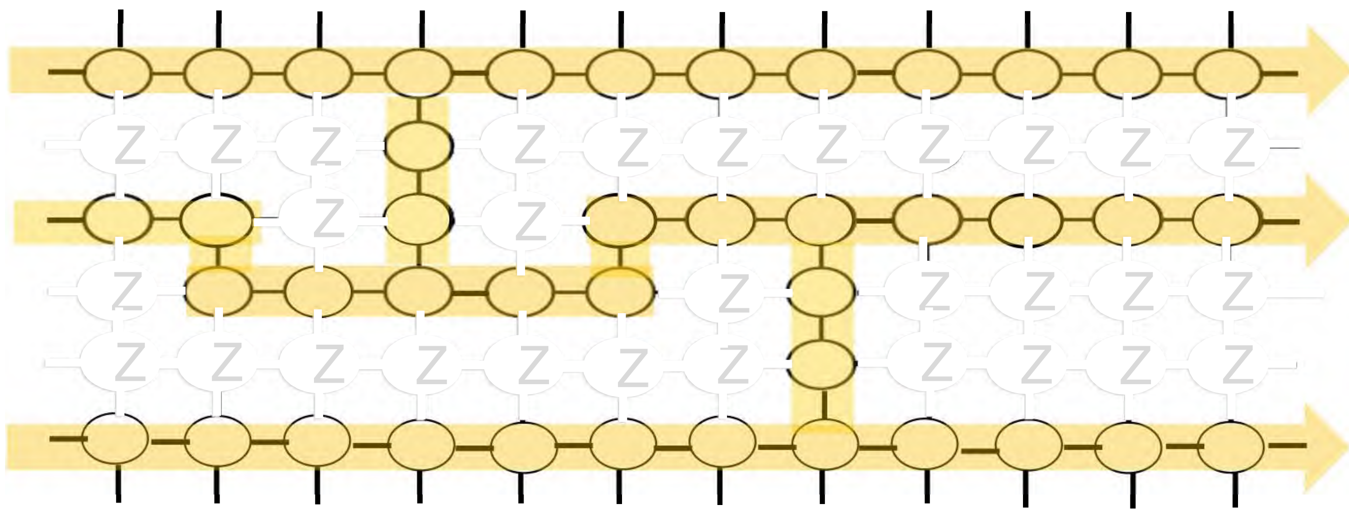
QC by Local Measurement

[Raussendorf & Brigel '01]

- First: carve out entanglement structure by local Pauli Z measurement







- Then:

  (1) Measurement along each wire simulates one-qubit evolution (gates)

  (2) Measurement near & on each bridge simulates two-qubit gate (CNOT)

→ 2D or higher dimensions are needed for universal QC

How much entanglement is needed?

[Gross, Flammia & Eisert '09; Bremner, Mora & Winter '09]

- States (n-qubit) possessing too much geometric entanglement E_g are not universal for QC (i.e if $E_g > n - \delta$)

$$E_g(|\Psi\rangle) = -\log_2 \max_{\phi \in \mathcal{P}} |\langle \phi | \Psi \rangle|^2 \qquad \mathcal{P} = \text{set of product states}$$

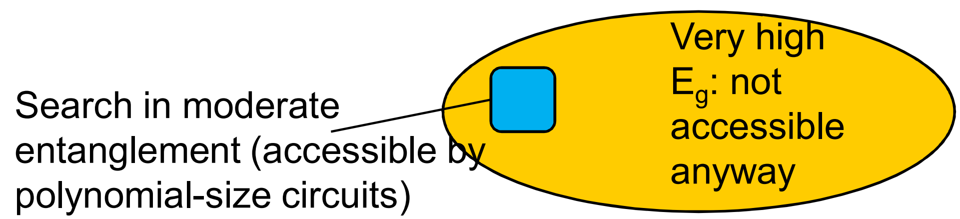
- Intuition: if state is very high in geometric entanglement, every local measurement outcome has low probability

  ➔ whatever local measurement strategy, the distribution of outcomes is so random that one can simulate it with a random coin (thus not more powerful than classical random string)

- Moreover, states with high entanglement are typical:

  those with $E_g < n - 2\log_2(n) - 3$ is rare, i.e. with fraction $< e^{-n^2}$

➔ Universal resource states are rare ☹



Key questions for MBQC

- Characterizing all resource states? Still open
- Can they be unique ground state with 2-body Hamiltonians with a finite gap? ➔ If so, create resources by cooling!

- Affleck-Kennedy-Lieb-Tasaki (AKLT) family of states [*AKLT* '87, '88]
  - 1D (not universal): [*Gross & Eisert et al.* '07, '10] [*Brennen & Miyake* '08]
  - 2D (universal): [*Miyake*'11] [*Wei, Affleck & Raussendorf* '11] [*Wei et al*. '13-'15]
    - Nonzero 2D gap still not proven (after 30 yrs) [see also Abdul-Rahman et al. 1901.09297; Pomata & Wei 1905.01275]
- Symmetry-protected topological states
  - 1D (not universal): [*Miyake'10, Miller&Miyake '15*] [*Else, Doherty & Bartlett '12*] [*Prakash & Wei* '15] [*Stephen et al.* '17, Raussendorf et al. '17]
  - 2D (universal, *but not much explored*): [*Miller & Miyake* '15] [*Poulsen Nautrup & Wei* '15]
    - Important progress for QC in entire symmetry-protected phases: [Raussendorf et al. PRL' 19, and Devakul & Williamson, PRA'18, Daniel, Alexander& Miyake (talk yesterday)]
- Thermal states (density matrices at finite T): some topologically protected [*Li et al* '11, *Fujii &Morimae* '12, *Fujii, Nakata, Ohzeki& Murao*'13, *Wei,Li&Kwek* '14 ']

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Valence-bond ground states of isotropic antiferromagnet

- AKLT (Affleck-Kennedy-Lieb-Tasaki) states/models
  - Importance: provide strong support for Haldane's conjecture on spectral properties of spin chains [*AKLT* '87,88]
  - Provide concrete example for symmetry-protected topological order [*Gu & Wen* '09, '11, …]

- States of spin S=1,3/2, 2,.. (defined on any lattice/graph)

➔ Unique* ground states of gapped# two-body isotropic Hamiltonians

$$H = \sum_{\langle i,j \rangle} f(\vec{S}_i \cdot \vec{S}_j)$$

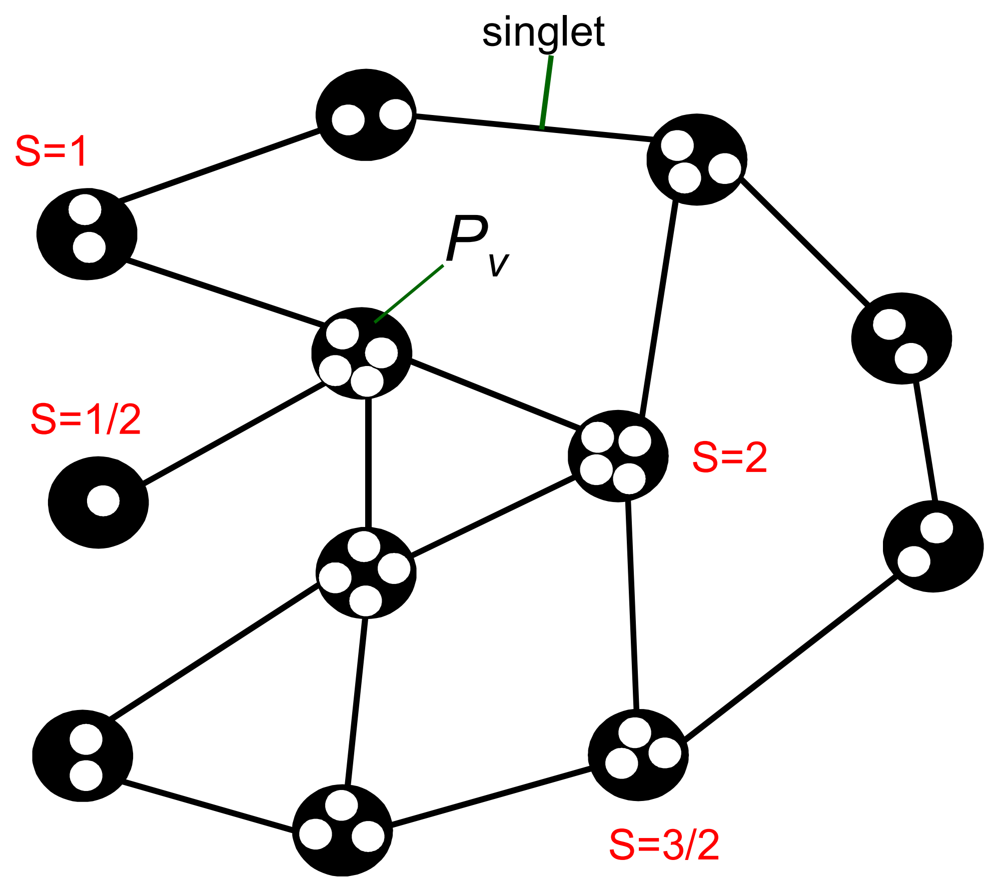
f(x) is a polynomial

e.g. 1D: S=1

$$H_{1D} = \sum_i \hat{P}^{(S=2)}_{i,i+1} = \frac{1}{2} \sum_{\text{edge}\,\langle i,j \rangle} \left[\vec{S}_i \cdot \vec{S}_j + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_j)^2 + \frac{2}{3} \right]$$

*w/ appropriate boundary conditions [Kennedy, Lieb & Tasaki '88]

(hybrid) AKLT state defined on any graph

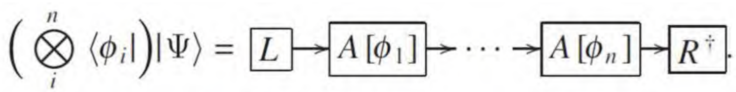


- # virtual qubits = # neighbors
- S= # neighbors / 2
- Physical spin Hilbert space = symmetric subspace of qubits

P_v = projection to symmetric subspace of n qubit ≡ spin n/2

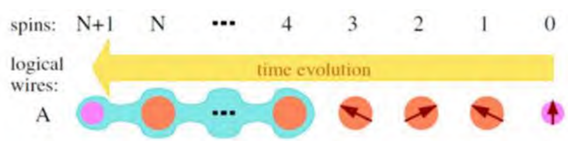
Warm up: 1D AKLT state for gates

- 1D spin-1 AKLT state can be used to implement arbitrary one-qubit gate
  - Using matrix-product representation: [*Gross & Eisert et al.* '07, '10]



$$A_{\alpha=x,y,z} = \sigma_\alpha$$

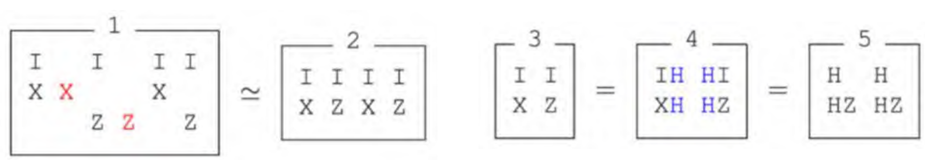
  - Using edge degrees of freedom: [*Brennen & Miyake* '08] [*Miyake'10*]



- Alternative view by reduction to 1D cluster state by local measurement

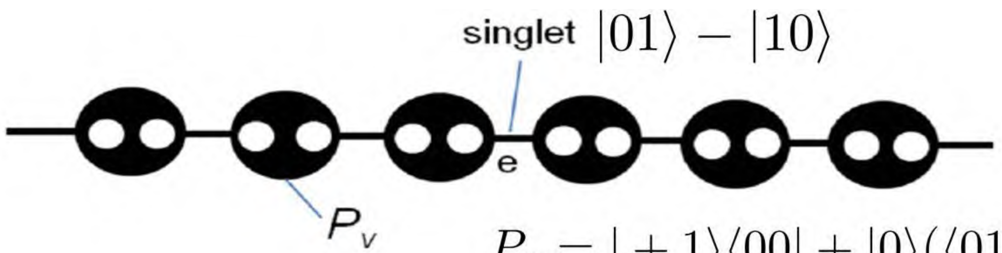
[*Chen, Duan, Ji & Zeng '10*]

  - Adaptive:



  - Fixed measurement: (see next) [*Wei, Affleck & Raussendorf '11*]

Converting 1D AKLT state to cluster state



$$P_v = |+1\rangle\langle 00| + |0\rangle(\langle 01| + \langle 10|)/\sqrt{2} + |-1\rangle\langle 11|$$

- Via fixed POVM ➔ *generalizable to 2D AKLT:*

[*Wei, Affleck & Raussendorf '11*]

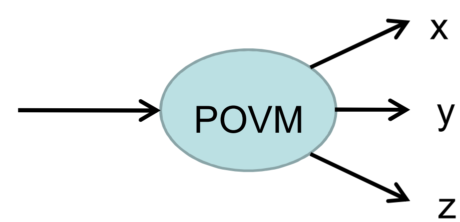
$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$$

$$F_x \sim |S_x = 1\rangle\langle S_x = 1| + |S_x = -1\rangle\langle S_x = -1| \sim |++\rangle\langle ++| + |--\rangle\langle --|$$

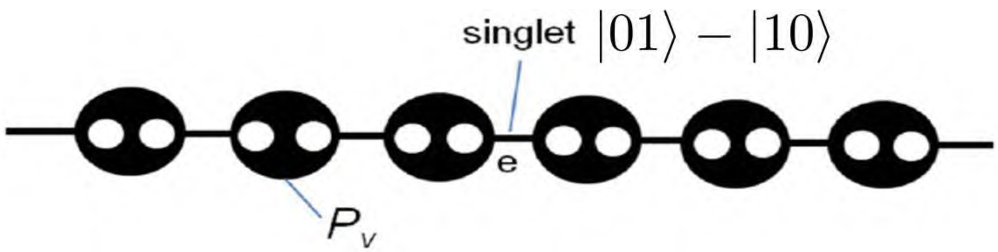
$$F_y \sim |S_y = 1\rangle\langle S_y = 1| + |S_y = -1\rangle\langle S_y = -1| \sim |i,i\rangle\langle i,i| + |-i,-i\rangle\langle -i,-i|$$

$$F_z \sim |S_z = 1\rangle\langle S_z = 1| + |S_z = -1\rangle\langle S_z = -1| \sim |00\rangle\langle 00| + |11\rangle\langle 11|$$

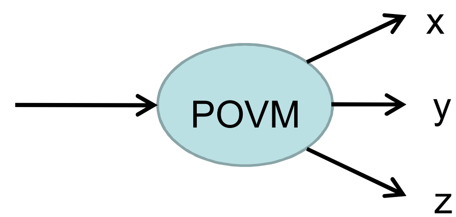
➔ *Outcome labeled by x,y, z:* $|\psi\rangle \to F_\alpha|\psi\rangle$ *projects to local two-level space*



POVM: 1D AKLT state ➔ cluster state



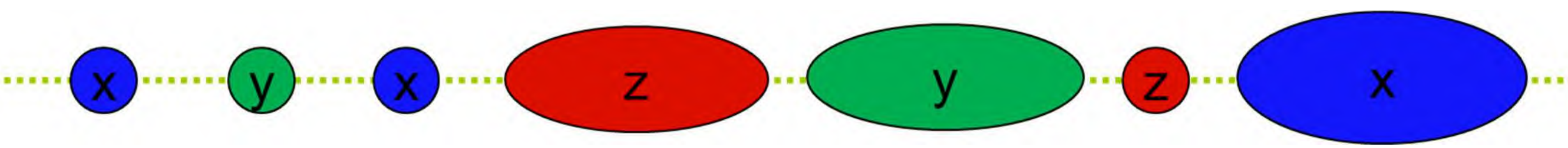
[*Wei, Affleck & Raussendorf '11, `12*]



- POVM: $F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$

  e.g. for the outcome (labeled x, y, z)

➔ the post-measurement state is an encoded 1D cluster state with graph:



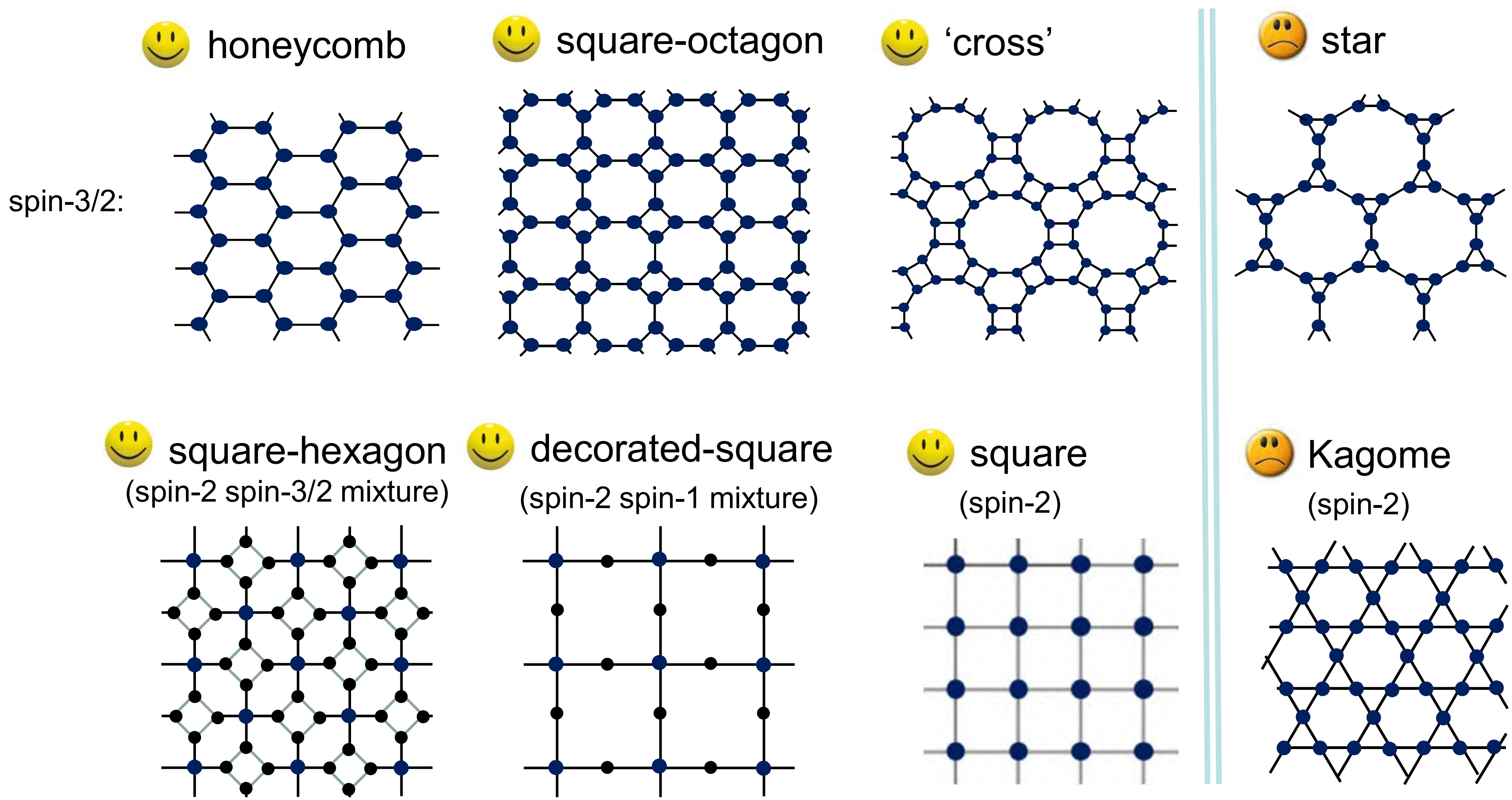
➔ 1 logical qubit = 1 domain = consecutive sites with same outcome

➔ This generalizes to some 2D AKLT states (with S ≤ 2)

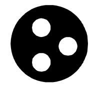
2D AKLT states for quantum computation?

- On various lattices

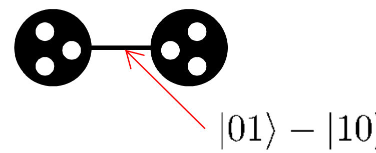
Miyake '11; ***Wei***, *Affleck & Raussendorf*, PRL '11
Wei, PRA '13, ***Wei***, *Haghnegahdar& Raussendorf, PRA '14*
Wei *& Raussendorf '15*



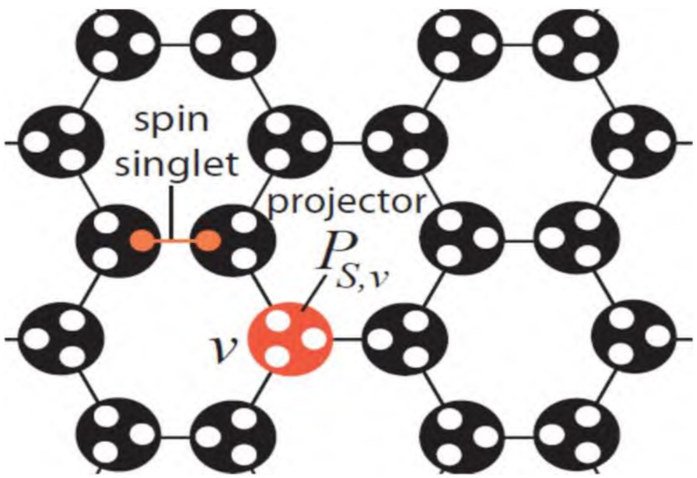
AKLT states on trivalent lattices

- Each site: three virtual qubits  ≡ spin 3/2 (in general: S= #nbr /2)

  ➔ physical spin = symmetric subspace of qubits

- Two virtual qubits on an edge form a singlet 

$$P = |3/2\rangle\langle 000| + |-3/2\rangle\langle 111| + |1/2\rangle\langle W| + |-1/2\rangle\langle \overline{W}|$$



$$|000\rangle \leftrightarrow \left|S = \frac{3}{2}, S_z = \frac{3}{2}\right\rangle$$

$$|111\rangle \leftrightarrow \left|\frac{3}{2}, -\frac{3}{2}\right\rangle$$

→ Effective qubit

$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow \left|\frac{3}{2}, \frac{1}{2}\right\rangle$$

$$|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow \left|\frac{3}{2}, -\frac{1}{2}\right\rangle$$

POVM for spin-3/2

$$F_z = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_z + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_z\right)$$

$$F_x = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_x + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_x\right)$$

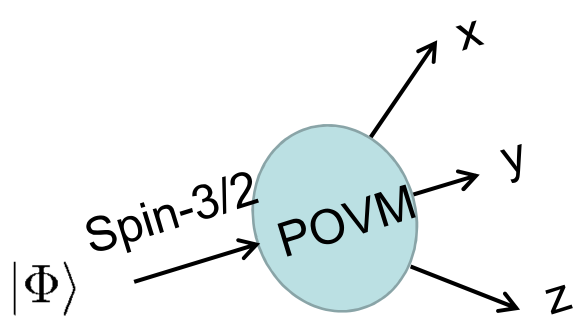
$$F_y = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_y + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_y\right)$$

[*Miyake* '11, ***Wei***,*Affleck & Raussendorf* '11]

Completeness:

$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$$

- POVM gives random outcome x, y and z at each site



$$|\Phi\rangle \longrightarrow F_{\alpha=x,y,\text{or }z}|\Phi\rangle$$

Tensor-network picture

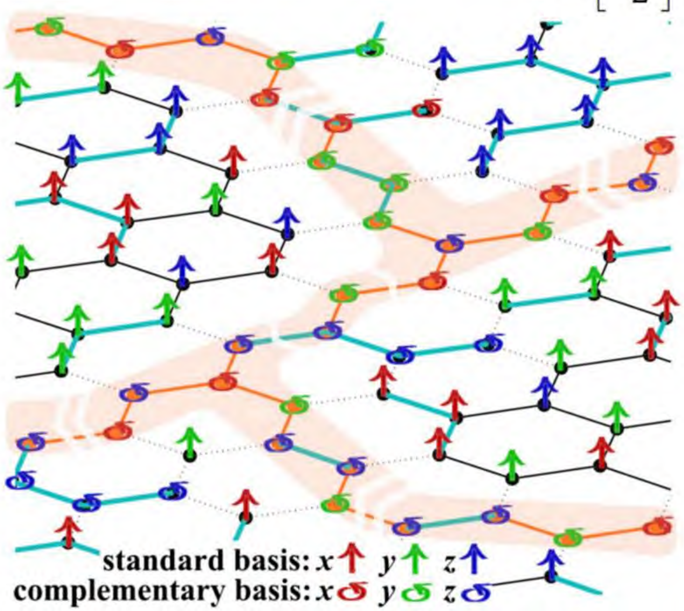
[*Miyake* '11]

- After POVM, each site effectively has two physical values

e.g. outcome z:

$$A_\perp\left[\frac{3^z}{2}\right] = -|0^z\rangle\langle 1^z| \otimes |0^z\rangle \qquad A_\top\left[\frac{3^z}{2}\right] = |0^z\rangle\langle 1^z| \otimes \langle 1^z|$$

$$A_\perp\left[-\frac{3^z}{2}\right] = |1^z\rangle\langle 0^z| \otimes |1^z\rangle \qquad A_\top\left[-\frac{3^z}{2}\right] = |1^z\rangle\langle 0^z| \otimes \langle 0^z|$$



- ✓ Further local measurements give rise to single- and two-qubit gates (in virtual bond space)
- ✓ Notion of computational backbone

Alternative: Reduction to 2D graph states

$$F_z = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_z + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_z\right)$$

$$F_x = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_x + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_x\right)$$

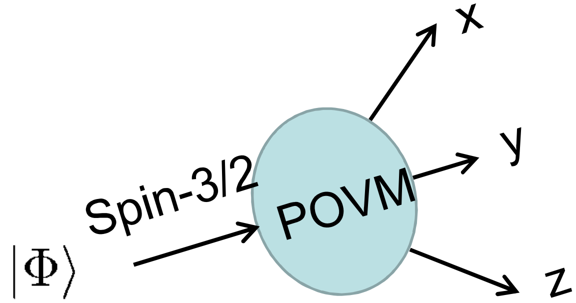
$$F_y = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_y + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_y\right)$$

[***Wei***, *Affleck & Raussendorf* '11
Miyake '11]

Completeness:

$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I$$

- POVM gives random outcome x, y and z at each site



$$|\Phi\rangle \longrightarrow F_{\alpha=x,y,\text{or } z}|\Phi\rangle$$

➔ Can show POVM on all sites converts AKLT to a graph state
(graph depends on random x, y and z outcomes)

Probability of POVM outcomes

- Measurement gives random outcomes, but what is the probability of a given set of outcomes?

$$P(\{\alpha(v)\}) \sim \langle \psi_{\mathrm{AKLT}} | \bigotimes_v F^\dagger_{\alpha(v)} F_{\alpha(v)} | \psi_{\mathrm{AKLT}} \rangle$$

- Can evaluate this using coherent states; alternatively use tensor product states

- Turns out to be a geometric object

$$P(\{\alpha(v)\}) \sim 2^{|V|-|\mathcal{E}|}$$

[***Wei***,*Affleck & Raussendorf*, PRL '11 & PRA '12]

Difference from 1D case: graph & percolation

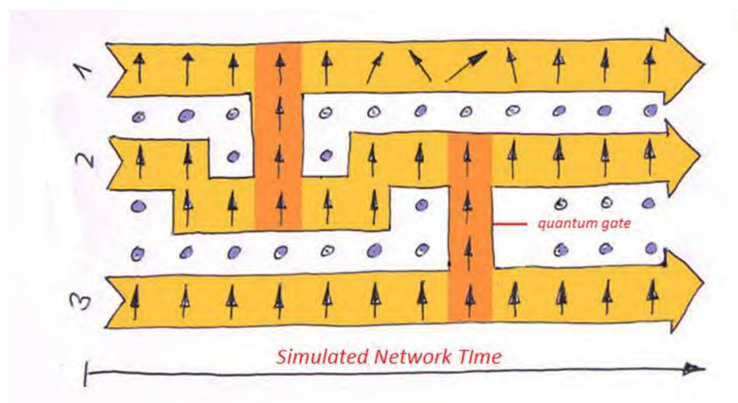
[***Wei***,*Affleck & Raussendorf* PRL'11]

1. What is the graph? which determines the graph state
➔ How to identify the graphs ?

   ✓ From these graphs we can 'cut out' the computational backbone

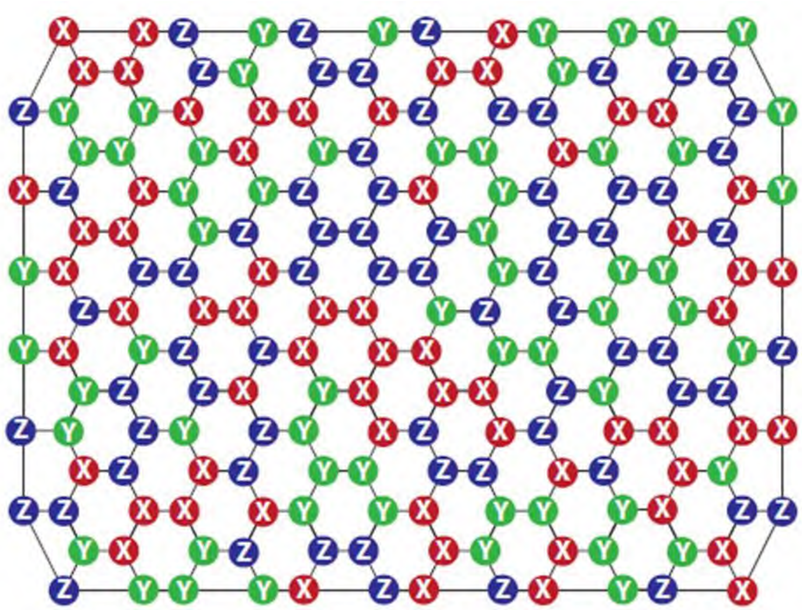
2. How do we know these graph states are universal?

   ✓ Percolation is the key

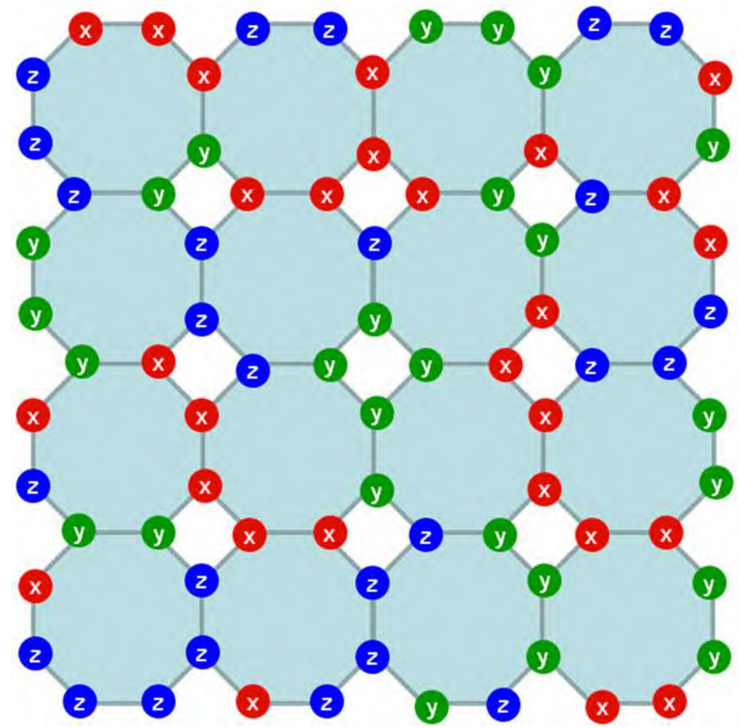


Recipe: construct graph for ‘the graph state’

- Examples: random POVM outcomes x, y, z



honeycomb

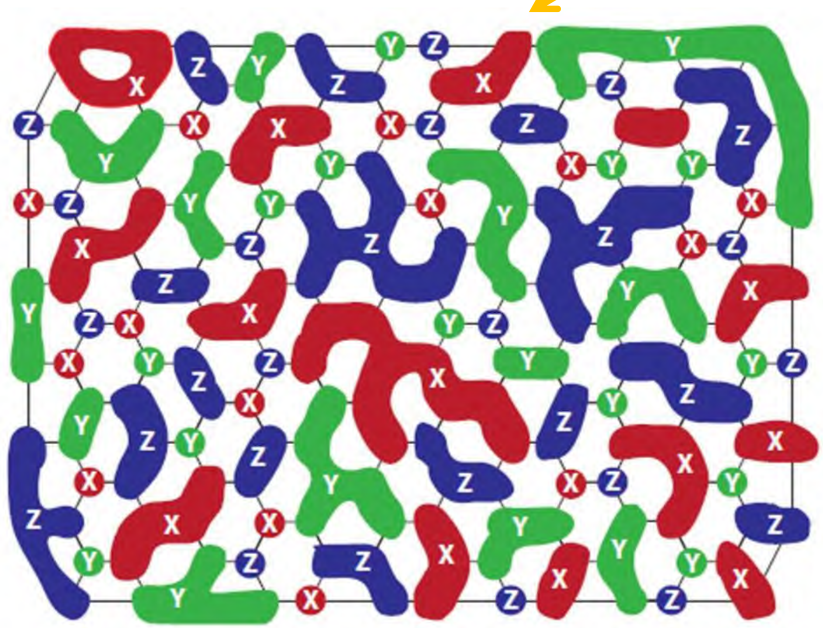


square octagon

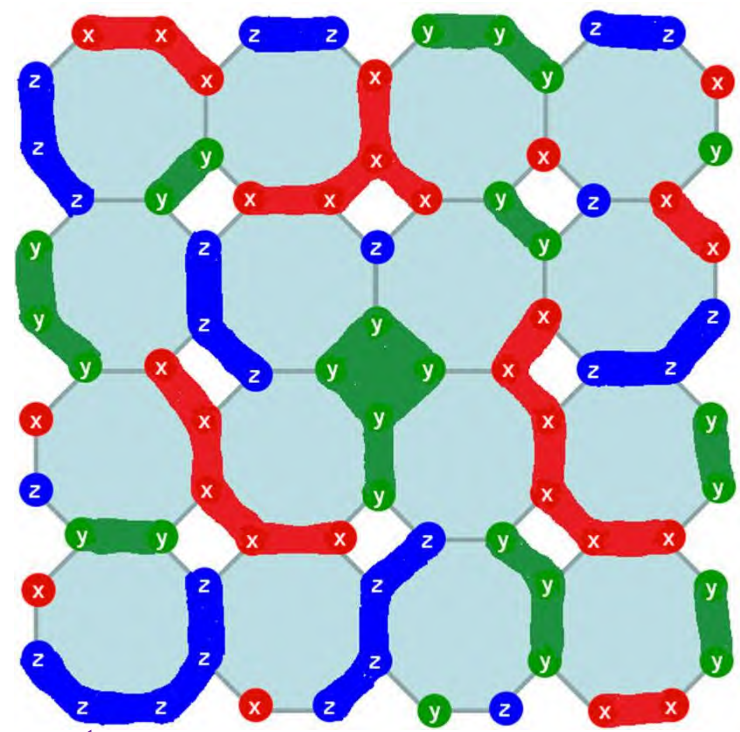
$$P(\{\alpha(v)\}) \sim 2^{|V|-|\mathcal{E}|}$$

Step 1: Merge sites to “domains”➔ vertices

- 1 domain = 1 logical qubit



honeycomb

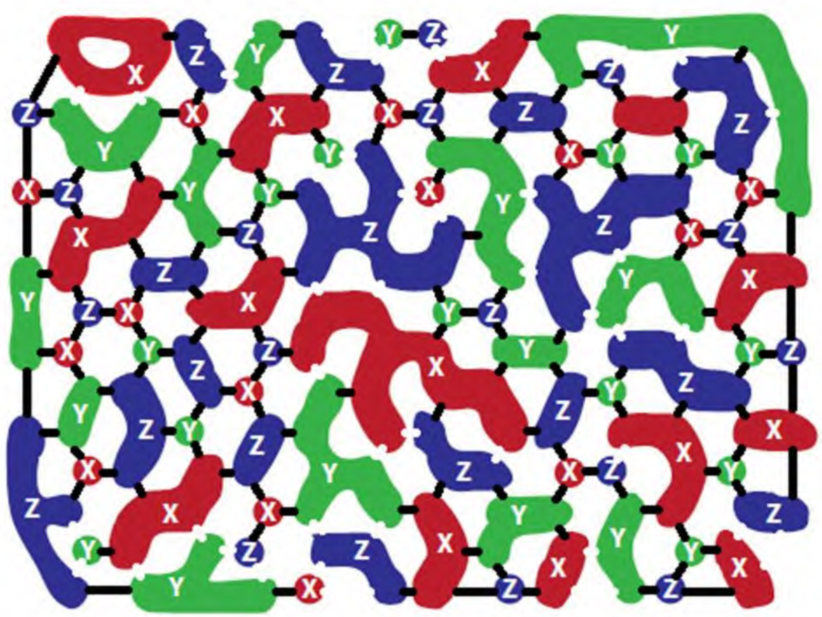


square octagon

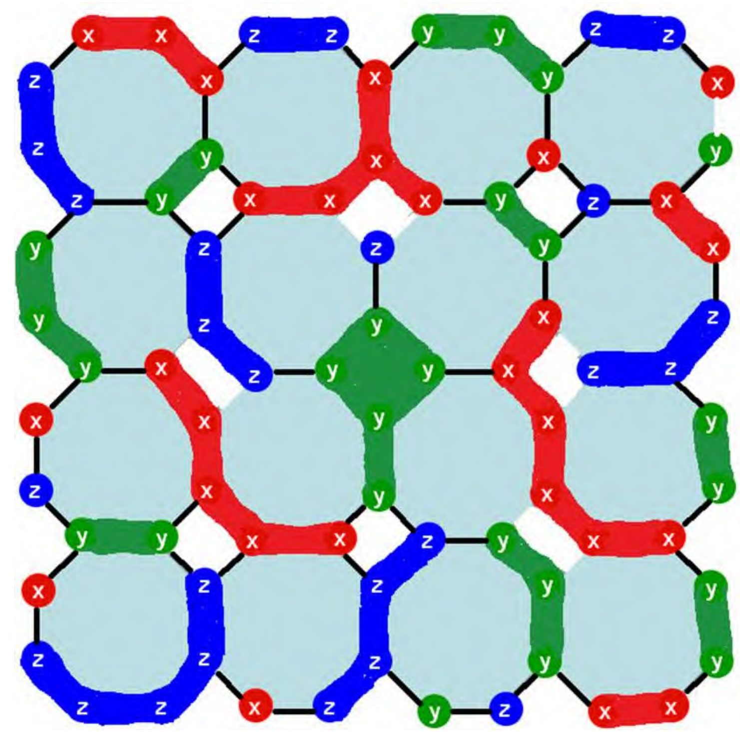
$|\uparrow\downarrow\uparrow\downarrow\uparrow\rangle$ $|\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$: encoding of a logical qubit

Step 2: edge correction between domains

- Even # edges = 0 edge, Odd # edges = 1 edge
  (due to $\sigma_z^2 = I$ in the C-Z gate)



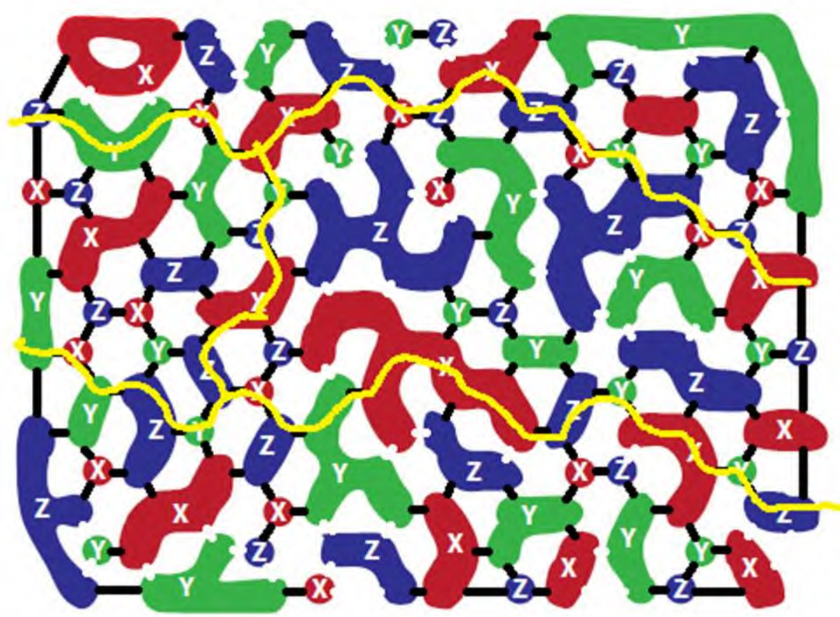
honeycomb

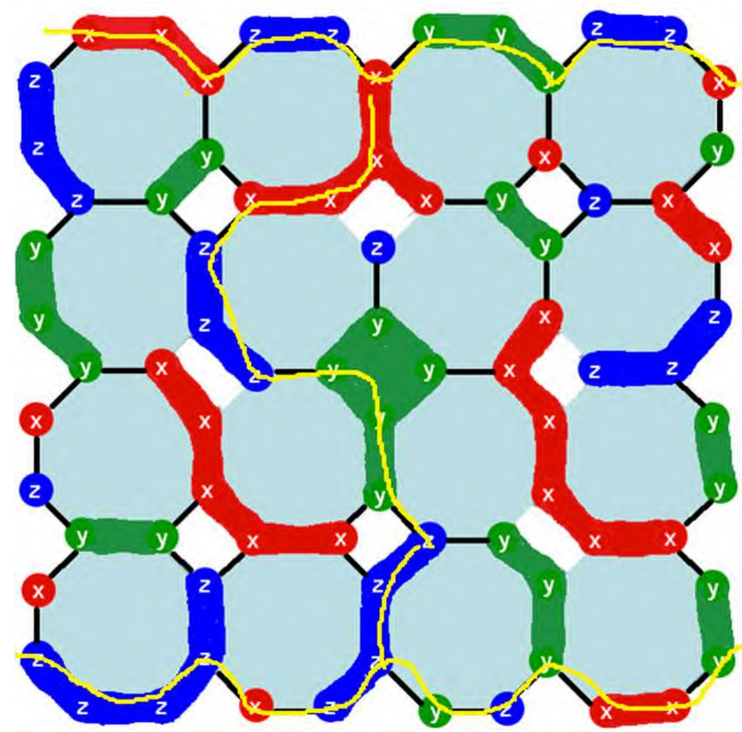


square octagon

Step 3: Check connections (percolation)

- Sufficient number of wires if graph is in supercritical phase (percolation)

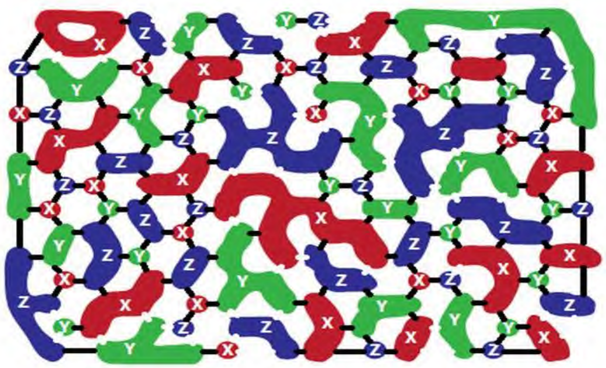


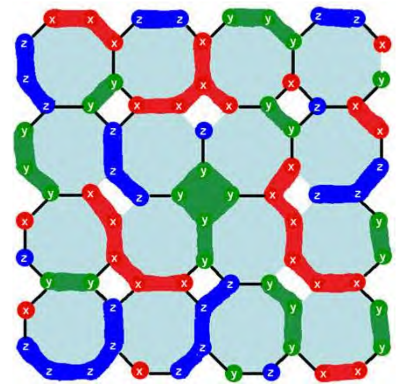


✓ Verified this for honeycomb, square octagon and cross lattices
➔ AKLT states on these are universal resources

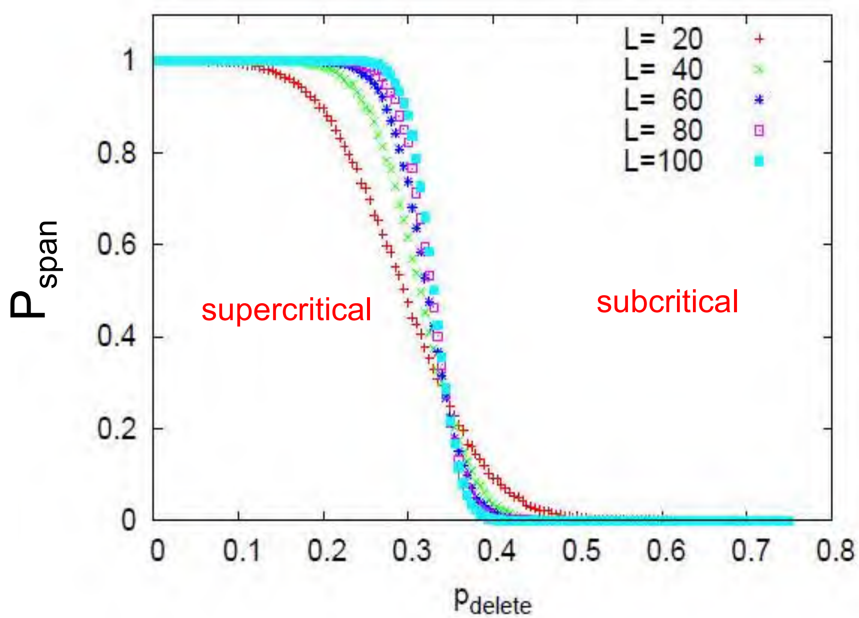
How robust is connectivity?

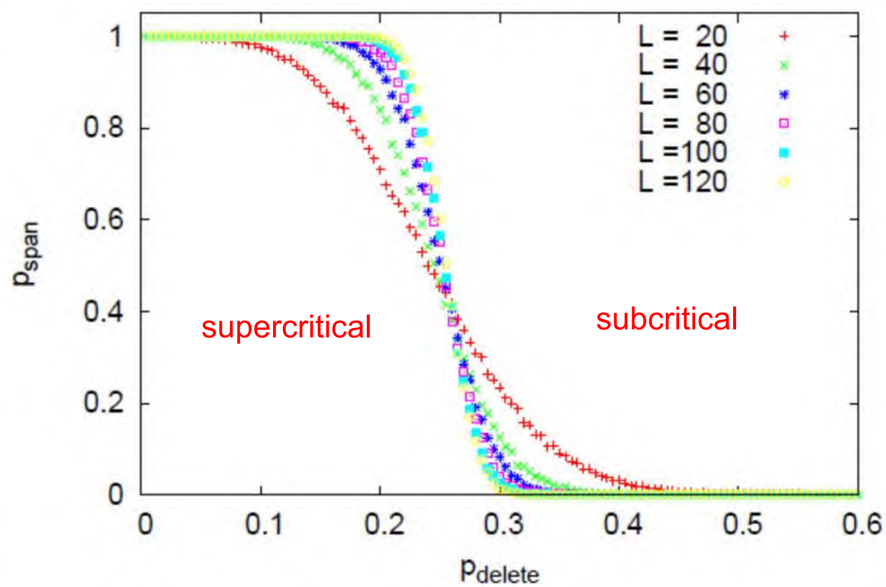
- Characterized by artificially removing domains to see when connectivity collapses (phase transition)



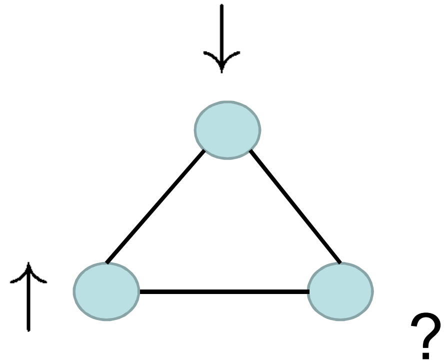


[*Wei* '13]

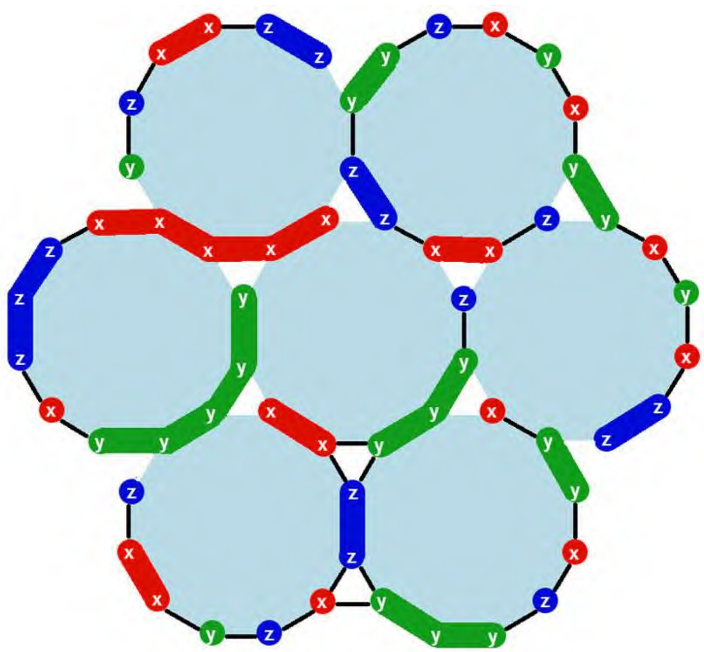




Frustration on star lattice

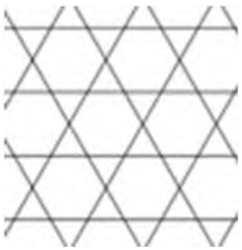


➔ Cannot have POVM outcome xxx, yyy or zzz on a triangle



➔ Consequences:

(1) Only 50% edges on triangles occupied
$< p_{th} \approx 0.5244$ of Kagome
→ disconnected graph



(2) Simulations confirmed: graphs not percolated

➔ AKLT on star likely NOT universal

Difficulty for spin-2

- Technical problem: trivial extension of POVM does NOT work!

$$F_z = \left|2\right\rangle\left\langle 2\right|_z + \left|-2\right\rangle\left\langle -2\right|_z$$
$$F_x = \left|2\right\rangle\left\langle 2\right|_x + \left|-2\right\rangle\left\langle -2\right|_x$$
$$F_y = \left|2\right\rangle\left\langle 2\right|_y + \left|-2\right\rangle\left\langle -2\right|_y$$

$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z \neq c \cdot I$$

➔ Leakage out of logical subspace (error)!

- Fortunately, can add elements K's to complete the identity

$$F_\alpha = \sqrt{\frac{2}{3}}\left(|S_\alpha = +2\rangle\langle S_\alpha = +2| + |S_\alpha = -2\rangle\langle S_\alpha = -2|\right)$$

[Wei, Haghnegahdar, Raussendorf '14]

$$K_\alpha = \sqrt{\frac{1}{3}}\left(|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\right) \qquad |\phi_\alpha^-\rangle \equiv \sqrt{\frac{1}{2}}\left(|S_\alpha = 2\rangle - |S_\alpha = -2\rangle\right)$$

$$\alpha = x, y, z$$

Completeness: $\sum_{\alpha=x,y,z} F_\alpha^\dagger F_\alpha + \sum_{\alpha=x,y,z} K_\alpha^\dagger K_\alpha = I$

Another difficulty: sample POVM outcomes

$$p(\{F,K\}) = \langle \text{AKLT}| \bigotimes_u F^\dagger_{\alpha(u)} F_{\alpha(u)} \bigotimes_v K^\dagger_{\beta(v)} K_{\beta(v)} |\text{AKLT}\rangle = ?$$ [*Wei, Raussendorf* '15]

- How to calculate such an *N*-body correlation function?

Lemma. If there exists a set Q (subset of D_K) such that $-\otimes_{\mu\in Q}(-1)^{|V_\mu|}X_\mu$ is in the stablizer group $\mathcal{S}(|G_0\rangle)$ of the state $|G_0\rangle$, then $p(\{F,K\})=0$. Otherwise,

$$p(\{F,K\}) = c\left(\frac{1}{2}\right)^{|\mathcal{E}|-|V|+2|J_K|-\dim\left(\ker(H)\right)},$$

where c is a constant.

- $|G_0\rangle \sim \bigotimes_v F_{\alpha(v)}|\text{AKLT}\rangle$
- D_K: set of domains having all sites POVM K
- $(H)_{\mu\nu} = 1$ if $\{\mathcal{K}_\mu, X_\nu\} = 0$, and $(H)_{\mu\nu} = 0$ otherwise

➔ Bottom line: can use Monte Carlo sampling

Local POVM: 5-level to (2 or 1)-level

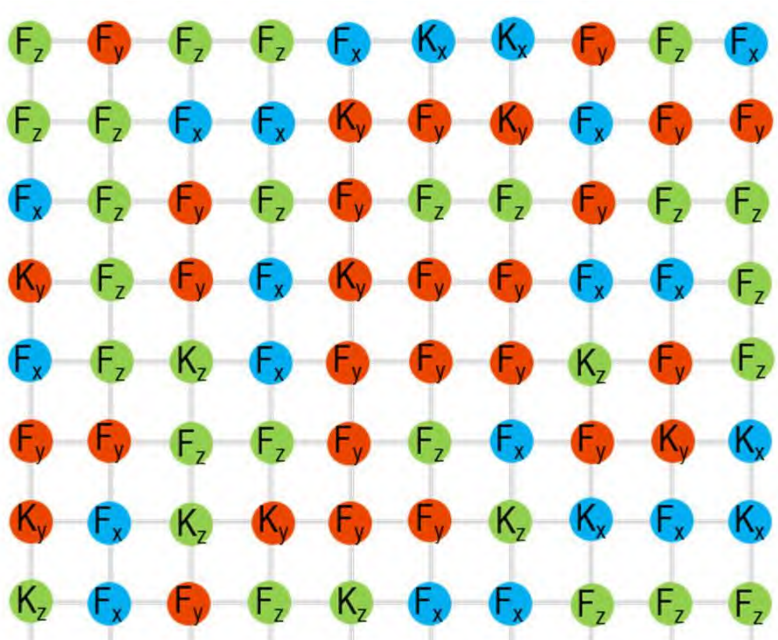
$$\begin{cases} F_\alpha = \sqrt{\frac{2}{3}}\left(|S_\alpha=+2\rangle\langle S_\alpha=+2| + |S_\alpha=-2\rangle\langle S_\alpha=-2|\right) \\ K_\alpha = \sqrt{\frac{1}{3}}\left(|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\right) = \frac{1}{\sqrt{2}}|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\,F_\alpha \\ \alpha = x, y, z \end{cases}$$

[*Wei*, *Haghnegahdar*, *Raussendorf* '14]

$$|\phi_\alpha^\pm\rangle \equiv \sqrt{\frac{1}{2}}\left(|S_\alpha=2\rangle \pm |S_\alpha=-2\rangle\right)$$

Completeness: $\sum_{\alpha=x,y,z} F_\alpha^\dagger F_\alpha + \sum_{\alpha=x,y,z} K_\alpha^\dagger K_\alpha = I$

- POVM gives random outcome F_x, F_y, F_z, K_x, K_y, K_z at each site



➔ Local action (depends on outcome):

$$|\Phi\rangle \longrightarrow F_{\alpha=x,y,\text{or } z}|\Phi\rangle$$

or

$$|\Phi\rangle \longrightarrow K_{\alpha=x,y,\text{ or z}}|\Phi\rangle$$

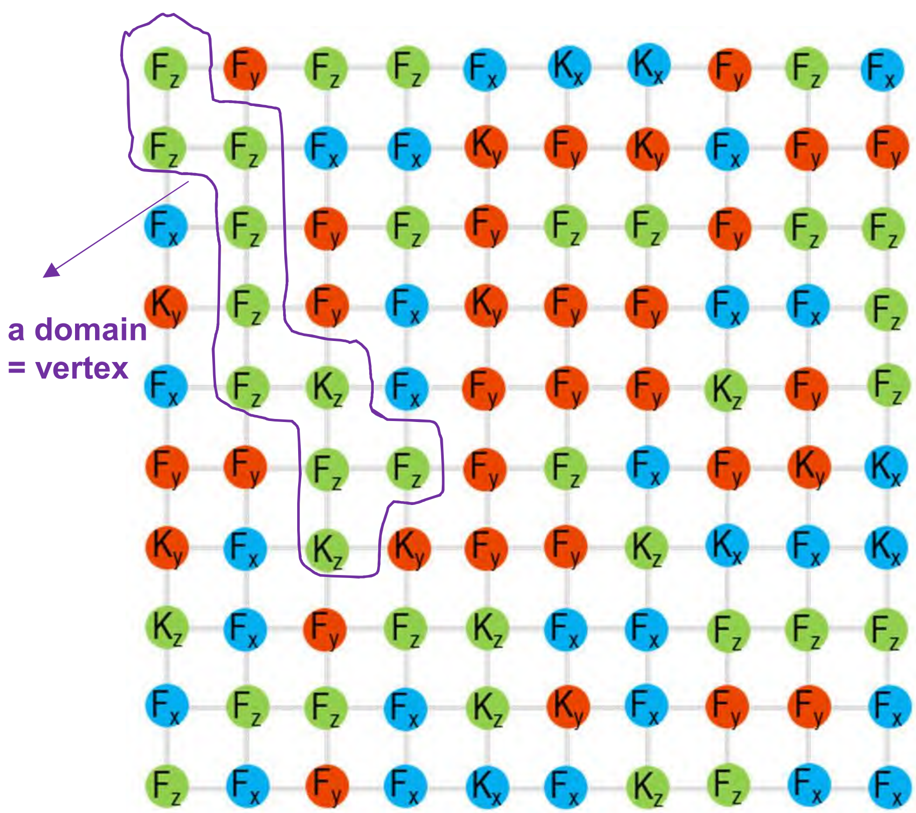
Post-POVM state: graph state

$$F_\alpha = \sqrt{\frac{2}{3}}\left(|S_\alpha=+2\rangle\langle S_\alpha=+2| + |S_\alpha=-2\rangle\langle S_\alpha=-2|\right)$$

[Wei, Haghnegahdar, Raussendorf '14]

$$K_\alpha = \sqrt{\frac{1}{3}}\left(|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\right) = \frac{1}{\sqrt{2}}|\phi_\alpha^-\rangle\langle\phi_\alpha^-|\,F_\alpha \qquad |\phi_\alpha^\pm\rangle \equiv \sqrt{\frac{1}{2}}\left(|S_\alpha=2\rangle \pm |S_\alpha=-2\rangle\right)$$

$$\alpha = x, y, z$$



- If *F* outcome on **all** sites
  ➔ a *planar* graph state

$$|G_0\rangle = \bigotimes_v F_{\alpha_v}^{(v)}|\mathrm{AKLT}\rangle$$

✓ **Vertex = a domain of sites with same color (x, y or z)**

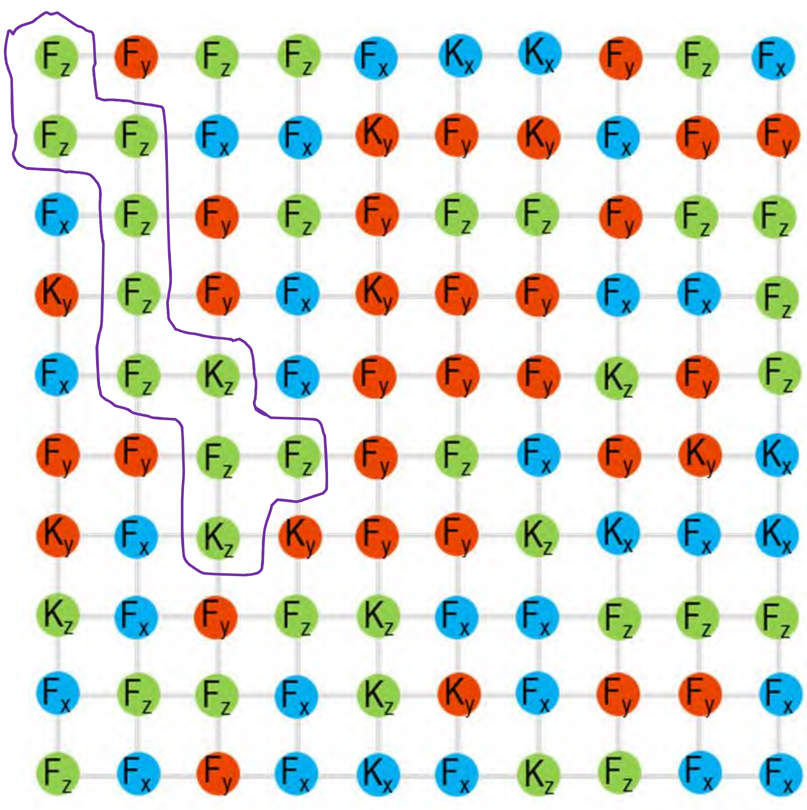
- *K* outcome = *F* followed by $\phi^\pm$ measurement (then *post-selecting* '-' result)

➔ Either
(1) shrinks domain size [trivial] or
(2) logical X or Y measurement [nontrivial]

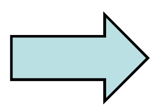
POVM ➔ Graph of the graph state

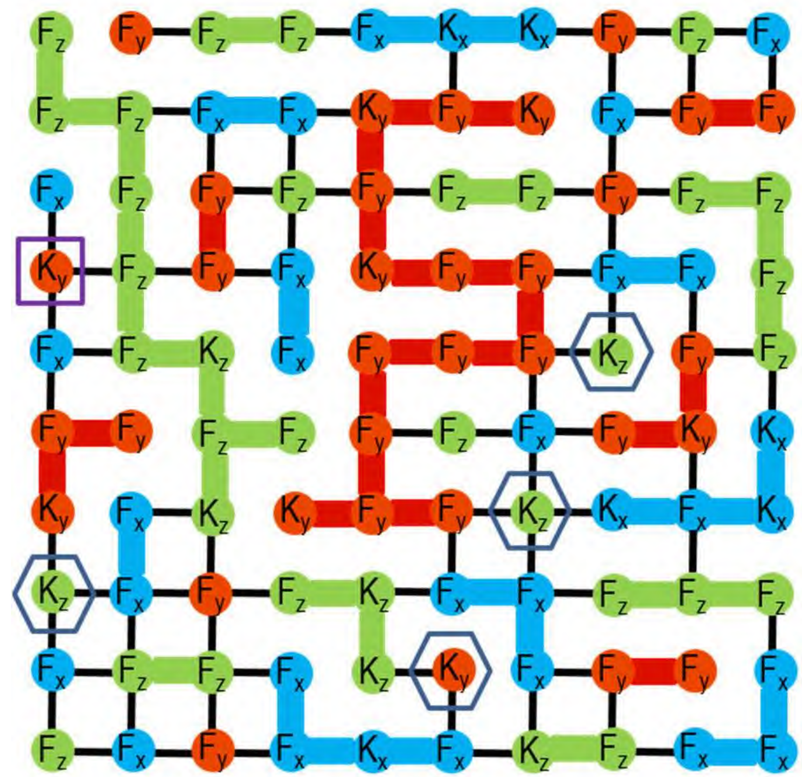
Vertex = domain = connected sites of same color
Edge = links between two domains (modulo 2)

$$|G_0\rangle = \bigotimes_v F_{\alpha_v}^{(v)} |\mathrm{AKLT}\rangle$$



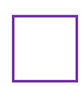
$$\bigotimes_v F_{\alpha_v}^{(v)}$$

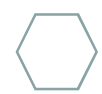




- Effect of nontrivial $K_\alpha = \frac{1}{\sqrt{2}} |\phi_\alpha^-\rangle\langle\phi_\alpha^-| F_\alpha$

  ➔ non-planar graph

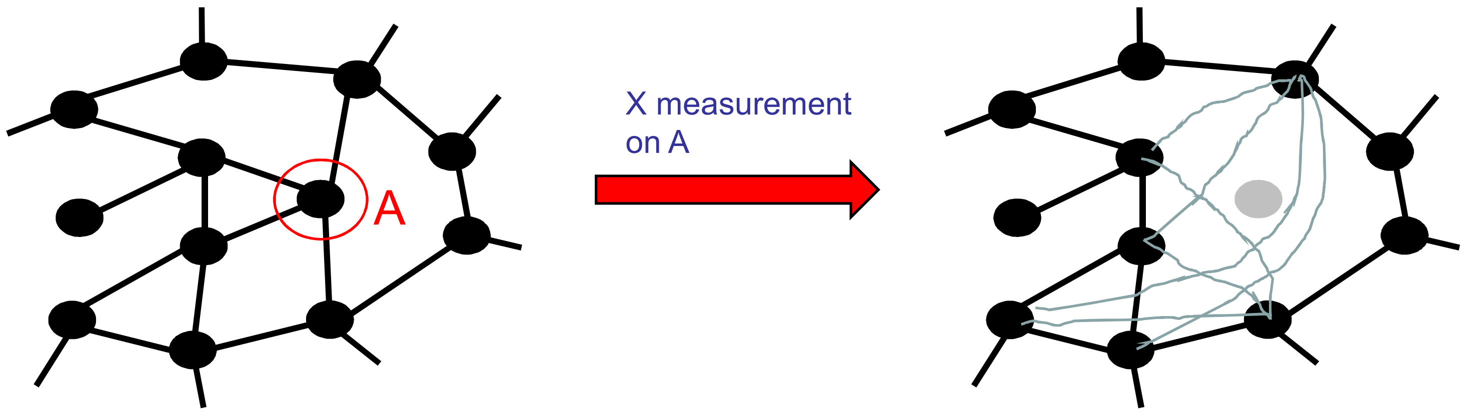
 :logical X measurement

 :logical Y measurement

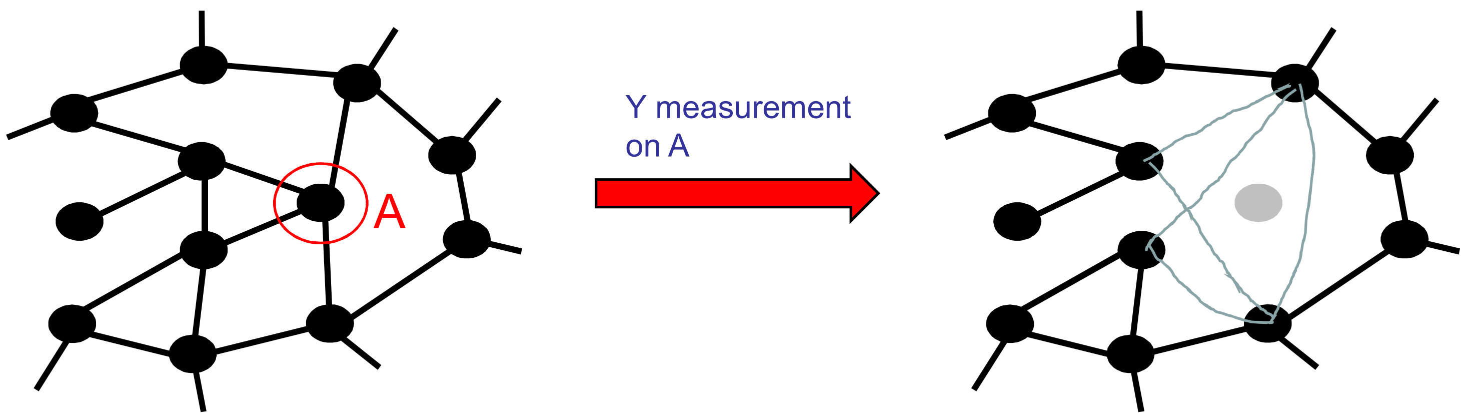
Non-planarity from X/Y measurement

[See e.g. *Hein et '06*]

X:



Y:

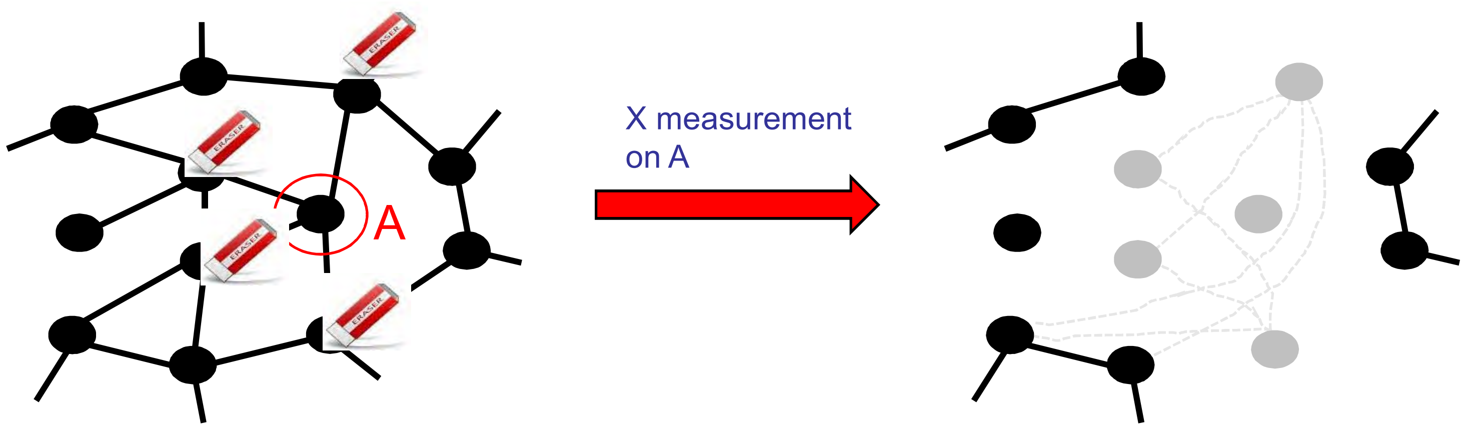


➔ Effect of X measurement is more complicated than Y measurement

Restore planarity: further measurement

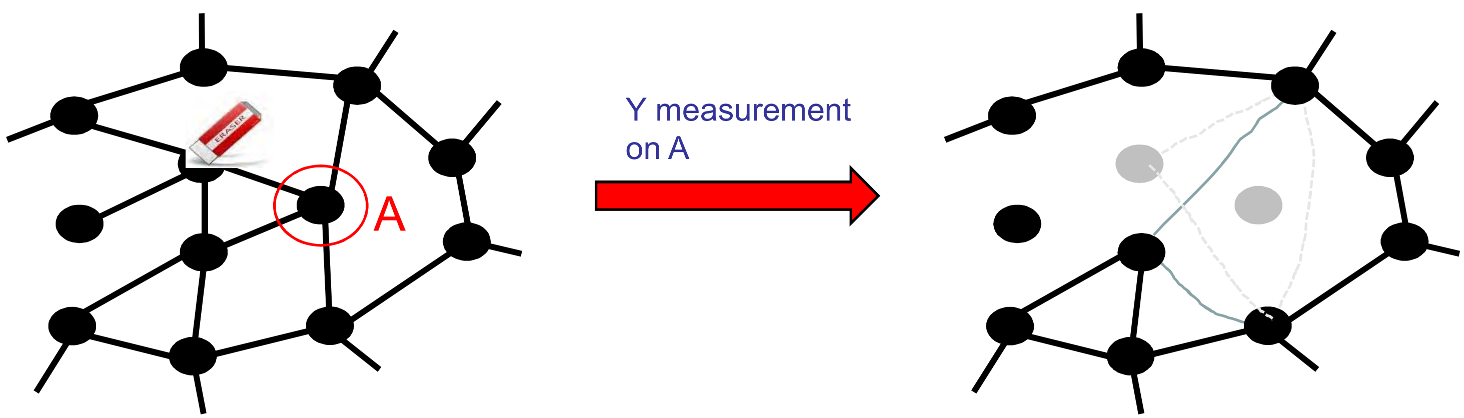
- Deal with non-planarity due to Pauli X measurement:
  remove all vertices surrounding that of X measurement (via Z measurement)

X:



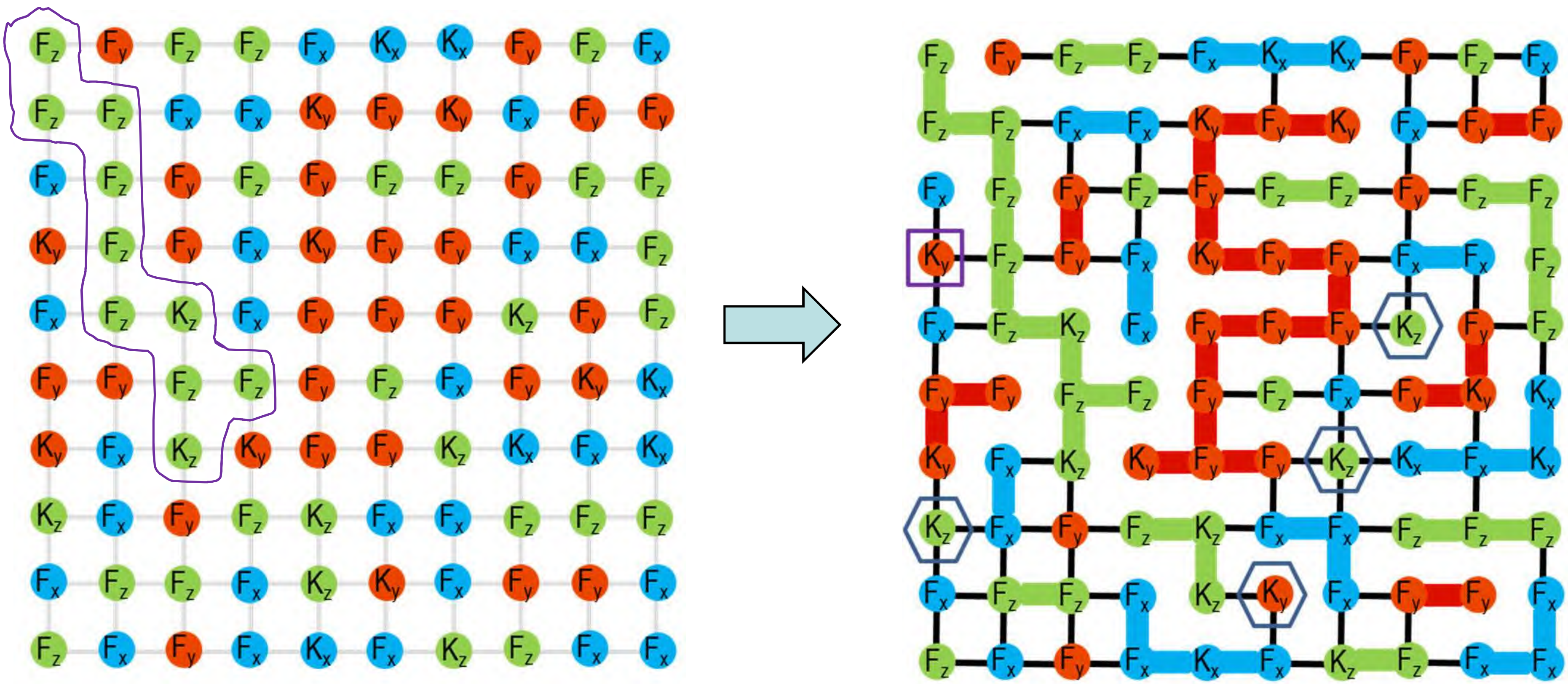
- Deal with non-planarity due to Pauli Y measurement:
  remove only subset of vertices surrounding that of Y measurement

Y:



POVM ➔ Graph of the graph state

Vertex = domain = connected sites of same color
Edge = links between two domains (modulo 2)



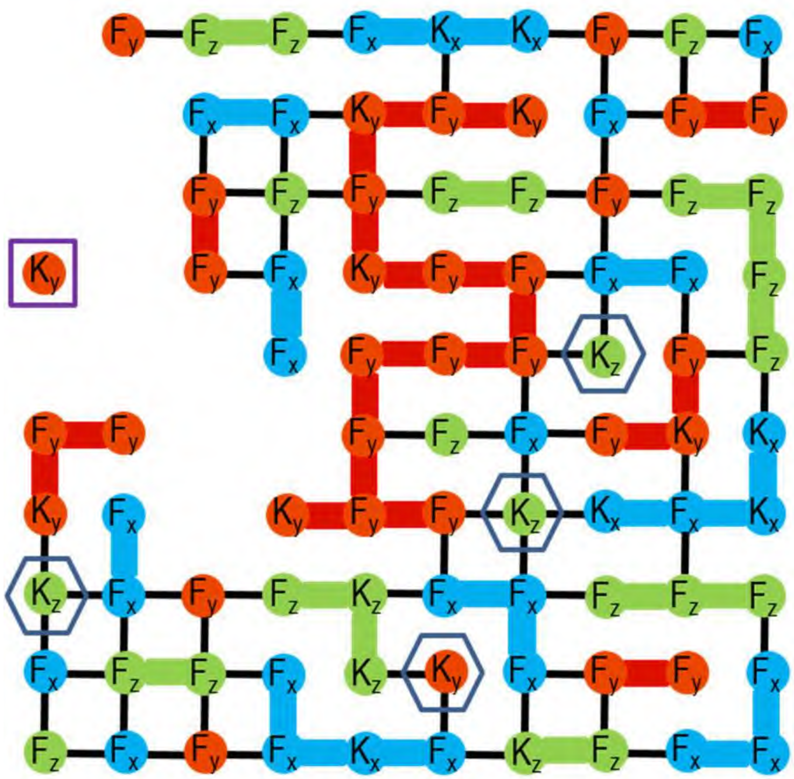
- Pauli X or Y measurement on planar graph state ➔ non-planar graph

□ :logical X measurement

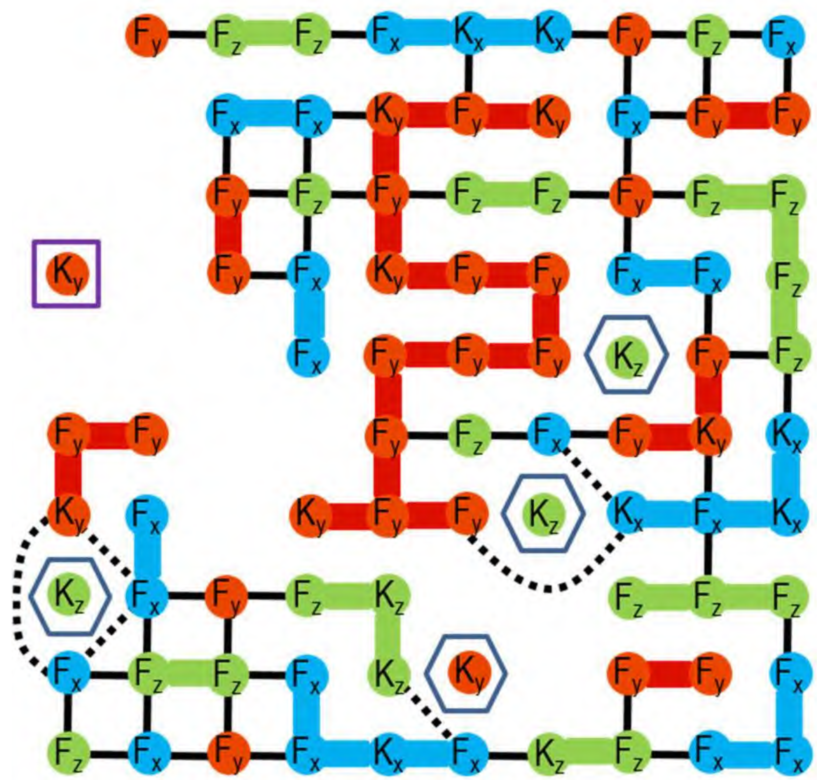
⬡ :logical Y measurement

Restore Planarity by
Another round of measurement

Deal with X measurement

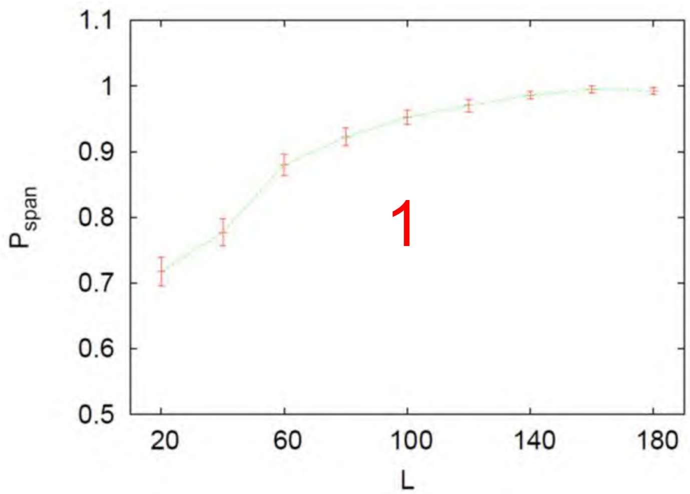


Deal with Y measurement

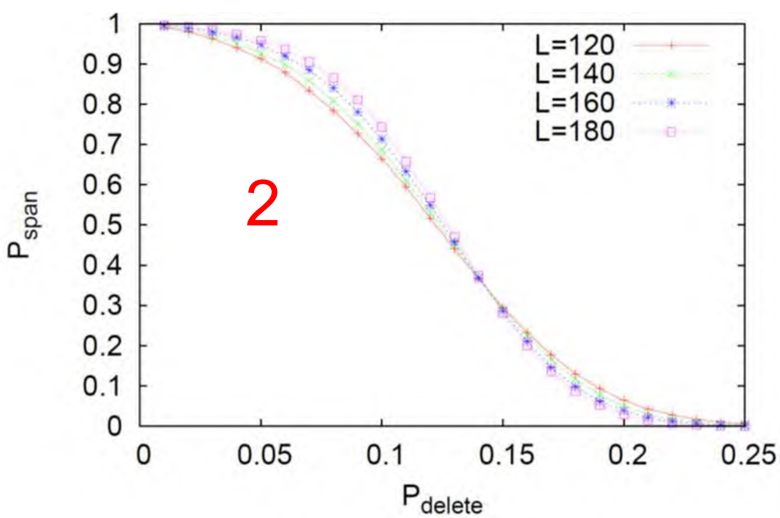


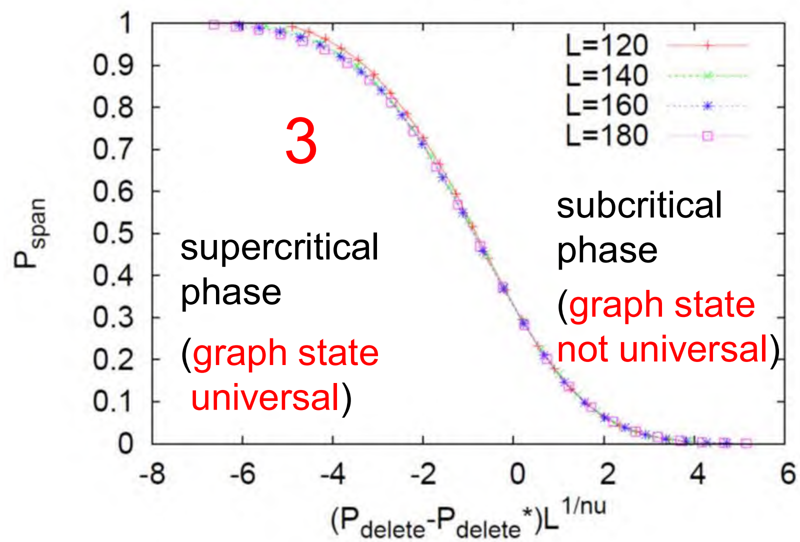
Examining percolation of typical graphs

(resulting from POVM and active logical Z measurement)



- ✓ 1. As system size N=L x L increases, exists a spanning cluster with high probability
- ✓ 2. Robustness of connectivity: finite percolation threshold (deleting each vertex with increasing probability)
- ✓ 3. Data collapse: verify that transition is continuous (critical exponent $\nu = 4/3$)





Spin-2 AKLT on square is universal for quantum computation

- Because the typical graph states (obtained from local measurement on AKLT) are universal ➔ hence AKLT itself is universal

- Difference from spin-3/2 on honeycomb: ***not all*** randomly assigned POVM outcomes are allowed
  ➔ weight formula is crucial

- If there are different spin magnitudes in the system, we can apply corresponding POVMs (for spin-1/2, we do nothing)

- **Emerging (partial) picture for AKLT family**:

  AKLT states involving spin-2 and other lower spin entities are universal if they reside on a 2D frustration-free lattice (e.g. w/o triangles) with any combination of spin-2, spin-3/2, spin-1 and spin-1/2

Outline

I. Introduction

II. AKLT models and states for universal quantum computation (in MBQC framework)

III. Nonzero gap for some 2D AKLT models

IV. Summary

AKLT Hamiltonians and gap(?)

- On honeycomb lattice

$$H = \sum_{\text{edge}\,\langle i,j\rangle} \hat{P}^{(S=3)}_{i,j} = \sum_{\text{edge}\,\langle i,j\rangle} \left[\vec{S}_i\cdot\vec{S}_j + \frac{116}{243}(\vec{S}_i\cdot\vec{S}_j)^2 + \frac{16}{243}(\vec{S}_i\cdot\vec{S}_j)^3 + \frac{55}{108}\right]$$

- Kennedy, Lieb & Tasaki (KLT) proved decay of correlation functions (including on square lattice): [KLT '88]

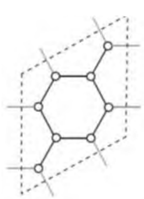
$$0 \leqslant (-1)^{|i-j|}\langle \boldsymbol{S}_i\cdot\boldsymbol{S}_j\rangle \leqslant C\exp(-|i-j|/\xi)$$

C, ξ const. >0

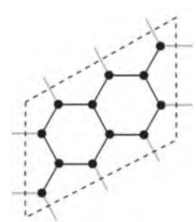
==> strongly suggests nonzero gap (no analytic proof after 30 yrs!)

==> they also showed ground state is unique

- Some example numerical values

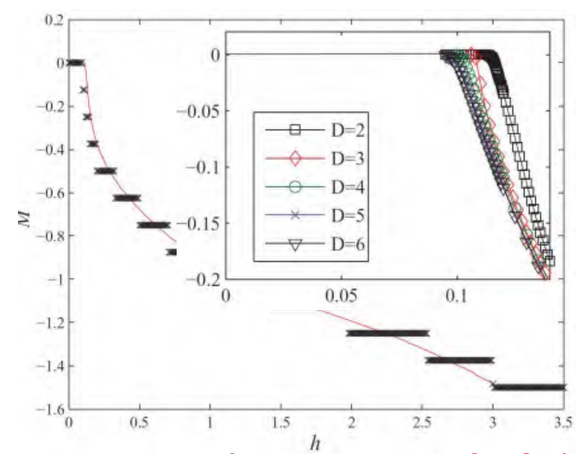


$\Delta_{2\times4} = 0.092029$



$\Delta_{3\times4} = 0.095345$

[Garcia-Saez,Murg & Wei '13]



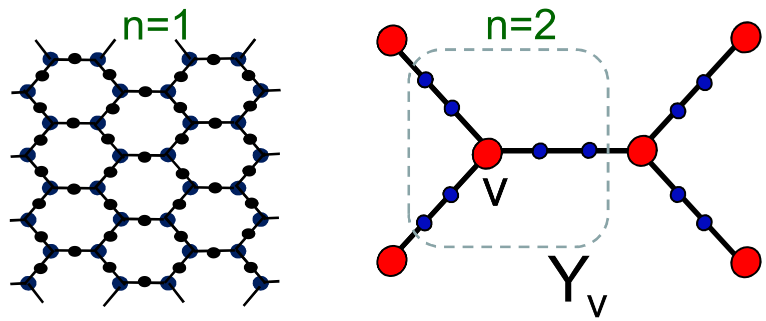
iPEPS tensor network ∞ system: Δ=0.10

[see also Vanderstraeten '15]

Progress in proving nonzero gap

- Decorating lattice Λ into $\Lambda^{(n)}$ by adding n spin-1 sites to each edge

$$H^{\text{AKLT}}_{\Lambda^{(n)}} = \sum_{e\in\mathcal{E}_{\Lambda^{(n)}}} P_e^{(z(e)/2)}$$



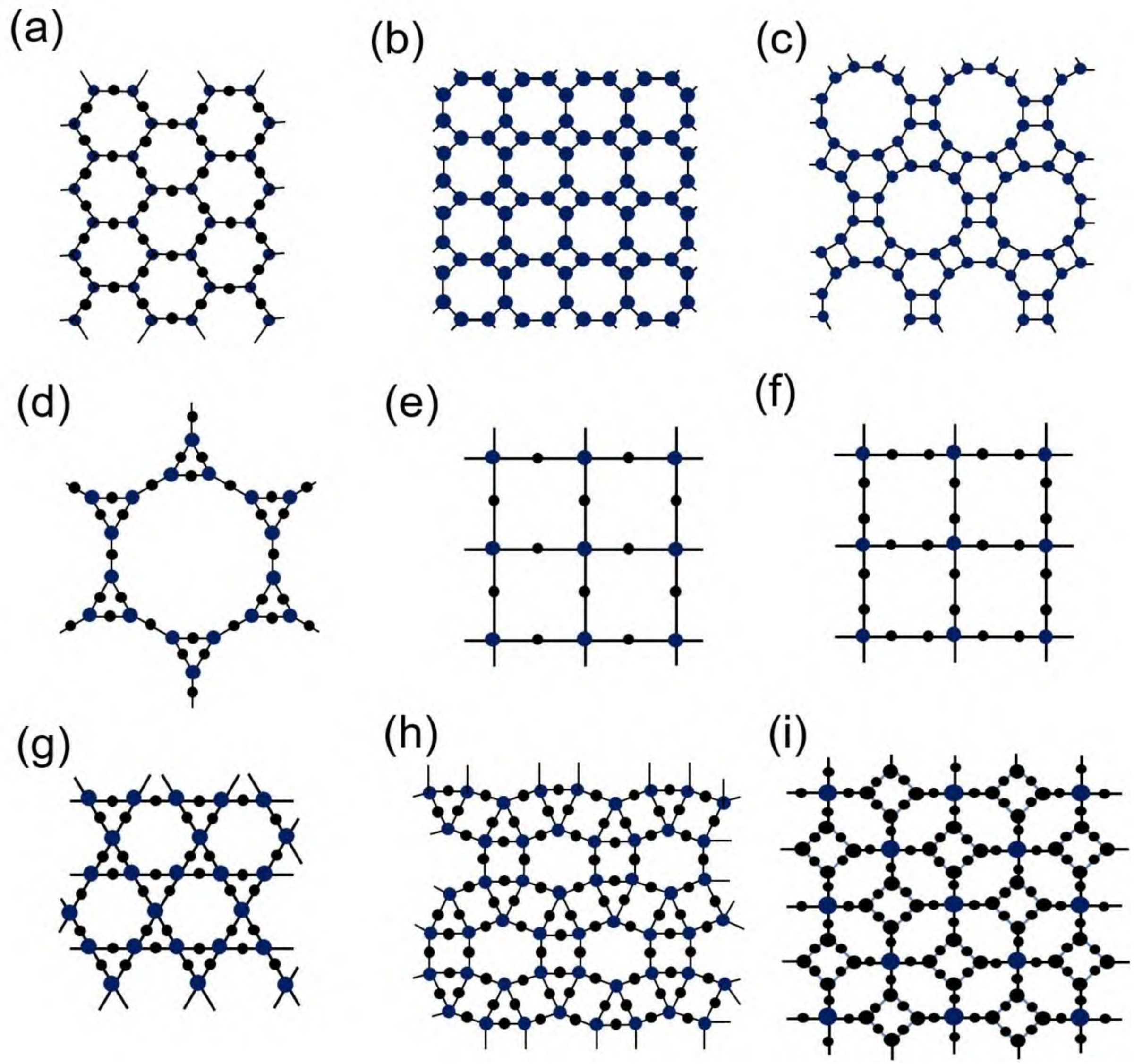
- Abdul-Rahman, Lemm, Luica, Nachtegaele & Young (ALLNY), arXiv:1901.09297

Theorem 2.2. *The spectral gap above the ground state of the AKLT model on the edge-decorated honeycomb lattice with $n \geq 3$ has a strictly positive lower bound uniformly for all finite volumes with periodic boundary conditions.*

✓ First analytic proof of nonzero gap for some 2D AKLT models ☺
(but not the undecorated honeycomb model)

❖ Nothing can be said about n=1 & 2 cases regarding spectral gap
Can we prove n=0 case?
What about other lattices? Decorated square lattices? Triangular?

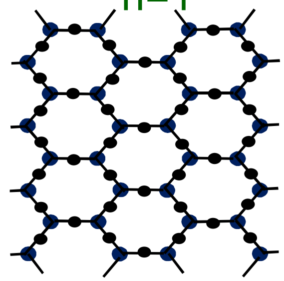
Other lattices

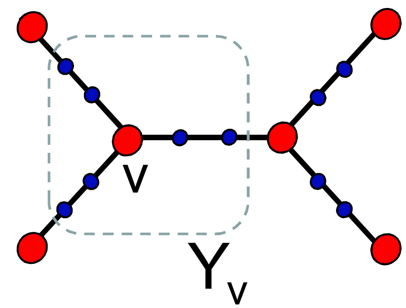


Ideas by ALLNY '19

- Decorating lattice Λ into $\Lambda^{(n)}$ by adding n spin-1 sites to each edge

$$H^{\mathrm{AKLT}}_{\Lambda^{(n)}} = \sum_{e \in \mathcal{E}_{\Lambda^{(n)}}} P_e^{(z(e)/2)}$$





- Also consider two modified H:

  [Abdul-Rahman et al. 1901.09297]

(1) $$H_Y \equiv \sum_{v \in \Lambda} h_v = \sum_{v \in \Lambda} \sum_{e \in \mathcal{E}_{Y_v}} P_e^{(z(e)/2)} \quad \Rightarrow H^{\mathrm{AKLT}}_{\Lambda^{(n)}} \le H_Y \le 2H^{\mathrm{AKLT}}_{\Lambda^{(n)}}$$

(2) $$\tilde{H}_{\Lambda^{(n)}} \equiv \sum_{v \in \Lambda} P_v, \quad P_v\text{: projection to range of } h_v$$

$$\Rightarrow \frac{\gamma_Y}{2} \tilde{H}_{\Lambda^{(n)}} \le H^{\mathrm{AKLT}}_{\Lambda^{(n)}} \le \|h_v\| \tilde{H}_{\Lambda^{(n)}}$$

γ_Y is the smallest nonzero eigenvalue of h_v

- They proved gap of (2) for n≥3 (hence lower bound on gap of AKLT models)

How to prove nonzero gap?

- Squaring H:

[Knabe '88, Fannes, Nachtergaele & Werner '92, …., Abdul-Rahman et al. 1901.09297]

$$(\tilde{H}_{\Lambda(n)})^2 = \tilde{H}_{\Lambda(n)} + \sum_{v\neq w}(P_vP_w + P_wP_v) \quad [\text{Note } P_v^2 = P_v]$$

$$\geq \tilde{H}_{\Lambda(n)} + \sum_{(v,w)\in\mathcal{E}_\Lambda}(P_vP_w + P_wP_v)$$

> Throwing out non-overlapping $P_v P_w \geq 0$

- Overlapping $P_v P_w$ can be non-positive.

But if we have: $P_vP_w + P_wP_v \geq -\eta(P_v + P_w)$

η>0 is smallest as possible

then we have

$$(\tilde{H}_{\Lambda(n)})^2 = \tilde{H}_{\Lambda(n)} - \sum_{(v,w)\in\mathcal{E}_\Lambda}(P_v + P_w)$$

$$\geq (1 - z\eta_n)\tilde{H}_{\Lambda(n)} = \gamma\tilde{H}_{\Lambda(n)} \quad \text{[z: coordination \#]}$$

- If $\gamma = (1-z\eta) > 0$, then there is a nonzero gap

Useful lemma to upper bound η

- [Fannes, Nachtergaele, Werner '92]:

  For two projectors E & F:

$$EF + FE \geq -\varepsilon(E+F) \qquad (\varepsilon \geq \eta \text{ in our case})$$

$$\varepsilon = ||EF - E \wedge F||$$

E ^ F : projection onto ran(E)= E$\mathcal{H}$ & ran(F)=F$\mathcal{H}$

  - Proof discussed later

- **(1-zε)** > 0 implies **γ** = **(1-zη)** > 0, then there is a nonzero gap
  - Want ε<1/z (z=3 for honeycomb)

[Abdul-Rahman et al. (ALLNY) 1901.09297]

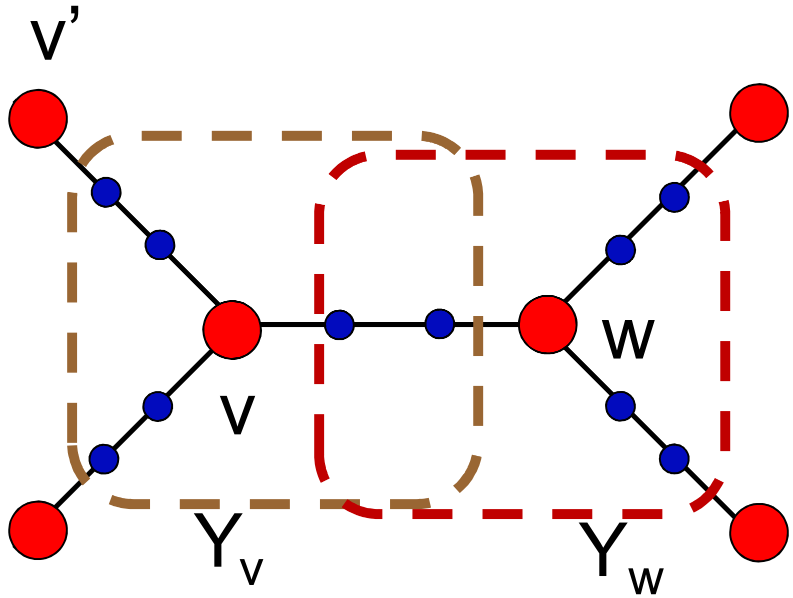
Proposition 2.1. *Let*

$$A_n = \frac{4}{3^n\left(1 - \frac{8(1+3^{-2n-1})}{3^n(1-3^{-2n})}\right)}.$$

Then, for all $n \geq 3$, the quantity ε_n defined in (2.7) satisfies

$$\varepsilon_n \leq A_n + A_n^2\left(1 + \frac{8(1+3^{-2n-1})^2}{3^n(1-3^{-2n})^2}\right) < 1/3. \tag{2.10}$$

Key point in upper bounding ε



- Use $\boldsymbol{E=I-P_v}$ (projection to local ground space supported on Y_v), $\boldsymbol{F=I-P_w}$ (projection to local ground space supported on Y_w) & $E \wedge F$ (projection to local ground space supported on $Y_v \cup Y_w$) in

$$\varepsilon = ||EF - E \wedge F|| = \sup \frac{|\langle \phi | EF - E \wedge F | \psi \rangle|}{||\phi||\,||\psi||}$$

$$\Rightarrow \epsilon = \sup \left\{ \frac{|\langle \phi, \psi \rangle|}{||\phi||\,||\psi||} \middle| \phi \in E\mathcal{H}, \psi \in F\mathcal{H}, \phi, \psi \perp E\mathcal{H} \cap F\mathcal{H} \right\}$$

- ALLNY 1901.09297 used tensor-network approaches (e.g. MPS) to give an upper bound on ε [No time for details here]

- n=1 case: EF - E ∧ F is operator roughly on size of 12 qubits, unfortunately ε≈0.4778 > 1/3; n=2 operator on ~ 20 qubits (not accessible); n=5 -> ~43.6 qubits

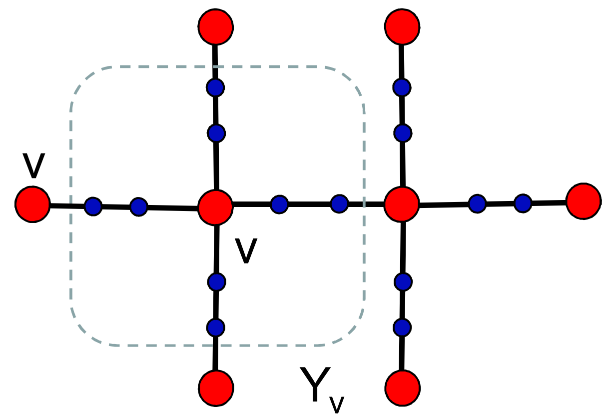
Our main results

[Pomata & Wei: 1905.01275]

- Analytically prove AKLT models on decorated square lattice (spin-2 + spin-1 decoration) are gapped for n ≥ 4

- Prove AKLT models on decorated mixed degree 3 & 4 lattices are gapped for n ≥ 4

  - Proof extends to lattices with same local structure: e.g. decorated square lattices gapped ↔ decorated kagome lattices gapped ↔ decorated diamond lattices gapped



- Reduce the effective size to obtain ε by exact diagonalization

n	deg. 3, e.g. honeycomb	deg. 4, e.g. square	mixed deg. 3&4	deg. 6
1	0.4778328889	0.5234369088	0.5001917602	0.6027622993
2	0.1183378500	0.1218467396	0.1200794787	0.1285855428
3	0.0384373228	0.0389033280	0.0386700977	
4	0.0124460198	0.0124961718	0.0124710706	
5	0.0041321990			

gapped

Useful lemma to upper bound η

- [Fannes, Nachtergaele, Werner '92]:

  For two projectors E & F:

$$EF + FE \geq -\varepsilon(E+F) \qquad (\varepsilon \geq \eta \ \text{ in our case})$$

$$\varepsilon = ||EF - E \wedge F||$$

E ∧ F : projection onto ran(E)= E$\mathcal{H}$ & ran(F)=F$\mathcal{H}$

  - Proof discussed later

- $(1-z\varepsilon) > 0$ implies $\gamma = (1-z\eta) > 0$, then there is a nonzero gap
  - Want $\varepsilon<1/z$ (z=3 for honeycomb)

[Abdul-Rahman et al. (ALLNY) 1901.09297]

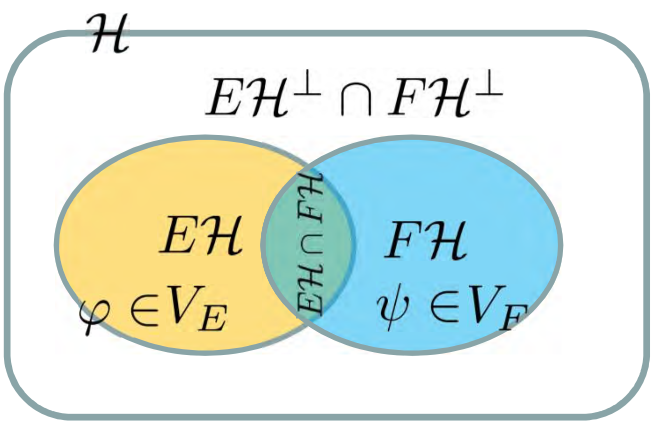
Proposition 2.1. *Let*

$$A_n = \frac{4}{3^n\left(1 - \frac{8(1+3^{-2n-1})}{3^n(1-3^{-2n})}\right)}.$$

Then, for all $n \geq 3$, the quantity ε_n defined in (2.7) satisfies

$$\varepsilon_n \leq A_n + A_n^2\left(1 + \frac{8(1+3^{-2n-1})^2}{3^n(1-3^{-2n})^2}\right) < 1/3. \tag{2.10}$$

Hilbert space and two projectors



E & F are projectors;
$V_E \equiv E\mathcal{H} \cap (E\mathcal{H} \cap F\mathcal{H})^\perp$
and similarly V_F do not include intersection

$$EF + FE \geq -\varepsilon(E+F)$$

$$\varepsilon = ||EF - E \wedge F||$$

- Consider eigenvalue equation α in [-1,1]:

$$(E+F)\Upsilon = (1-\alpha)\Upsilon$$

  - If α= -1, $\Upsilon \in E\mathcal{H} \cap F\mathcal{H}$
  - If α= 1, $\Upsilon \in E\mathcal{H}^\perp \cap F\mathcal{H}^\perp$

- If α in (-1,1), unique decomposition

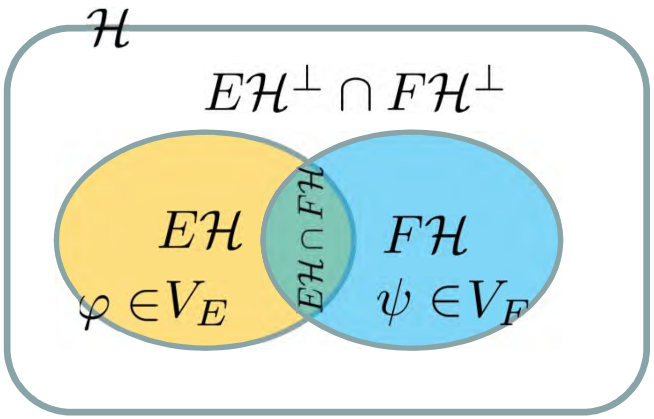
$$\Upsilon = \varphi + \psi \quad (\varphi \in V_E \,\&\, \psi \in V_F)$$

and $F\varphi = -\alpha\psi, \; E\psi = -\alpha\varphi$ (can prove this)

hence $(EF+FE)\Upsilon = -\alpha(1-\alpha)\Upsilon$

- So $\varepsilon = \max_{\text{eigen}\,\alpha \neq 1} \alpha$

Proving $\varepsilon = ||EF - E \wedge F||$



E & F are projectors; V_E and V_F do not include intersection

- E ∧ F projects onto $E\mathcal{H} \cap F\mathcal{H}$
- If α in (-1,1),

  $(E+F)\Upsilon = (1-\alpha)\Upsilon$

  has unique decomposition

  $\Upsilon = \varphi + \psi \quad F\varphi = -\alpha\psi, \; E\psi = -\alpha\varphi$

- Then $(EF - E \wedge F)\psi = EF\psi = -\alpha\varphi$

  (can show φ & ψ have same norm)

$||EF - E \wedge F|| \geq |\alpha|$

hence
$$\begin{aligned} \varepsilon &= \max_{\text{eigen } |\alpha| \neq 1} \alpha \\ &= ||EF - E \wedge F|| \end{aligned}$$

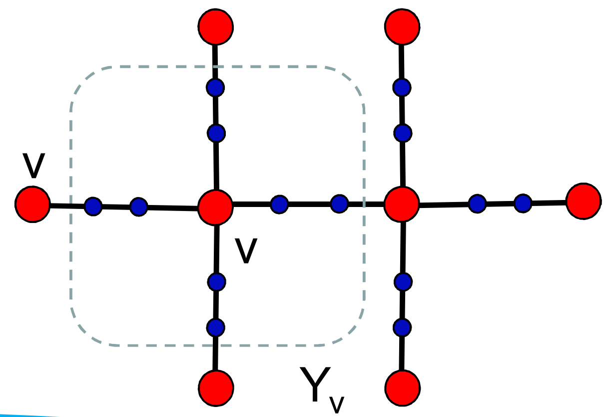
Our main results

[Pomata & Wei: 1905.01275]

- Analytically prove AKLT models on decorated square lattice (spin-2 + spin-1 decoration) are gapped for n ≥ 4

- Prove AKLT models on decorated mixed degree 3 & 4 lattices are gapped for n ≥ 4

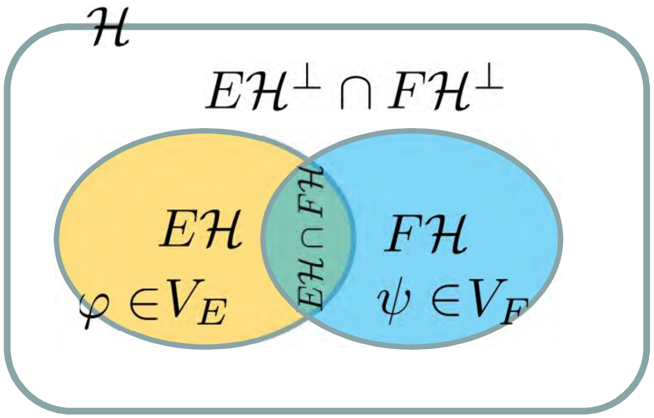
  - Proof extends to lattices with same local structure: e.g. decorated square lattices gapped ↔ decorated kagome lattices gapped



- Reduce the effective size to obtain ε by exact diagonalization

n	deg. 3, e.g. honeycomb	deg. 4, e.g. square	mixed deg. 3&4	deg. 6
1	0.4778328889	0.5234369088	0.5001917602	0.6027622993
2	0.1183378500	0.1218467396	0.1200794787	0.1285855428
3	0.0384373228	0.0389033280	0.0386700977	
4	0.0124460198	0.0124961718	0.0124710706	
5	0.0041321990			

Reducing Hilbert space size



E & F are projectors; V_E and V_F do not include intersection

- Consider a projector A satisfies:
  (1) $AE = EA = E$ (so $E\mathcal{H} \in A\mathcal{H}$)
  (2) $AF = FA$ (commute)

- If α in (-1,1)\{0}, $(E+F)\Upsilon = (1-\alpha)\Upsilon$
  then $A\Upsilon = \Upsilon$ (spectrum preserved)

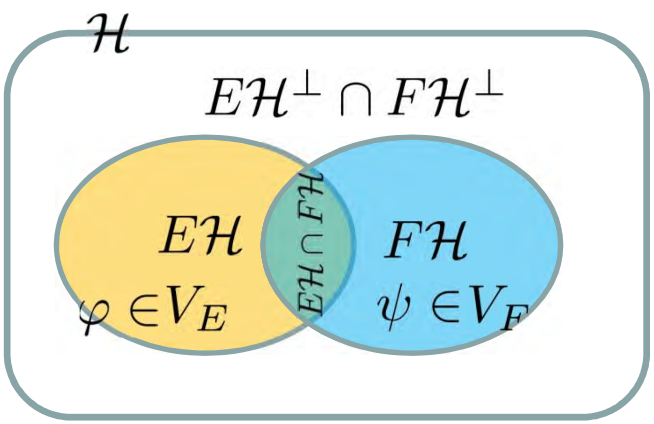
  $$FE\psi = -\alpha F\varphi = \alpha^2\psi$$
  $$(\alpha \neq 0) \rightarrow A\psi = \alpha^{-2}AFE\psi = \psi$$

- SVDecompose $A = U_A^\dagger U_A$ so $U_A : \mathcal{H} \rightarrow \mathcal{H}'$ (smaller space) $U_A U_A^\dagger = I_{\mathcal{H}'}$

  "Smaller projectors": $E' = U_A E U_A^\dagger$ $F' = U_A F U_A^\dagger$

  but preserve the norm $\varepsilon = ||EF - E \wedge F|| = ||E'F' - E' \wedge F'||$

Eigenvalue max α is preserved



E & F are projectors;
V_E and V_F do not include intersection

- Decompose $A = U_A^\dagger U_A, \quad U_A U_A^\dagger = I'$
  so $U_A : \mathcal{H} \to \mathcal{H}'$ ($E' = U_A E U_A^\dagger$)
- Consider

$$(E' + F')\Upsilon' = (1-\alpha)\Upsilon'$$
$$\Rightarrow \; U_A(E+F)U_A^\dagger \Upsilon' = (1-\alpha)\Upsilon'$$
$$\Rightarrow (E+F)U_A^\dagger \Upsilon' = (1-\alpha)U_A^\dagger \Upsilon'$$

==> spectrum (1-α) is preserved

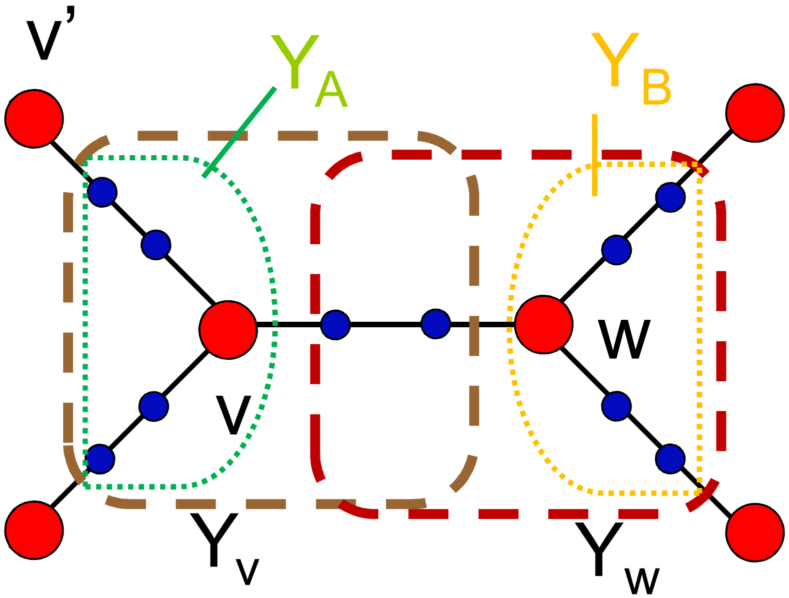
$$\varepsilon = ||EF - E \wedge F|| = ||E'F' - E' \wedge F'||$$

- Can further reduce dimension if exists projector B:
  (1) $BF = FB = F$ (2) $BE = EB$

then $B' \equiv U_A B U_A^\dagger = U_B^\dagger U_B$ ($E'' \equiv U_B E' U_B^\dagger$)

$$\varepsilon = ||EF - E \wedge F|| = ||E'F' - E' \wedge F'|| = ||E''F'' - E'' \wedge F''||$$

Numerical procedure



- Obtain $E=I-P_v$ via tensor Ψ of Y_v by SVD w.r.t. $\mathcal{H}_{\rm phys} \otimes \mathcal{H}_{\rm virt}$

$$\Psi = WsV^\dagger \Rightarrow E = WW^\dagger \equiv U_E^\dagger U_E$$

- Similarly for $F=I-P_w$, A and B
- Define $E' \equiv U_E'^\dagger U_E'$, $F' \equiv U_F'^\dagger U_F'$

  where $U_E' \equiv U_E U_A^\dagger$ $\quad U_F' \equiv U_F U_B^\dagger$

- Calculate smallest eigenvalue $1-\varepsilon$ of $E'+F'$
- If $\varepsilon < 1/z$, then the model is gapped
- Reduction: for a pair of vertices of degrees z & z': E+F acts on space of dimension $(z+1)(z'+1)3^{(z+z'-1)n}$, but E'+F' acts on reduced dimension $2^{(z+z'+2)}3^n$.

e.g. z=z'=3, n=5 --> reduction from 43.6 to 15.9 qubits

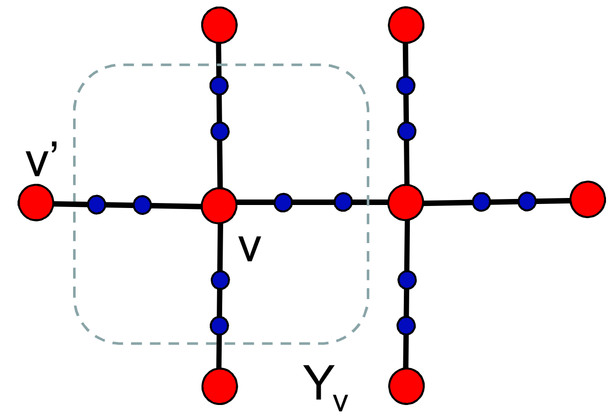
Improved lower bound on gap

- Consider re-arrangement of H:

$$H^{\rm AKLT}_{\Lambda^{(n)}} = \sum_{v\in\Lambda} h'_{Y;v}$$

$$h'_{Y;v} = \sum_{e\in\mathcal{E}_{Y_v}\setminus\mathcal{E}_v} \frac{1}{2} P_e^{(z(e)/2)} + \sum_{e\in\mathcal{E}_v} P_e^{(z(e)/2)}$$

$$\Rightarrow \Delta_Y \tilde{H}_{\Lambda^{(n)}} \le H^{\rm AKLT}_{\Lambda^{(n)}} \le \|h'_{Y;v}\| \tilde{H}_{\Lambda^{(n)}}$$



$\mathcal{E}_v$: the set of edges incident on v

$\Delta_Y(n)$: smallest nonzero eigenvalue of $h'_{Y;v}$

$$\Rightarrow \text{gap}(H^{\rm AKLT}_{\Lambda^{(n)}}) \ge \gamma(n) \equiv \Delta_Y(n)(1 - z\varepsilon_n),$$

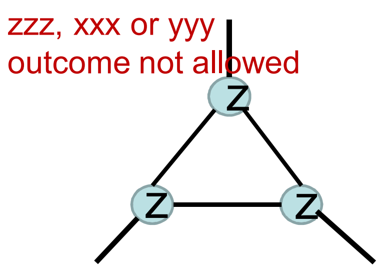
n	$\Delta_Y(n)$ for deg. 3	gap lower bound $\gamma(n)$	$\Delta_Y(n)$ for deg. 4	gap lower bound $\gamma(n)$
1	0.283484861		0.170646233	
2	0.239907874	0.154737328	0.197934811	0.101463966
3	0.207152231	0.183265099		

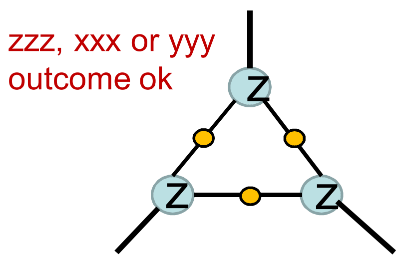
- Observation: naive extrapolation of lower bound from n=3 & n=2 linearly [1] to n=1: $\gamma(1)\approx 0.1262096$, [2] to n=0: $\gamma(0)\approx 0.097682$ cf. iPEPS: $\Delta=0.10$

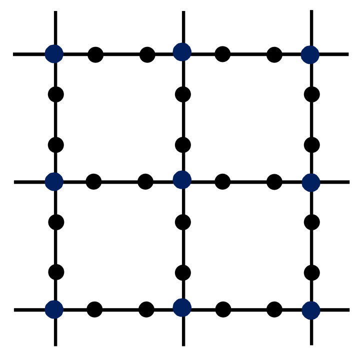
Discussions

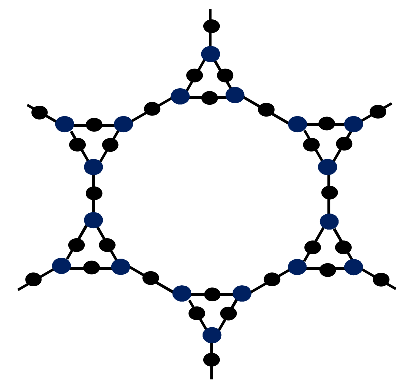
- Decoration of spin-1 sites make the AKLT state more likely to be universal
  - Short 1D AKLT wire between neighboring undecorated sites

- Decoration removes the frustration feature of measurement:









- Decoration weakens/removes Néel order: e.g. on 3D cubic lattice

[Parameswaran, Sondhi & Arovas '09]: AKLT state on cubic lattice is Néel ordered

- AKLT model gapless, but
  --> adding decoration make the decorated model gapped (at least for n=2 sites per edge)
  --> weakens tendency toward long-range order

Discussions: “deformation”

- Can consider deformed AKLT states and investigate phase diagrams

[Niggemann, Klümper& Zittartz ‘97,’00, Hieida,Okunishi& Akutsu ‘99, Darmawan, Brennen, Bartlett ‘12, Huang, Wagner, Wei’16, Huang,Pomata,Wei ‘18]

- Example on square lattice:

$$H(\vec{a}) \equiv \sum_{\langle i,j\rangle} D(\vec{a})_i^{-1} \otimes D(\vec{a})_j^{-1}\, h_{ij}^{(\mathrm{AKLT})}\, D(\vec{a})_i^{-1} \otimes D(\vec{a})_j^{-1}$$

- deformation:

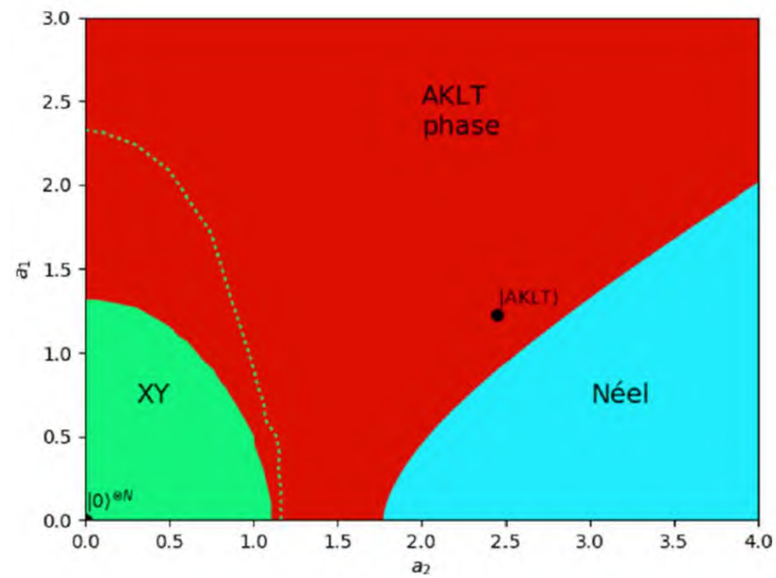
$$\begin{aligned} D(a_1,a_2) \;=\; & \frac{a_2}{\sqrt{6}}(|S_z=2\rangle\langle S_z=2| + |S_z=-2\rangle\langle S_z=-2|) \\ & + \frac{2a_1}{\sqrt{6}}(|S_z=1\rangle\langle S_z=1| + |S_z=-1\rangle\langle S_z=-1|) \\ & + |S_z=0\rangle\langle S_z=0| \end{aligned}$$

- ground state:

$$|\Psi(\vec{a})_{\mathrm{deformed}}\rangle \propto D(\vec{a})^{\otimes N}|\psi_{\mathrm{AKLT}}\rangle$$

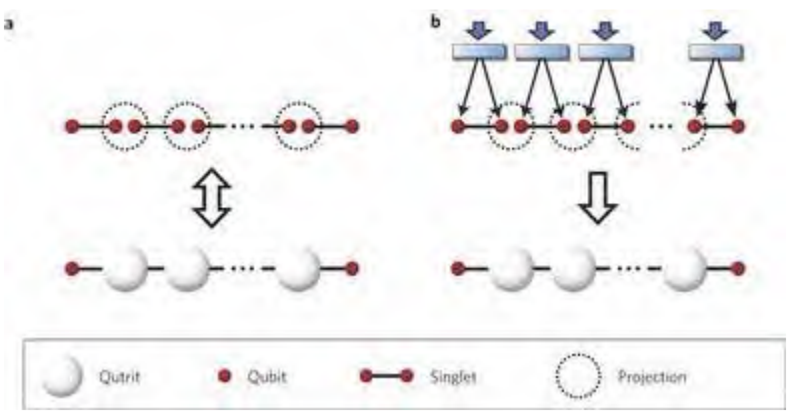
$$|\Psi_{\mathrm{AKLT}}\rangle = |\Psi(a_1=\sqrt{6}/2, a_2=\sqrt{6})\rangle$$

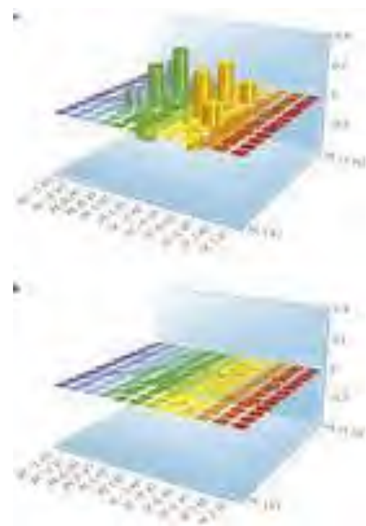
[Huang,Pomata,Wei ’18]

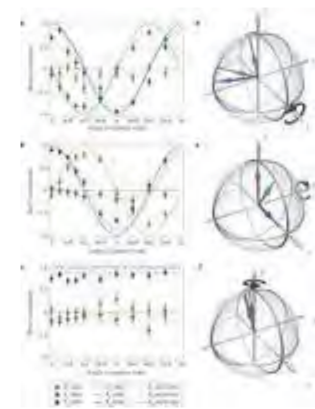


Discussion: Realizations of 1D AKLT state

- Resch's group: photonic implementation (Nature Phys 2011)

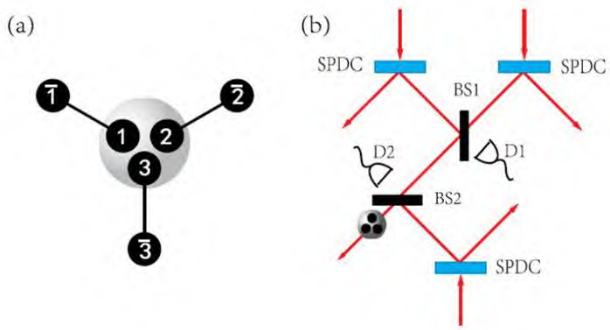


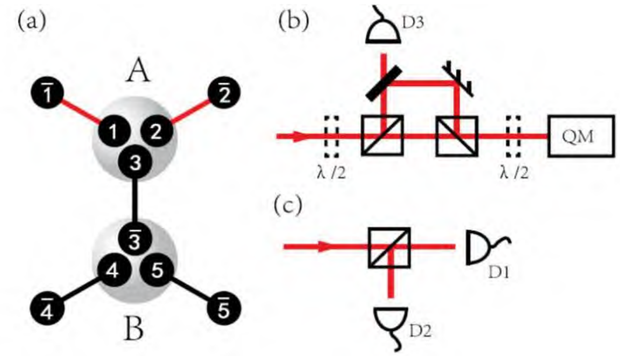




Discussion: creating 2D AKLT states?

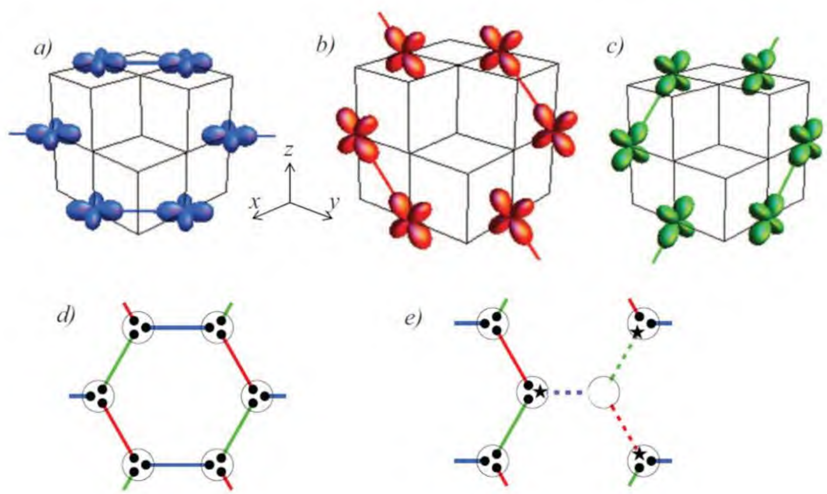
- Liu, Li and Gu [JOSA B 31, 2689 (2014)]





- Koch-Janusz, Khomskii & Sela [PRL 114, 247204 (2015)]

t_{2g} electrons in Mott insulator



Summary and open questions

- Discussed AKLT family of states for universal measurement-based QC
- Discussed how to establish nonzero gap for AKLT models on decorated lattices

- Universal MBQC using AKLT states with higher spins S>2?
- Using AKLT for QC but without the “preprocessing” POVM?
- What is essential symmetry that stabilizes the AKLT phase? Can the entire phase be universal resource?
- Proving nonzero gap for AKLT models on honeycomb and square lattices?

[see also Lemm, Sandvik & Yang 1904.01043 for gap on hexagonal chain]