TOPOLOGICAL STRING ENTANGLEMENT

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Veronika Hubeny, Roji Pius, MR [1905.09890]

MOTIVATION

- Holographic entanglement proposals are well understood in the regime of planar, strongly coupled field theories, which translates to the hierarchy:

$$\ell_{\mathrm{AdS}} \gg \ell_s \gg \ell_P$$

- Perturbation theory in inverse powers of coupling lead to geometric (though not extremal) surfaces

$$g^2_{YM}\, N \sim \left(\frac{\ell_{\mathrm{AdS}}}{\ell_s}\right)^4$$

- Quantum gravitational effects however lead to quantum entanglement across the bulk extremal surface

$$N \sim \left(\frac{\ell_{\mathrm{AdS}}}{\ell_P}\right)^4$$

- Effectively, our understanding of how between geometry arises from entanglement in field theory is limited to a small corner of parameter space. We should be asking for more…

MOTIVATION

- Explore the connection between entanglement and geometry in classical string theory, when $\ell_{\rm AdS} \sim \ell_s \gg \ell_P$.

- For now, focus on topological context where lack of dynamics is compensated by the intrinsic tractability of the theories.

- Open/closed topological string duality provides a concrete context where we can explore what is the topological closed string quantity which captures Chern-Simons topological entanglement .

- More precisely, seek a topological analog of generalized gravitational entropy, and ask how the replica construction in Chern-Simons ports across to the closed topological string.

- We'll argue for a natural replica construction in the closed topological string and the notion of an *entangling brane* in topological string theory.

Act I

The topological open/closed string duality

OPEN/CLOSED TOPOLOGICAL STRING DUALITY

- The open/closed topological string duality relates the worldvolume theory of topological D-branes with a closed topological string theory.
- Worldvolume theory of A-model D-branes on the *deformed conifold:* Chern-Simons gauge theory on $\mathbf{S}^3$ with $SU(N)_k$.
- Dual closed string theory is the topological A-model on the *resolved conifold* geometry.

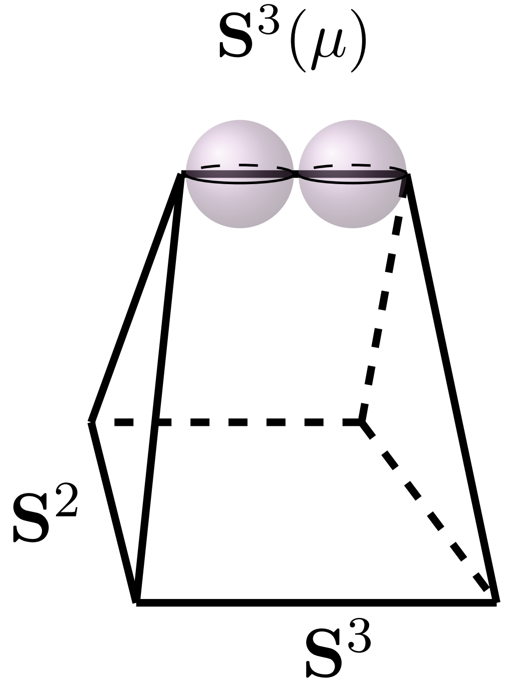
$$\mathcal{Z}_c(\mathcal{R}) = \mathcal{Z}_{\rm CS}(\mathbf{S}^3)$$

$$g_s = \frac{2\pi}{k+N}\,, \qquad t = i\,\frac{2\pi\,N}{k+N} = i\,\lambda\,.$$

Gopakumar, Vafa '98
Ooguri, Vafa '99, '02

- The A-model string is independent of the complex structure of the target space but depends on Kahler structure. It computes topological invariants associated with holomorphic worldsheets into the target CY 3-fold.

TARGET SPACE GEOMETRIES



$T^*\mathbf{S}^3$ Deformed conifold

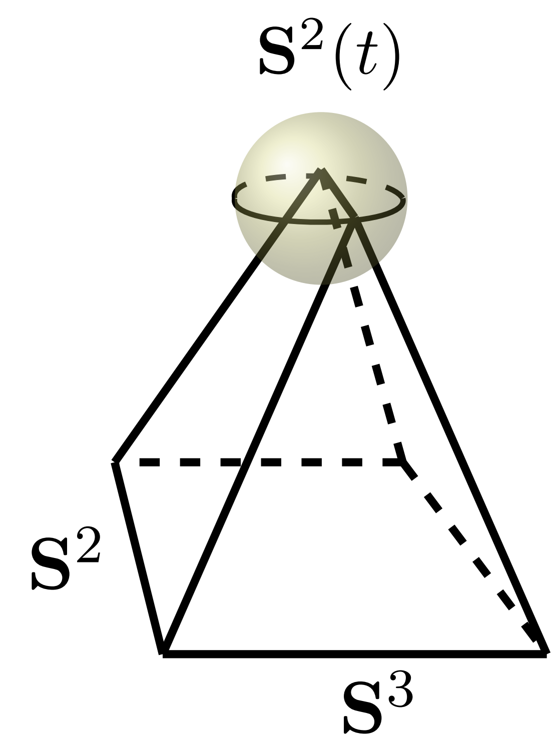
$$\sum_{a=1}^{4} \zeta_a^2 = \mu^2$$

$$\sum_{a=1}^{4} \left(|q_a|^2 - |p_a|^2\right) = \mu^2 \;\; \& \;\; \sum_{a=1}^{4} q_a\, p_a = 0$$

Resolved conifold

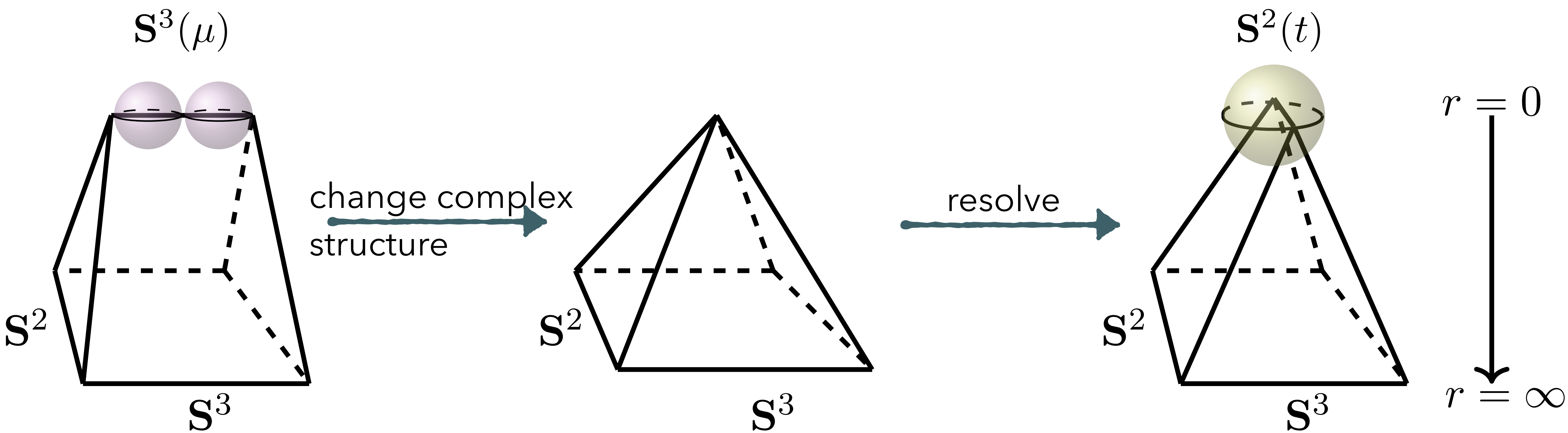
$$\mathcal{R} \equiv \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$$

$$|\xi_1|^2 + |\xi_4|^2 - |\xi_2|^2 - |\xi_3|^2 = t$$



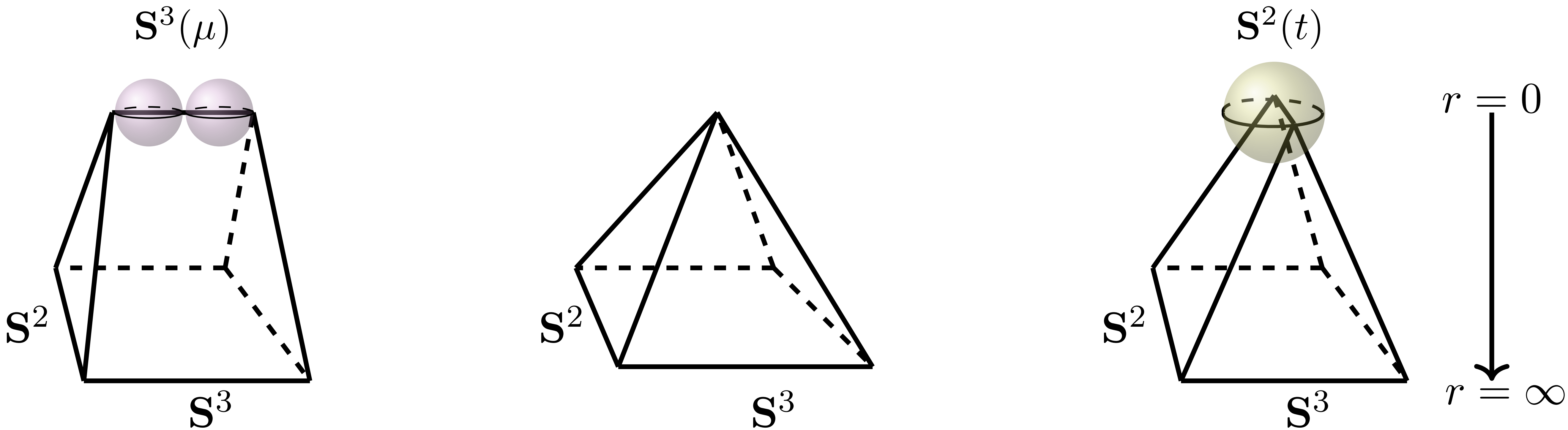
GEOMETRIC TRANSITION

- The geometric transition between the deformed and resolved conifolds forms the backbone of the open/closed string duality.



- *Heuristically*, view the open string theory as living on D-branes wrapping the three-sphere of the deformed conifold at large radius. The geometric transition is tantamount to dissolving the branes into flux which results in the resolved conifold target.

OPEN/CLOSED DUALITY



Chern-Simons (CS) = full open string field theory $\quad \mathcal{Z}_o(T^*\mathbf{S}^3) = \mathcal{Z}_{\mathrm{CS}}(\mathbf{S}^3)$

Witten '92

Closed strings on deformed conifold are trivial $\quad \mathcal{Z}_c(T^*\mathbf{S}^3) = 1$

$$\mathcal{Z}_o(T^*\mathbf{S}^3)\,\mathcal{Z}_c(T^*\mathbf{S}^3) = \mathcal{Z}_c(\mathcal{R}) \quad \Longrightarrow \quad \mathcal{Z}_{\mathrm{CS}}(\mathbf{S}^3) = \mathcal{Z}_c(\mathcal{R})$$

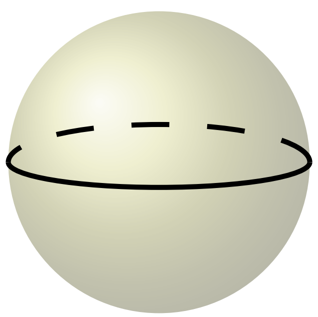
Act II

Topological entanglement in Chern-Simons

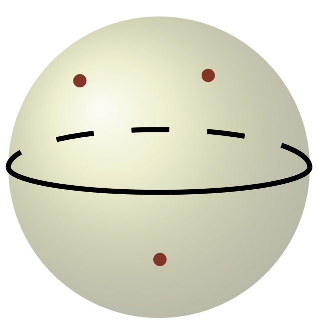
CHERN-SIMONS STATE SPACE

- The state space of Chern-Simons can be obtained by canonical quantization on $\Sigma_g \times \mathbb{R}$.

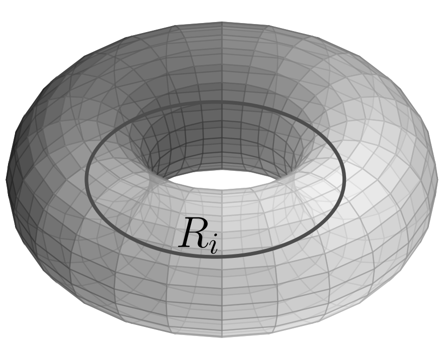
Witten '89



unmarked two-sphere has a 1 dimensional state space. prepared by functional integral over a three-ball.

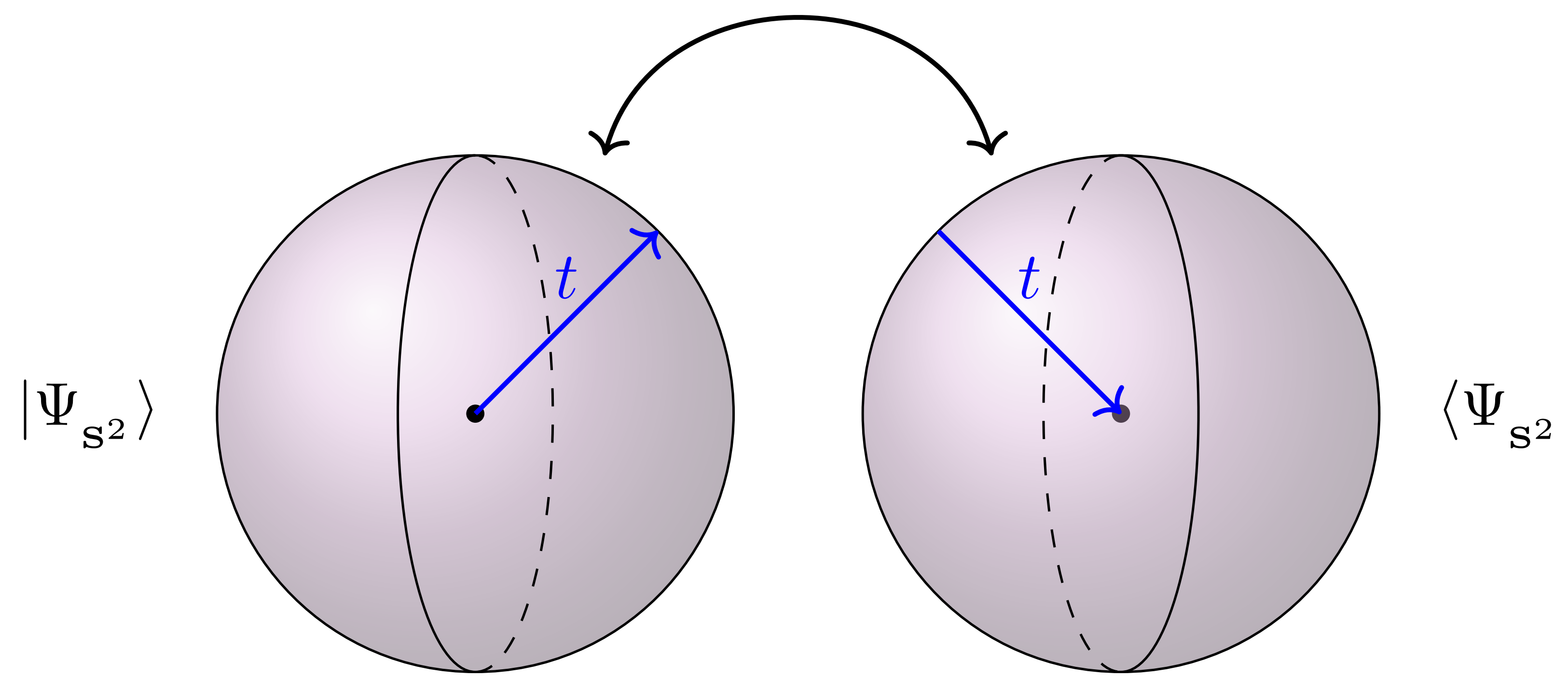


marked two-sphere: state space labelled by representations at marked points



torus states labeled by integrable highest weight representations. Can be prepared by functional integral over solid torus with Wilson line in the representation along non-contractible cycle.

CHERN-SIMONS ON THREE-SPHERE

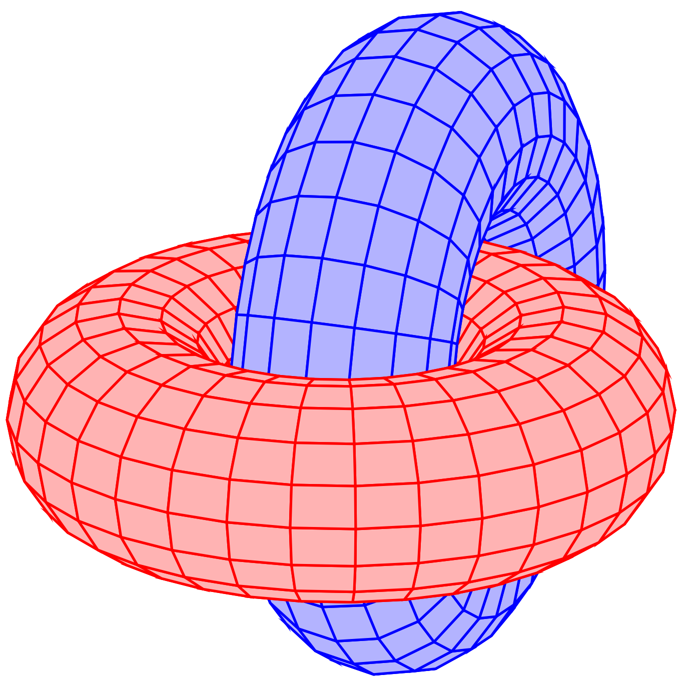


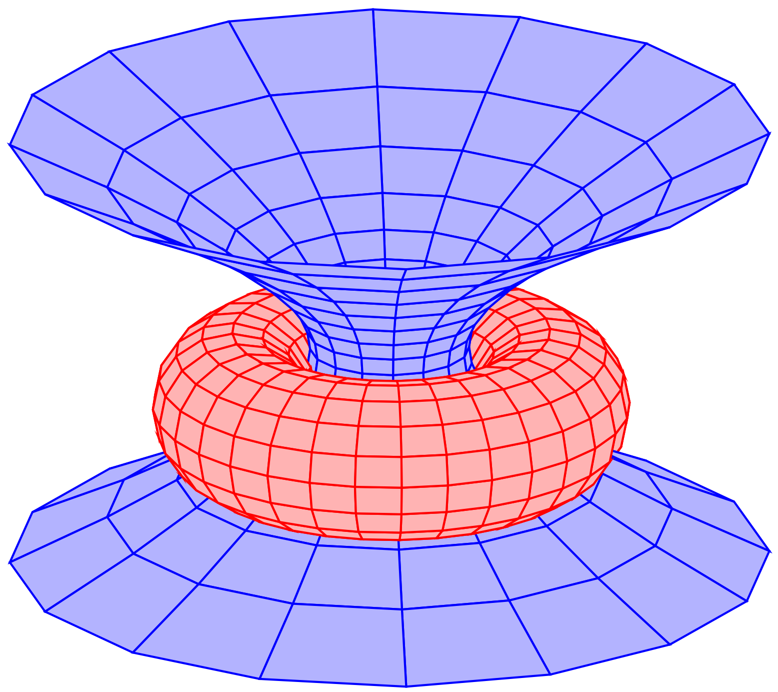
$$\langle\Psi_{\mathbf{S}^2}|\Psi_{\mathbf{S}^2}\rangle = \mathcal{Z}_{\text{CS}}(\mathbf{S}^3)$$

The state on a two-sphere can be prepared by functional integral over a three-ball. Useful to view time as the radial coordinate in the three-ball.

Gluing the balls for the ket and bra together we recover the partition function of theory on the three-sphere.

HEEGARD SPLITTING FOR OTHER STATES

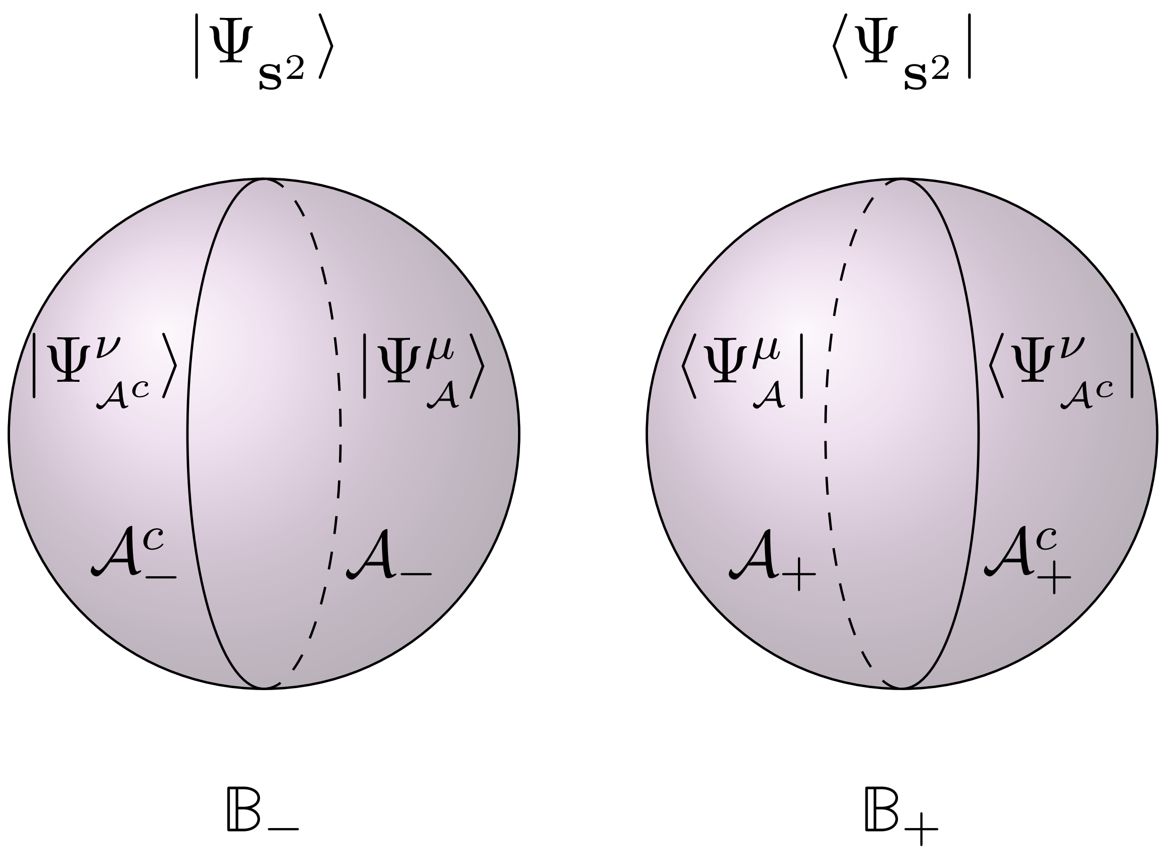




There exist other decompositions of the three-sphere using Heegard splitting. A topological three-sphere can be decomposed into two solid torii, glued on their boundaries after an S-transform to swap the cycles.

Using this decomposition we can build torus states by performing the functional integral on the solid torii.

CHERN-SIMONS REDUCED DENSITY MATRIX



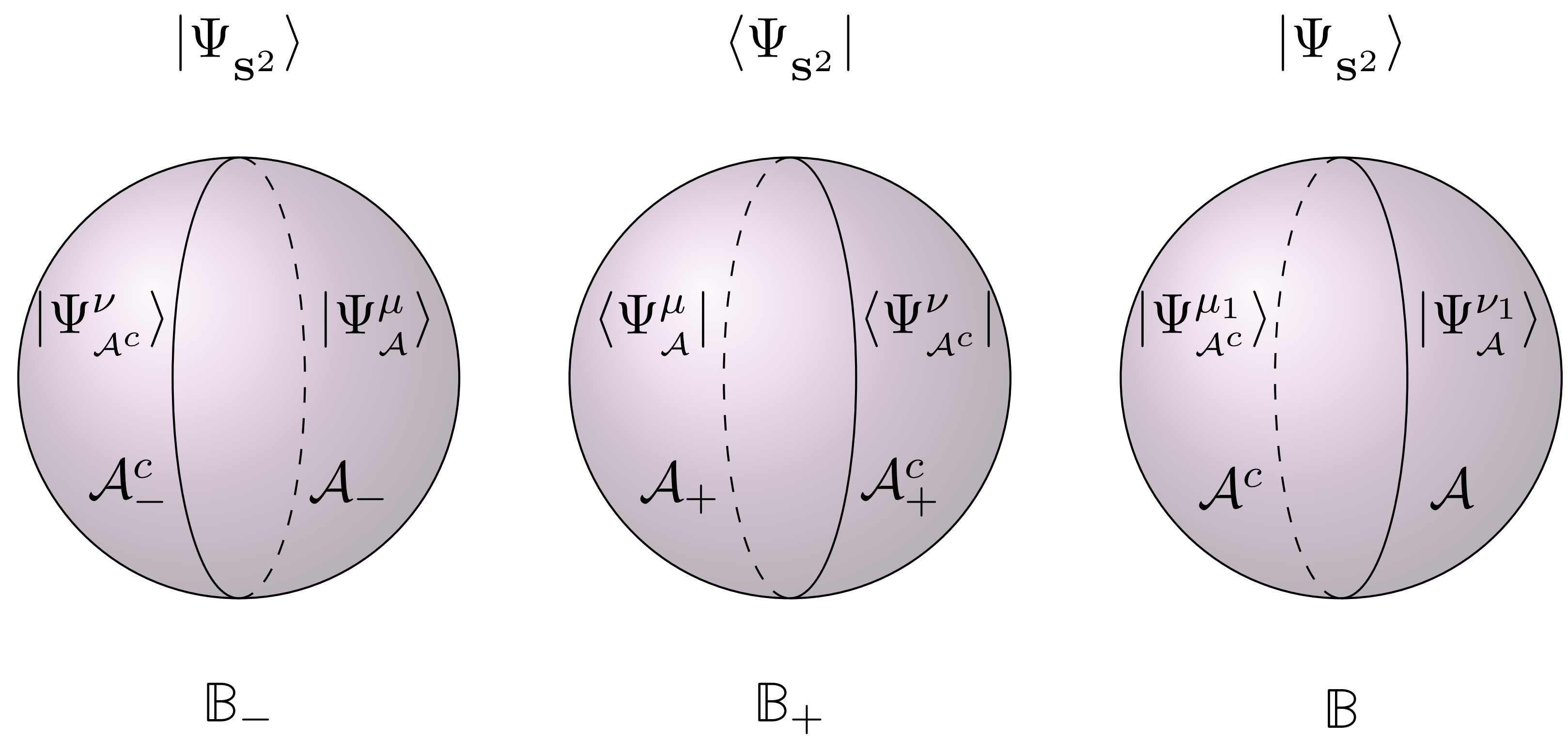
Spatial partition of the state on two-sphere by diving the boundary into region+ complement across an entangling surface.

$$|\Psi_{\mathbf{S}^2}\rangle = \sum_{\mu,\nu} \mathfrak{c}_{\mu\nu} |\Psi^{\mu}_{\mathcal{A}}\rangle \otimes |\Psi^{\nu}_{\mathcal{A}^c}\rangle,$$

$$\sum_{\mu,\nu} |\mathfrak{c}_{\mu\nu}|^2 = \mathcal{Z}_{\mathrm{CS}}(\mathbf{S}^3)$$

$$\rho_{\mathcal{A}} = \sum_{\nu,\mu_1,\mu_2} \mathfrak{c}_{\mu_1\nu}\, \mathfrak{c}^*_{\mu_2\nu}\, |\Psi^{\mu_1}_{\mathcal{A}}\rangle\langle\Psi^{\mu_2}_{\mathcal{A}}|$$

CHERN-SIMONS ENTANGLEMENT



$$\rho_{\mathcal{A}} |\Psi_{\mathbf{S}^2}\rangle = |\Psi_{\mathbf{S}^2}\rangle$$

$$\mathrm{Tr}_{\mathcal{A}}\left({\rho_{\mathcal{A}}}^q\right) = \mathcal{Z}_{\mathrm{CS}}(\mathbf{S}^3)$$

Replica computation follows by sequential gluing of balls, made easy, by the simple action of the reduced density matrix on the two-sphere state.

$$S_{\mathcal{A}} = \lim_{q\to 1} \frac{1}{1-q}\left[\log \mathrm{Tr}_{\mathcal{A}}\left({\rho_{\mathcal{A}}}^q\right) - q\ \log \mathrm{Tr}_{\mathcal{A}}\left(\rho_{\mathcal{A}}\right)\right] = \log \mathcal{Z}_{\mathrm{CS}}(\mathbf{S}^3)$$

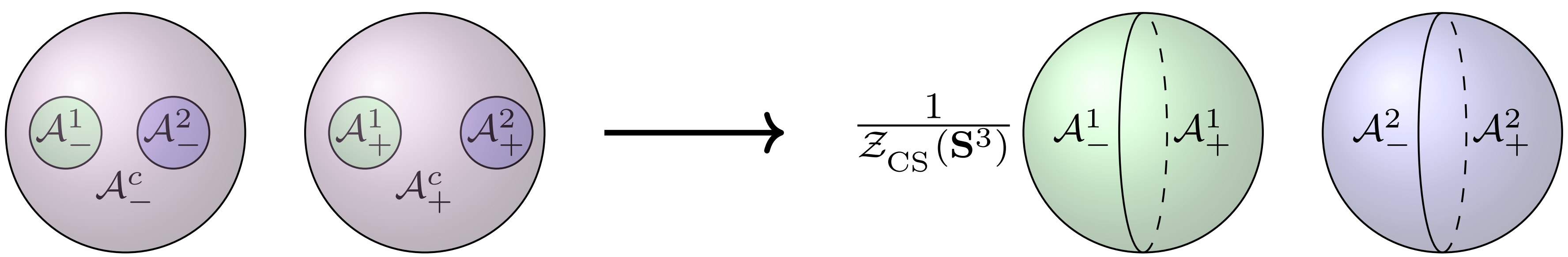
Dong, Fradkin, Leigh, Nowling ’08

FEATURES OF CHERN-SIMONS ENTANGLEMENT

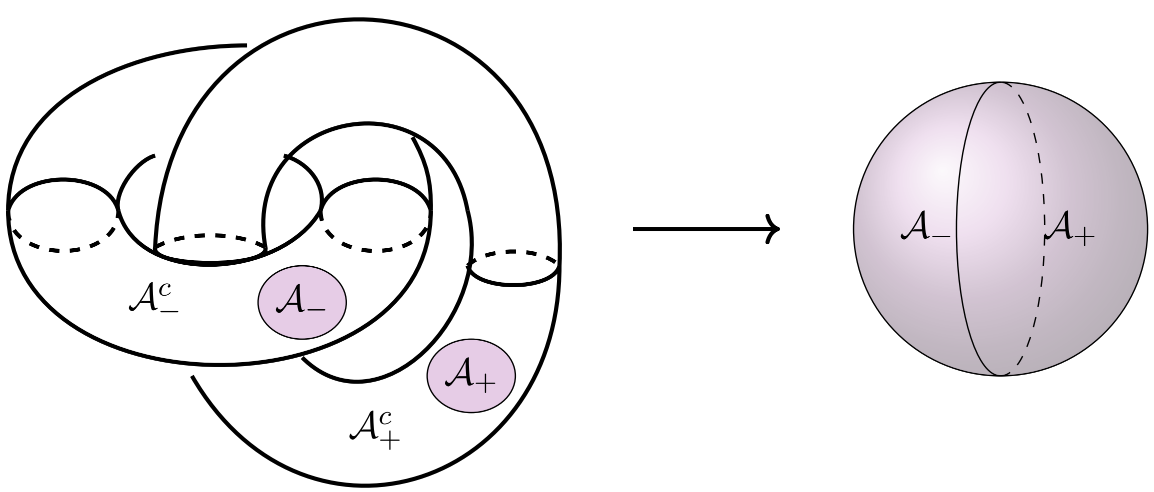
- The topological entanglement has a flat spectrum which can explicitly be mapped to the quantum dimension.

Kitaev, Preskill '05
Levin, Wen '05

- Can have multiple components for our region: the entanglement is then proportional to the number of entangling interfaces: $M \log \mathcal{Z}_{\mathrm{CS}}(\mathbf{S}^3)$



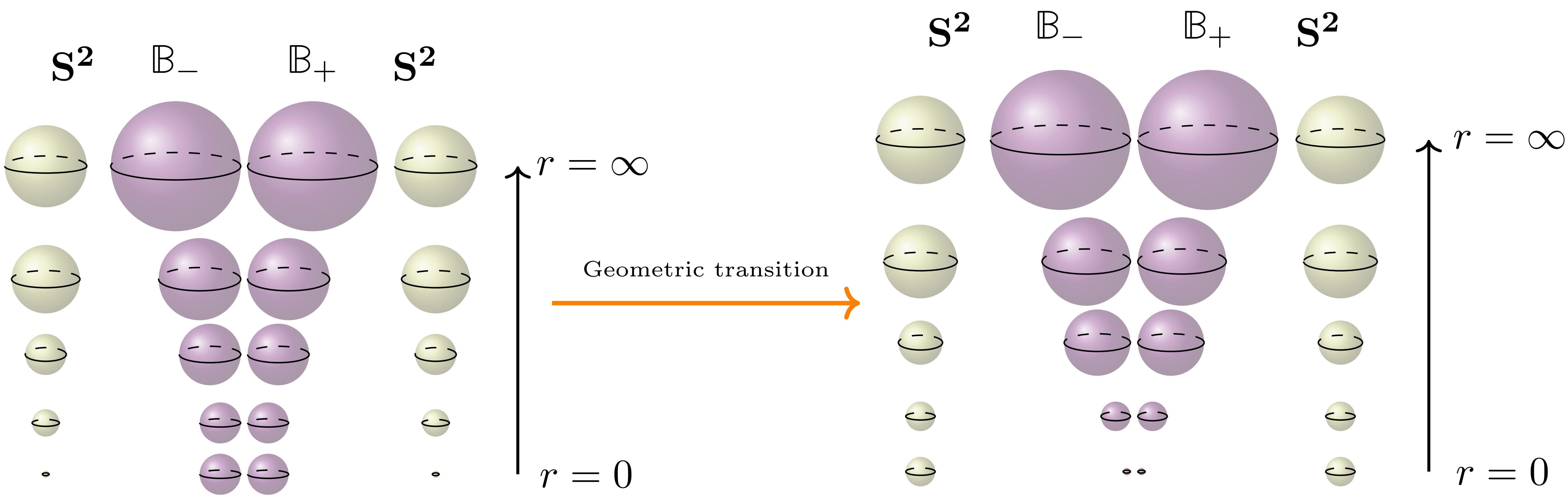
- Similar results for higher genus states without Wilson lines.



Act IIa

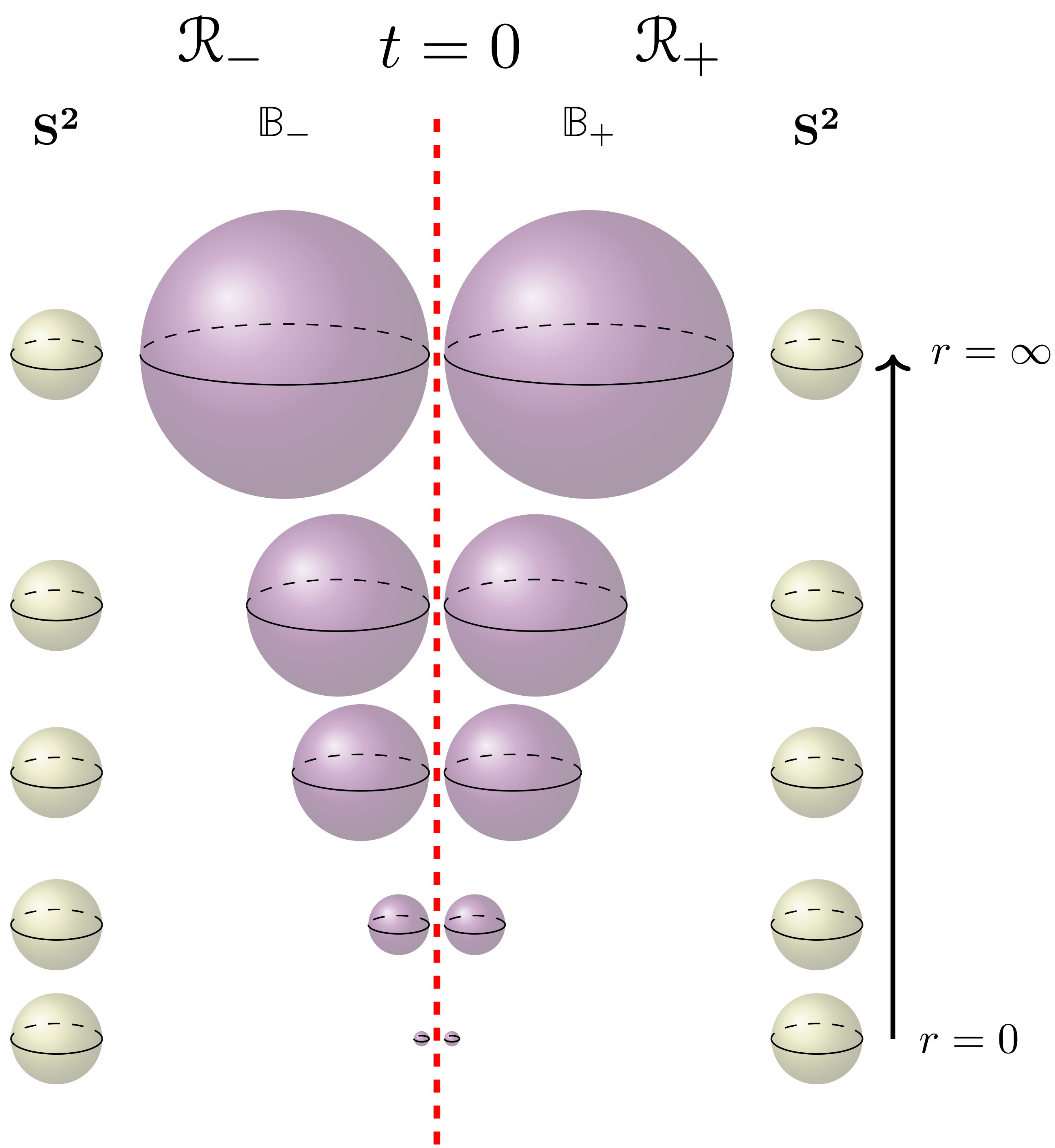
Topological string state space

TOPOLOGICAL STRING STATE SPACE



In the closed string description, employ the Heegard splitting of the three-sphere again to expose constant time hypersurfaces.

TOPOLOGICAL STRING STATE SPACE



$$\langle \Phi_{\mathcal{R}} | \Phi_{\mathcal{R}} \rangle = \mathcal{Z}_c(\mathcal{R})$$

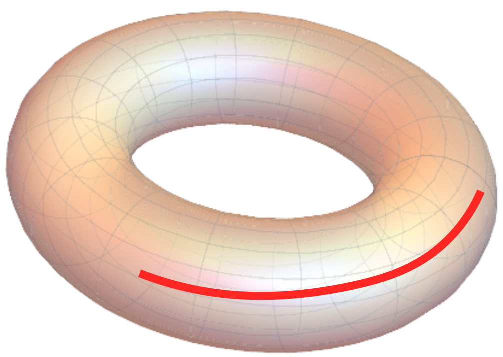
NB: Cauchy surface does not slice through closed string worldsheets (holomorphic maps wrapping homology two-cycles).

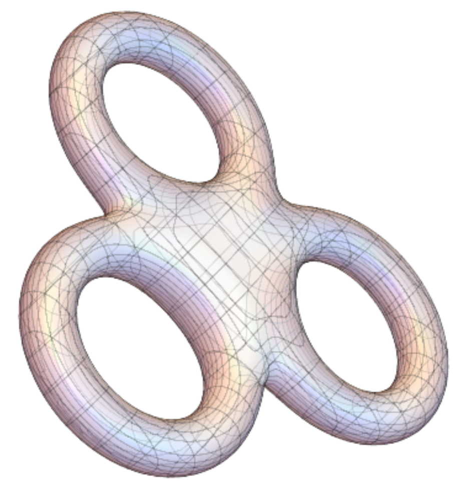
Interlude

Cosmic branes and generalized gravitational entropy

INTERLUDE: GENERALIZED GRAVITATIONAL ENTROPY

- Replica construction in QFT involves computing partition functions on a q-fold branched cover of the original background, branched at the entangling surface.





- When $\ell_{\rm AdS} \gg \ell_s \gg \ell_P$ the bulk spacetime can be obtained using a saddle point solution of the quantum gravity path integral, with boundary conditions provided by the branched cover.

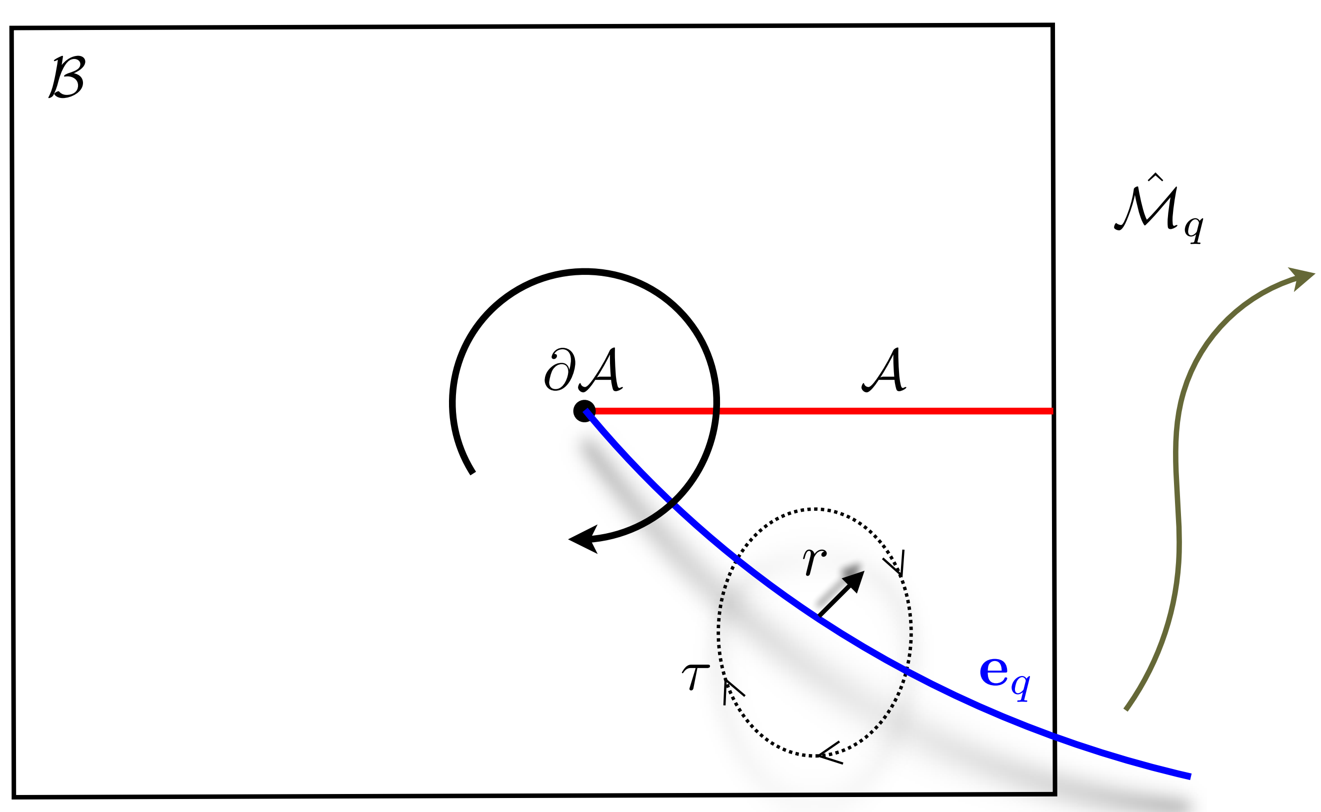
- Gravity affords a major simplification for computing the von Neumann entropy: the analytic continuation taking q to unity is made simple.

Lewkowycz, Maldacena '13

KINEMATICS

- Assume bulk saddles are replica symmetric and construct the orbifold

$$\hat{\mathcal{M}}_q = \mathcal{M}_q/\mathbb{Z}_q$$
$$\partial\hat{\mathcal{M}}_q = \mathcal{B}_q/\mathbb{Z}_q = \mathcal{B}$$



locus of orbifold singularities

opening angle $\dfrac{2\pi}{q}$

cosmic brane of tension

$$T_q = \frac{1}{4G_N^{(d+1)}}\,\frac{q-1}{q}$$

- On-shell action we want is simply related $I[\mathcal{M}_q] = q\, I[\hat{\mathcal{M}}_q]$
- Subtleties arising from global topology, replica symmetry breaking….

Hartman, Haehl, Marolf, Maxfield, MR '14

Belin, Maloney, Matsuura '13

DYNAMICS

✦ Gravitational analytic continuation: dial the tension of cosmic brane!

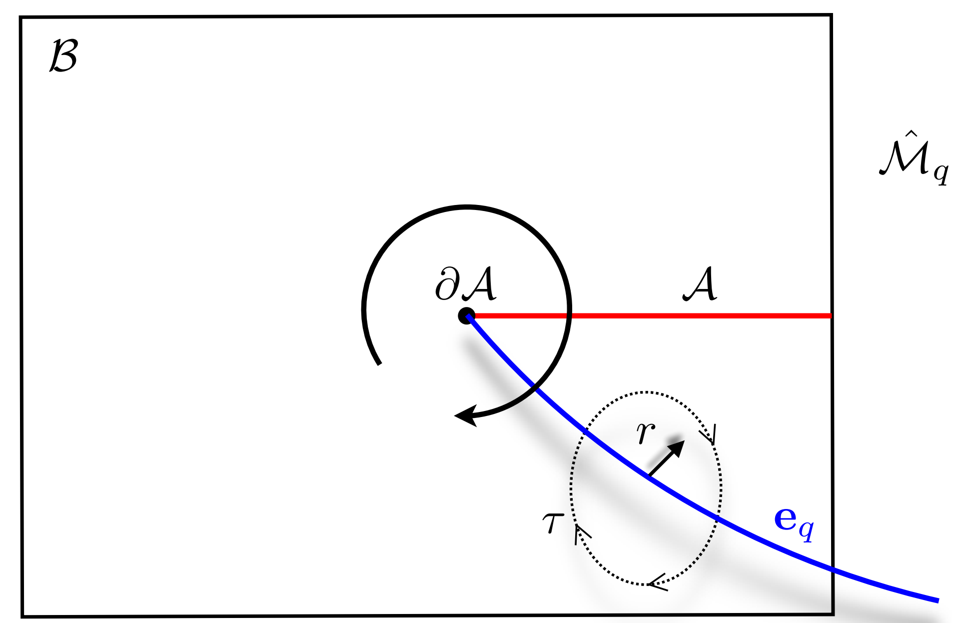
✦ Local analysis of eom in the vicinity of singular locus gives RT prescription.

$$ds^2 = \left(q^2\, dr^2 + r^2\, d\tau^2\right) + \left(\gamma_{ij} + 2\, K^x_{ij}\, r^q \cos\tau + 2\, K^t_{ij}\, r^q \sin\tau\right) dy^i\, dy^j + \cdots$$

$$\lim_{q\to 1} \mathbf{e}_q \to \mathcal{E}_\mathcal{A}\,, \qquad \mathcal{E}_\mathcal{A} \in \mathcal{M} \text{ with } \; t=0,\; K^x = 0 \qquad K^t = 0 \text{ by symmetry}$$

$$\partial_q I[\hat{\mathcal{M}}_q] = \int_{\mathbf{e}_q(\epsilon)} \Theta((g_q)_{AB}, \partial_q (g_q)_{AB})$$

$$\partial_q I[\hat{\mathcal{M}}_q] = \frac{\text{Area}(\mathbf{e}_q)}{4\, q^2 G_N}$$



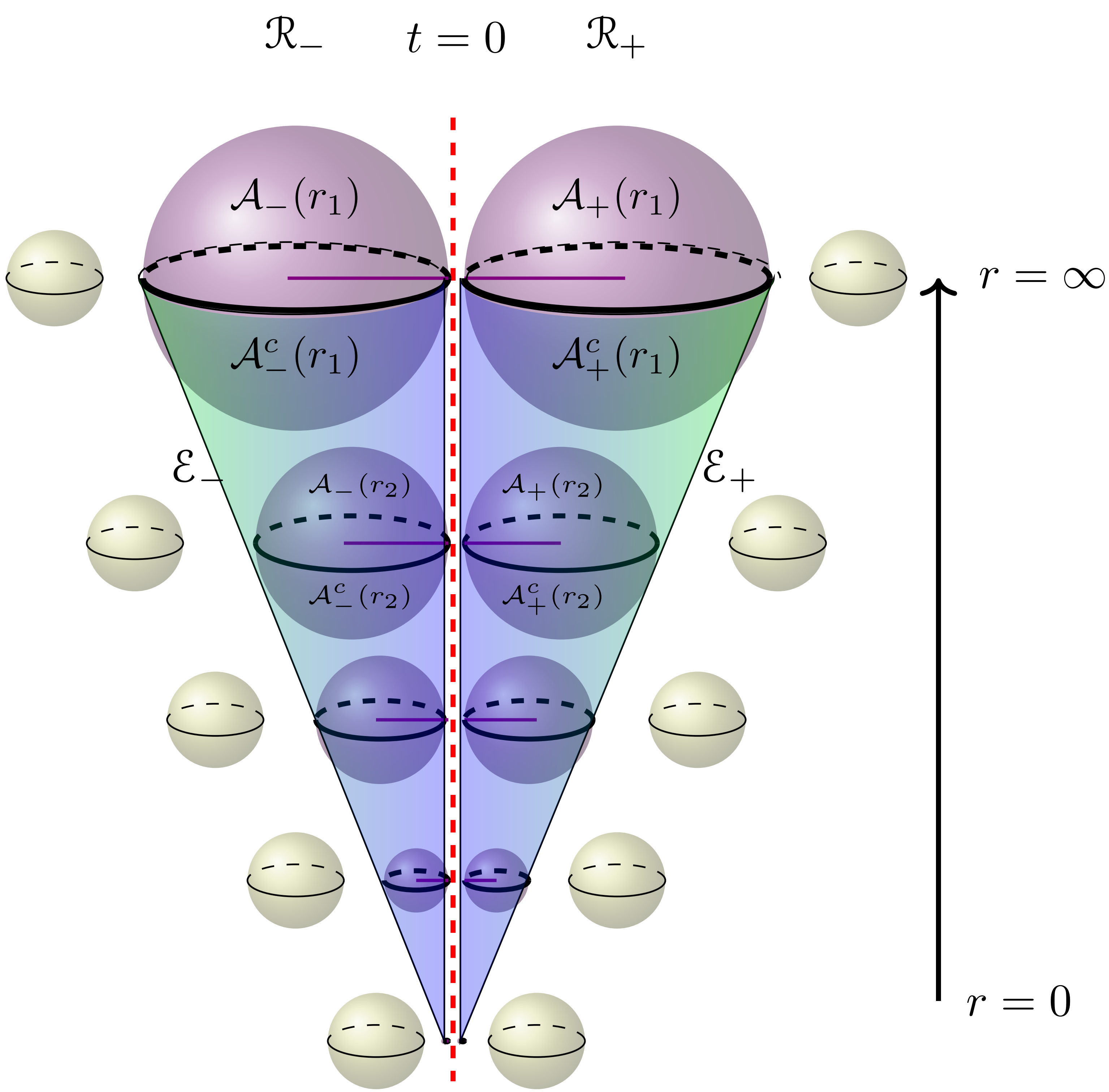
✦ The cosmic brane is the natural continuation of the entangling surface ends up as the bulk spatial separatrix in the gravity regime (quantum corrections are bulk entanglement across this surface).

Faulkner, Lewkowycz, Maldacena '13

Act IIb

Topological string entanglement

TOPOLOGICAL STRING DENSITY MATRIX



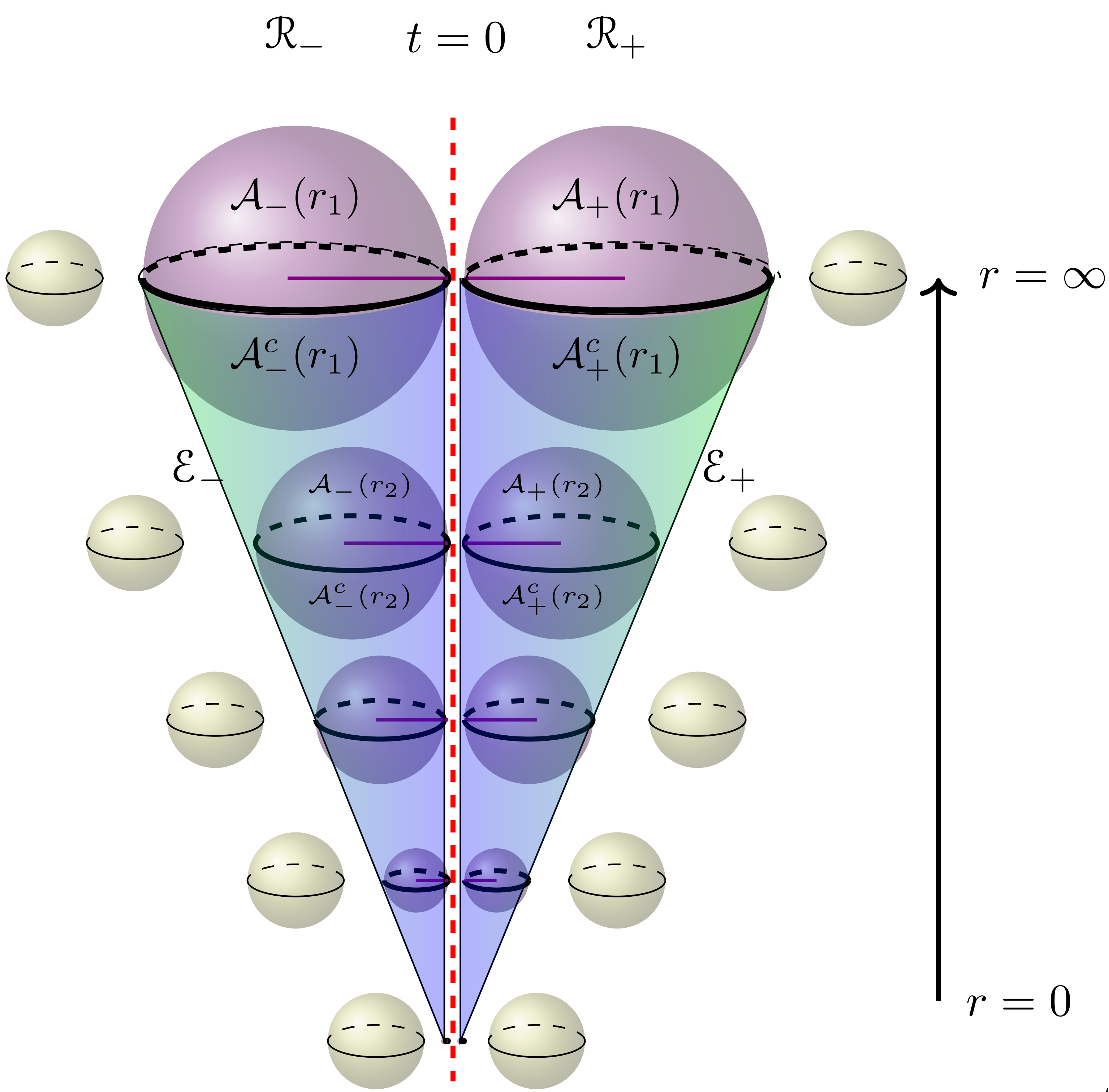
$$\mathcal{A} = \left(\bigcup_{r \geq 0} \mathcal{A}(r) \right) \times \mathbf{S}^2(1)$$

$$\mathcal{E} = \left(\bigcup_{r > 0} \mathbf{S}^1(r) \right) \times \mathbf{S}^2(1)$$

The *entangling brane* provides a spatial separatrix for the bipartition.

cf., Donnelly, Wong '16, '18

TOPOLOGICAL STRING DENSITY MATRIX



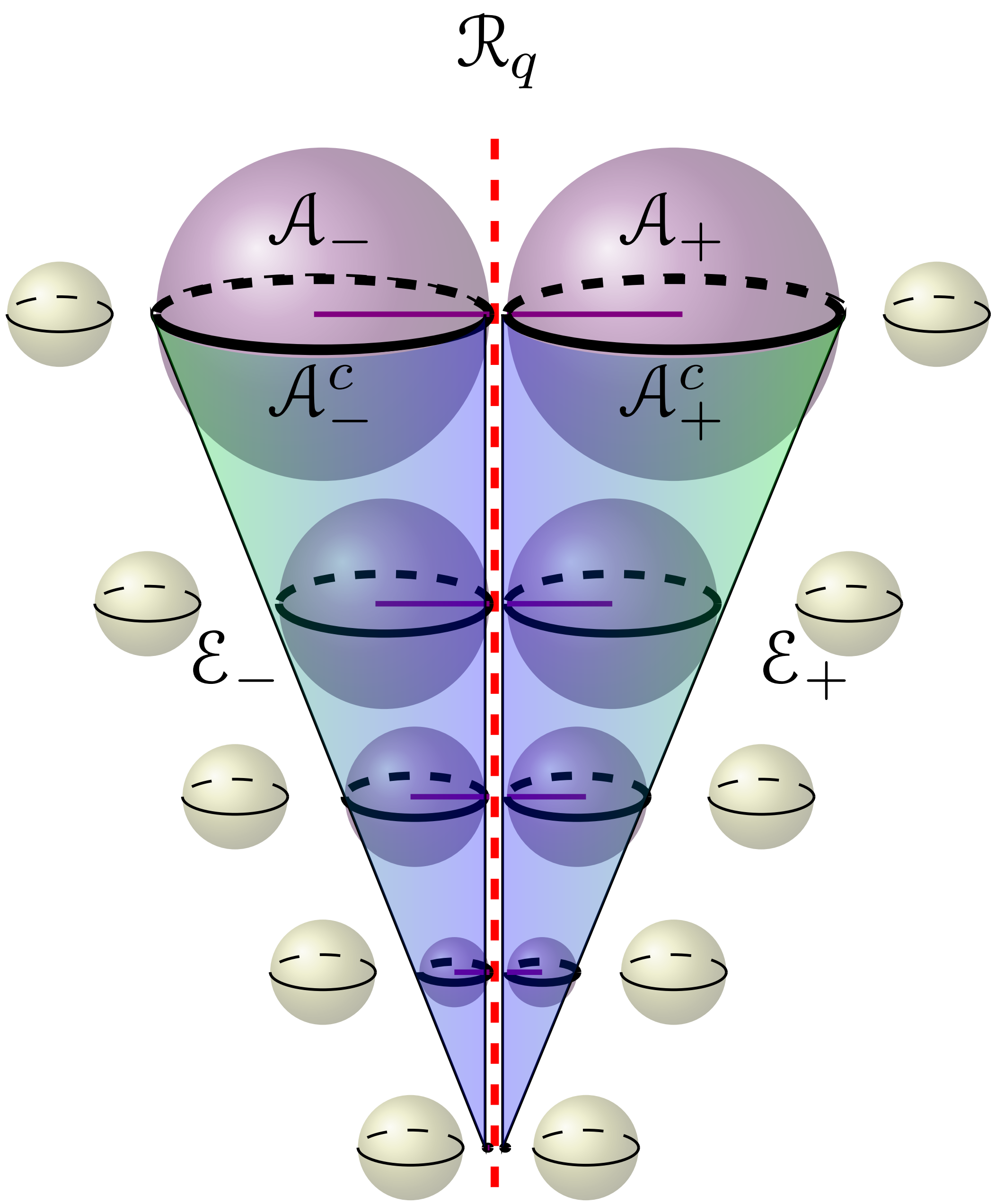
The *entangling brane* does not slice through any of the closed string worldsheets.

One can construct the topological string density matrix by gluing over the complement region.

Replica construction proceeds by cyclically gluing the target spaces together and accounting for normalization.

$$S_{\mathcal{A}}^{(q)} = \frac{1}{1-q}\left(\log \mathcal{Z}[\mathcal{X}^{(q)}] - q \log \mathcal{Z}[\mathcal{X}]\right)$$

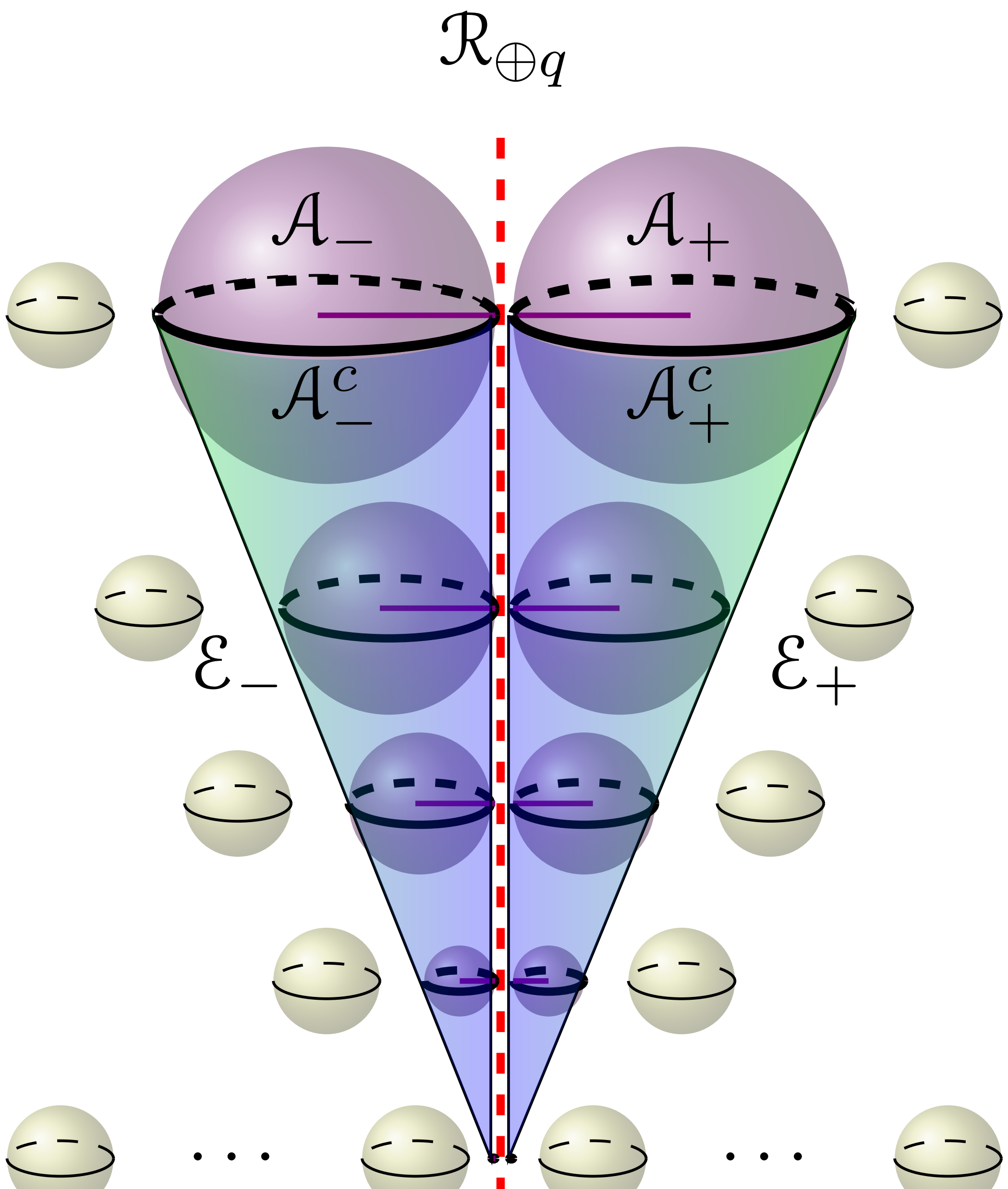
REPLICA GEOMETRY



q-fold gluing of the geometry computing the density matrix produces a geometry with a single non-contractible two-sphere (the spheres from each replica copy get successively identified).

$$\mathcal{Z}_c(\mathcal{R}_q) = \mathcal{Z}_c(\mathcal{R}).$$

DENSITY MATRIX NORMALIZATION



Amusing to compute the normalization by a topological string analog of the generalized entropy.

Construct a new geometry whose time-cycle is comparable to that of the replica geometry.

This geometry has q non-contractible two-spheres.

$$\mathcal{Z}_c(\mathcal{R}_{\oplus q}) \;=\; \mathcal{Z}_c(\mathcal{R})^q$$

CLOSED STRING RENYI ENTROPY

- Combining the results for the partition functions on the replica geometry and the normalization factor we recover the expected flat entanglement.

$$
\begin{aligned}
S_{\mathcal{A}}^{(q)} &= \frac{1}{1-q} \log \left[\mathcal{Z}_c(\mathcal{R}_q) - \log \mathcal{Z}_c(\mathcal{R}_{\oplus q})\right] \\
&= \frac{1}{1-q} \log \left[\mathcal{Z}_c(\mathcal{R}) - q \, \log \mathcal{Z}_c(\mathcal{R})\right] \\
&= \log \mathcal{Z}_c(\mathcal{R})
\end{aligned}
$$

- The difference between the two manifolds entering the above is localized at the tip of the geometry (it cares about the number of non-contractible two-spheres).

Act III

Wilson line states

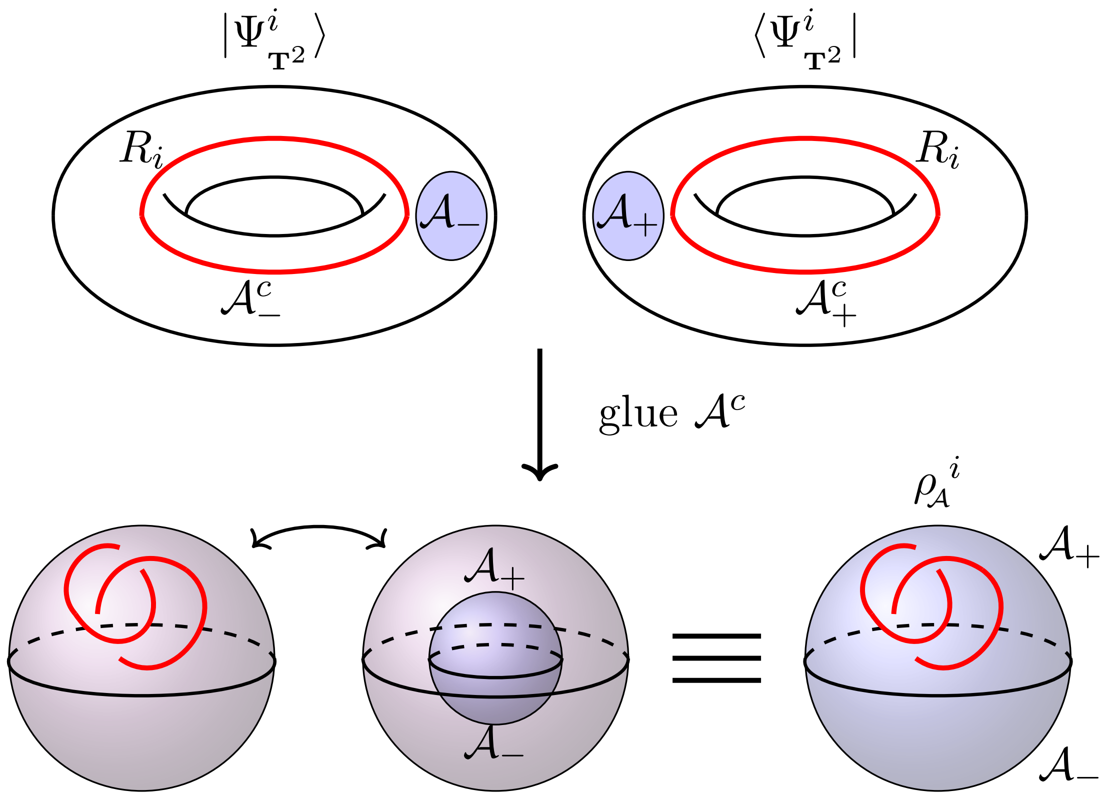
WILSON LINES AND THEIR DUAL LAGRANGIANS

- Consider Wilson lines wrapping a knot/link inside the three-sphere and decompose using Heegard splitting to obtain other states.
- Wilson lines can be `spacelike' or `timelike' and may or may not interfere with the spatial bipartitioning.
- Given a Wilson line in some representation, in the closed string description we have probe D-branes wrapping a non-compact Lagrangian cycle.

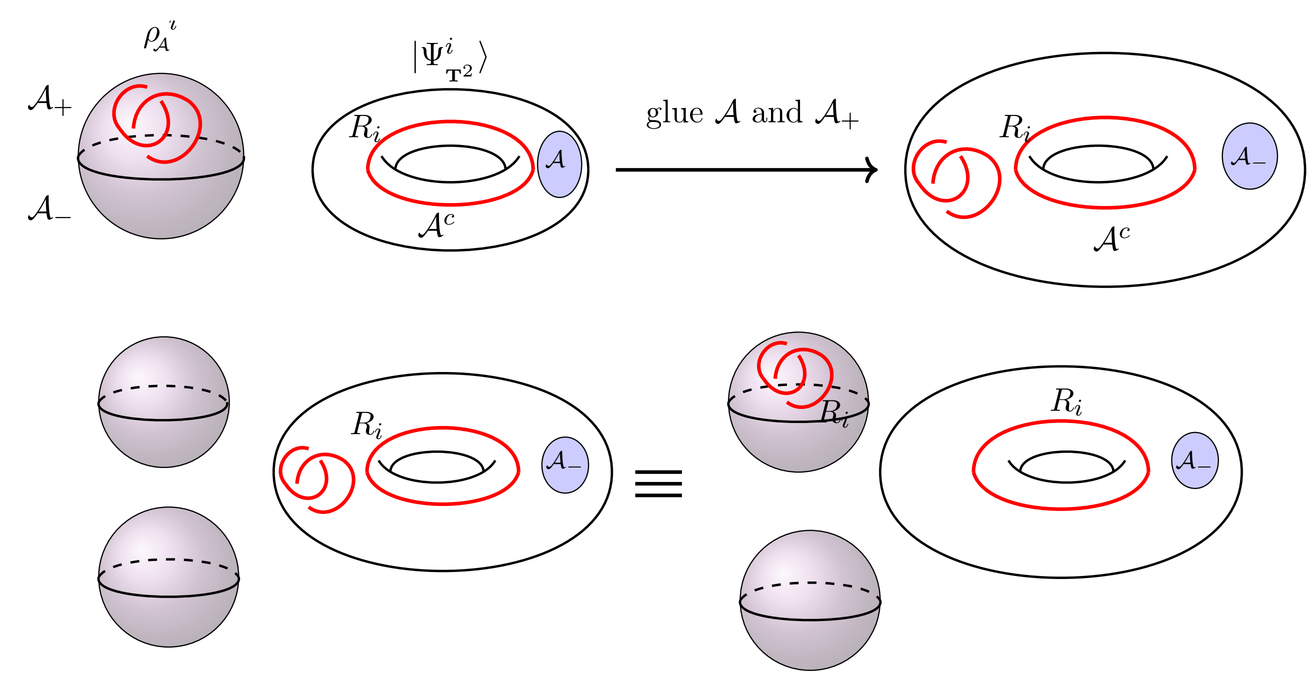
Ooguri, Vafa '99

- This cycle is anchored on the Wilson line in a large radius three-sphere (at the bottom of the cone), and wraps the equator of the non-contractible two-sphere, extending all along in the radial direction.

STATES WITH WILSON LINES I



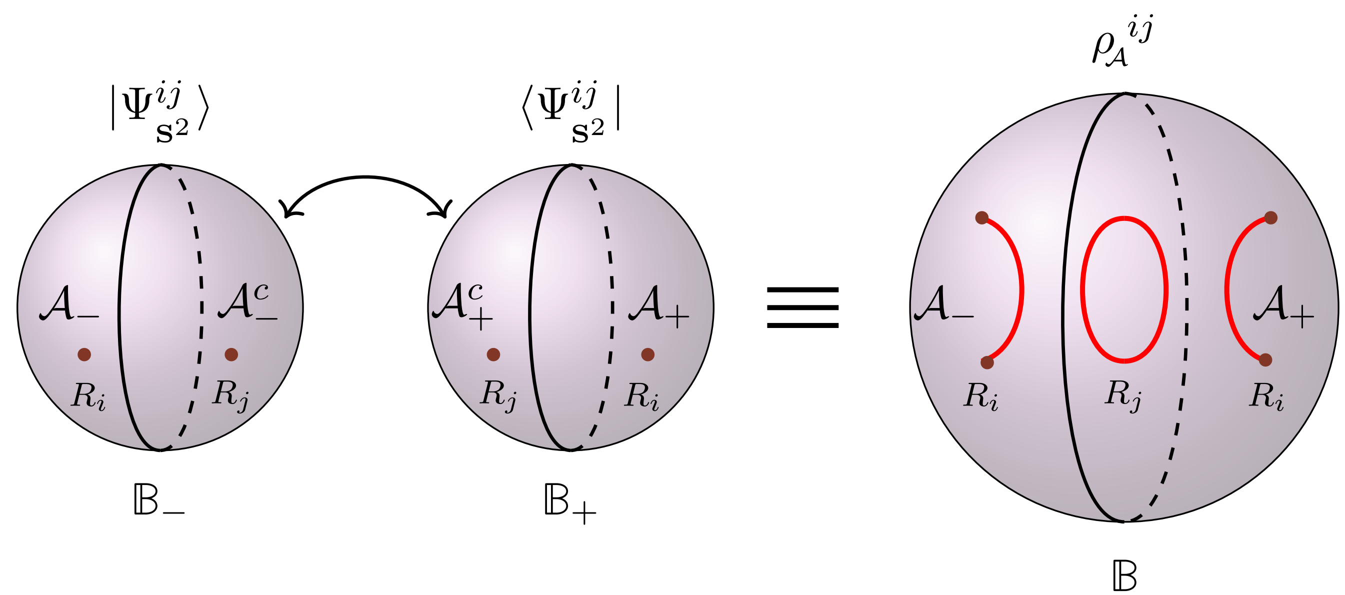
- Spacelike Wilson lines which are not bothered by the bipartitioning are straightforward. The dual Lagrangians are localized in the one piece of the Cauchy surface and play a spectator role.



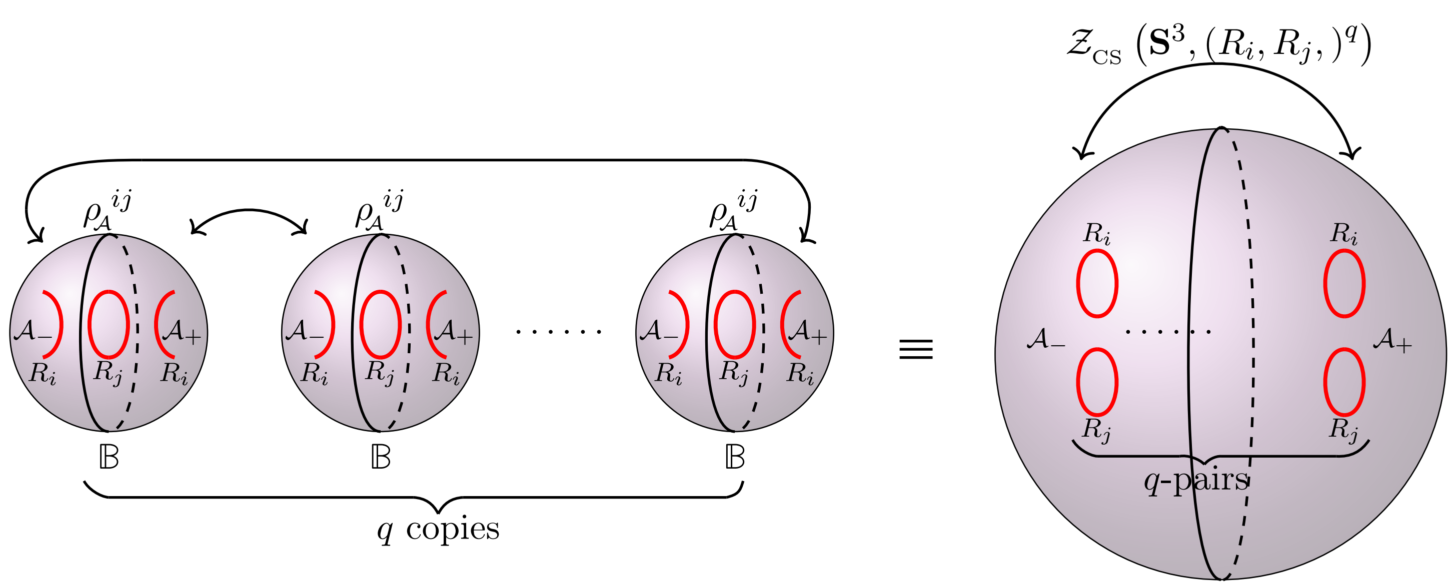
$$\rho_{\mathcal{A}}{}^i\,|\Psi^i_{\mathbf{T}^2}\rangle = \frac{\mathcal{Z}_{\text{CS}}(\mathbf{S}^3, L; R_i)}{\mathcal{Z}_{\text{CS}}(\mathbf{S}^3)}\,|\Psi^i_{\mathbf{T}^2}\rangle$$

$$S^{(q)}_{\mathcal{A}}(\rho_{\mathcal{A}}{}^i) = \frac{1}{1-q}\left[\log\frac{\mathcal{Z}_{\text{CS}}(\mathbf{S}^3, L; R_i)^q}{\mathcal{Z}_{\text{CS}}(\mathbf{S}^3)^{q-1}} - q\,\log \mathcal{Z}_{\text{CS}}(\mathbf{S}^3, L; R_i)\right] = \log \mathcal{Z}_{\text{CS}}(\mathbf{S}^3)$$

STATES WITH WILSON LINES IIA

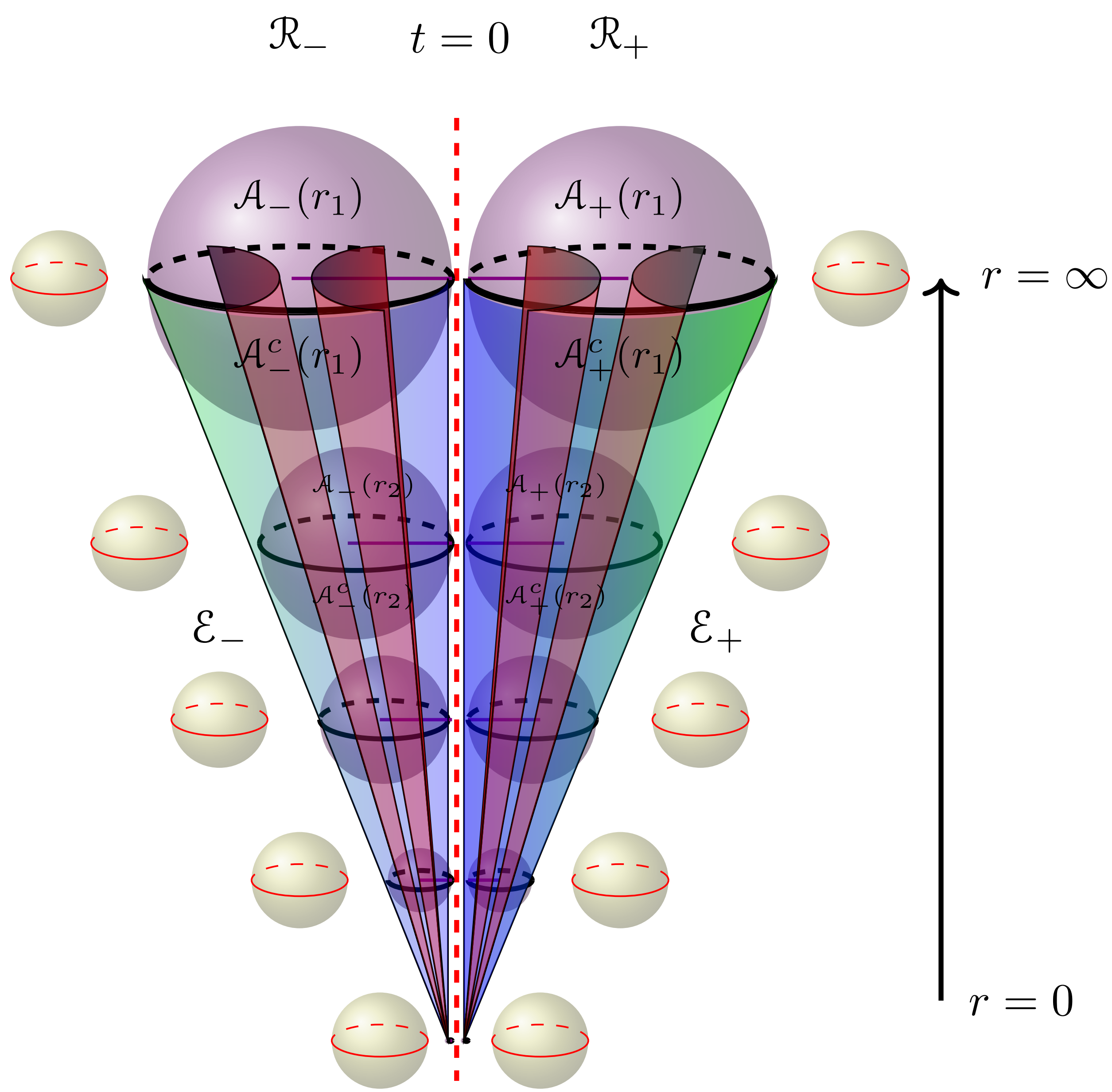


- Timelike Wilson lines may be sliced by the bipartioning leaving behind marked points carrying representation data.



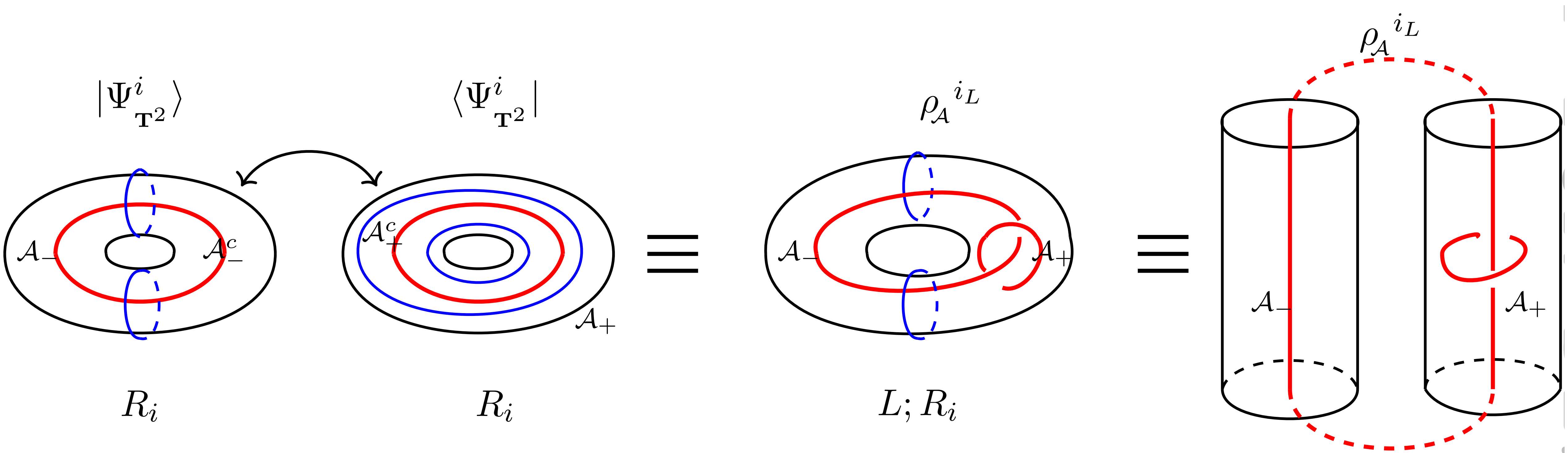
$$S^{(q)}_{\mathcal{A}}(\rho_{\mathcal{A}}{}^{ij}) = \frac{1}{1-q} \log \left[\frac{\mathcal{Z}_{\text{CS}}(\mathbf{S}^3, (R_i, R_j)^q)}{(\mathcal{Z}_{\text{CS}}(\mathbf{S}^3, R_i, R_j))^q} \right] = \log \mathcal{Z}_{\text{CS}}(\mathbf{S}^3)$$

STATES WITH WILSON LINES IIB



- In the closed string description we would now have the Cauchy surface slice through the dual Lagrangian cycles.
- However, the entangling brane does not interfere with the Lagrangian cycles; the marked points/planes are either in the region or in its complement.

STATES WITH WILSON LINES III



$$S^{(q)}_{\mathcal{A}}(\rho_{\mathcal{A}}{}^{iL}) = \frac{1}{1-q}\left[\log \mathcal{Z}_{\rm CS}(\mathbf{S}^3, L^{(q)}; R_i) - q\, \log \mathcal{Z}_{\rm CS}(\mathbf{S}^3, L; R_i)\right] = 2\, \log \mathcal{Z}_{\rm CS}(\mathbf{S}^3, \bar{R}_i)$$

- Spacelike Wilson lines may be cut by the biparition in the interior. In this case we can get non-trivial answers for the entropies as surgery induces non-trivial linking of the Wilson lines.

- The Lagrangian cycles will end up being linked, but since the entangling brane doesn't interfere with them, we anticipate recovering the Chern-Simons answer from the closed string.

Act IV

Summary

SUMMARY

- Topological entanglement in topological closed string theory can be meaningfully defined, by suitably uplifting three-manifold surgery techniques.
- There is a natural notion of an *entangling brane* which plays the dual role of a cosmic brane and a bulk entangling surface.
- For the examples studied, the entangling branes do not interfere with the closed string worldsheets, so one is able to make progress without detailed understanding of the closed topological string Hilbert space.
- Topological entanglement is the glue that allows surgery without reference to detailed geometric features. This explains the flatness of the entanglement spectrum, for it can only depend on the number of pieces being glued, with a degrees of freedom count for each gluing (the quantum dimension).

cf., Donnelly, Wong '16, '18

OPEN QUESTIONS

- The flat entanglement spectrum is a consequence of lack of penalty (i.e., no backreaction) in the closed string description. Should we therefore interpret tensor network approaches as at best capturing topological aspects of AdS/CFT with no insight into the dynamical aspects?

- Is the fact that there is no preferred geometric/topological construct that computes the entanglement entropy in topological string theory an indication that the physical string theory would likewise in the classical limit admit no simple RT/HRT type prescription, viz., *entanglement maps to entanglement and there is no semiclassical part of it that is suitably geometrized when* $\ell_{\rm AdS} \sim \ell_s \gg \ell_P$*?*