Entanglement negativity in many-body quantum systems

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[Jonah Kudler-Flam and SR
arXiv:1808.00446 and work in progress]

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Partial transpose and entanglement negativity

- Partial transpose of a density operator ρ:

$$\langle e_i^{(A)} e_j^{(B)} | \rho^{T_B} | e_k^{(A)} e_l^{(B)} \rangle := \langle e_i^{(A)} e_l^{(B)} | \rho | e_k^{(A)} e_j^{(B)} \rangle$$

  where $|e_i^{(A,B)}\rangle$ is the basis of $\mathcal{H}_{A,B}$.

- Entanglement negativity and logarithmic negativity:

$$\mathscr{N}(\rho) := (||\rho^{T_B}||_1 - 1)/2 = \sum_{\lambda_i < 0} |\lambda_i|,$$
$$\mathscr{E}(\rho) := \log ||\rho^{T_B}||_1 = \log(2\mathscr{N}(\rho) + 1)$$

Entanglement in mixed states?

- How to quantify quantum entanglement between A and B when $\rho_{A\cup B}$ is *mixed* ? E.g., finite temperature, A, B are parts of a bigger system.
- The entanglement entropy is an entanglement measure only for pure states. For mixed states, it is not monotone under LOCC.
- Positive partial transpose (PPT) criterion.
  [Peres (96), Horodecki-Horodecki-Horodecki (96), Eisert-Plenio (99), Vidal-Werner (02), Plenio (05) ...]

Partial transpose and quantum entanglement

- EPR pair: $|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10| - |01\rangle\langle 10| - |10\rangle\langle 01|]$$

- Partial transpose:

$$\rho^{T_2} = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10| - \underline{|00\rangle\langle 11|} - \underline{|11\rangle\langle 00|}]$$

- Entangled states are badly affected by partial transpose: Negative eigenvalues:

$$\{\lambda_i\} = Spec(\rho^{T_B}) = \{1/2, 1/2, 1/2, -1/2\}$$

- C.f. For a classical state: $\rho = \frac{1}{2}[|00\rangle\langle 00| + |11\rangle\langle 11|] = \rho^{T_B}$

Partial transpose and entanglement negativity

- (Logarithmic) Entnglement negativity:

$$\mathcal{N}(\rho) := \sum_{\lambda_i<0} |\lambda_i| = (||\rho^{T_B}||_1 - 1)/2,$$
$$\mathscr{E}(\rho) := \log(2\mathscr{N}(\rho) + 1) = \log ||\rho^{T_B}||_1.$$

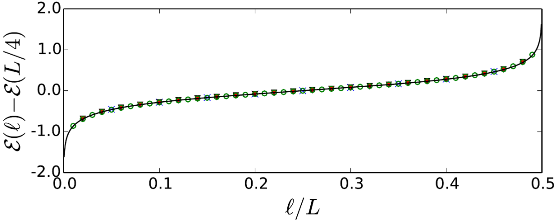
- Quantum entanglement measure for mixed state (monotone under LOCC).
- For mixed states, negativity extracts quantum correlations only.
- Computable.
- Negativity can zero even when the state is entangled (does not detect bound entanglement but only distillable one).

Applications of entanglement negativity to many-body physics?

- Entanglement negativity in CFTs, and TFTs
  [Calabrese-Cardy-Tonni(12-), Castelnovo (13), Lee-Vidal (13), Wen-Chang-SR, Wen-Matsuura-SR(16), and many others...]
- Partial transpose and topological invariants in SPT phases
  [Pollmann-Turner (12),Shapourian-Shiozaki-SR (16)]
- Experimental measurements? [Elben et al (18-19); Cornfield et al (18)]

Negativity in 2d CFT

- The logarithmic negativity for two adjacent intervals of equal length ℓ in free fermion chain

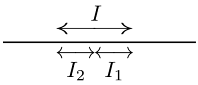


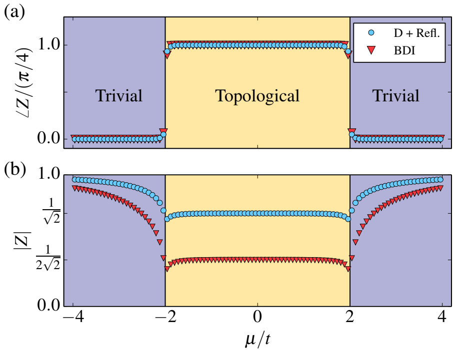
- The numerical result (points) using the free fermion formula [Shapourian-Shiozaki-SR(17)] agrees with the CFT result (solid line) [Calabrese-Cardy-Tonni]. $\mathcal{E} = \frac{c}{4} \ln \tan \frac{\pi \ell}{L}$

Partial transpose and topological invariant

[Shiozaki-Shapourian-SR (16)]

- Topological invariant of 1d topological superconductor (the Kitaev chain) $\mathrm{Tr}(\rho_I \rho_I^{T_1}) \sim e^{2\pi i \nu/8}$.





Holographic description of entanglement negativity?

Basic proposal

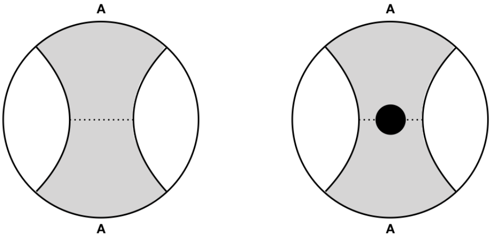
Negativity = Minimal entanglement wedge cross section with back reaction

[Jonah Kudler-Flam and SR, arXiv:1808.00446]

C.f. other proposal [Chaturvedi-Malvimat-Sengupta (16); Jain-Malvimat-Mondal-Sengupta (17)]

Entanglement wedge

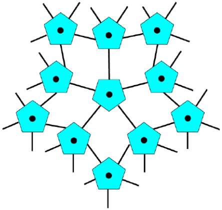
- Entanglement wedge = the bulk region corresponding to the reduced density matrix on the boundary [Headrick et al (14), Jafferis-Suh (14), Jafferis-Lewkowycz-Maldacena-Suh (15), ...]

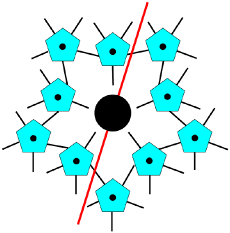


- Entanglement of purification [Takayanagi-Umemoto(17), Nguyen-Devakul-Halbasch-Zaletel-Swingle (17)]
- Odd entropy [Tamaoka (18)]
- Reflected entropy [Dutta-Faulkner (19)]

Perfect tensor holographic error correcting code

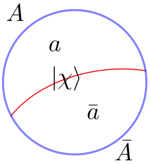
- Tensor network acts as an error correcting code encoding "bulk" logical qubits into "boundary" physical qubits
- Captures many aspects of holography; black holes, bulk reconstruction, subregion duality, holographic entanglement entropy, etc.

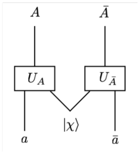




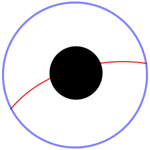
[Almheiri-Dong-Harlow(15), Harlow(17), Pastawski-Yoshida-Harlow-Preskill(15), Hayden et al (16)]

- Computed entanglement negativity in a tensor network model of holographic duality (using the language of QEC).





- Entanglement entropy: $S(\rho_A) = S(\chi_A) + S(\rho_a)$
- Entanglement negativity: $\mathscr{E}(\rho_{A\bar{A}}) = S_{1/2}(\chi_A) + \mathscr{E}(\rho_{bulk})$
- With BH in bulk, negativity avoids horizon



Full-fledged holography

- No back reaction for the von Neumann entropy
- Back reaction for Rényi entropy
- For pure state, negativity $=$ Rényi $1/2$
- How well do we know about the back reaction? How much control do we have?

Rényi entropy and back reaction

- Rényi entropy S_n: [Dong (16)]

$$n^2 \frac{\partial}{\partial n}\left(\frac{n-1}{n} S_n\right) = \frac{\text{Area}(\text{cosmic brane}_n)}{4G_N}$$

Cosmic brane has a tension $T_n = (n-1)/(4nG_N)$

- Conjecture: For "spherical" configurations [C.f. Hung-Myers-Smolkin-Yale (11), Rangamani-Rota (14)]

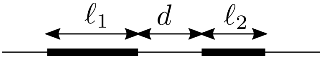
$$\mathscr{E} = \mathcal{X} \frac{E_W}{4G_N}$$

- For (1+1)d CFT (with spherical entangling surface) $\mathcal{X} = 3/2$,

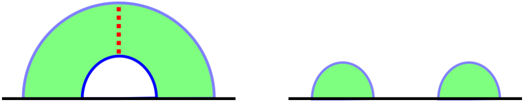
$$\mathscr{E} = \frac{3}{2} \frac{E_W}{4G_N}$$

Disjoint intervals at zero temperature

- Negativity for two disjoint intervals:



- Minimal entanglement wedge cross section E_W



$$E_W = \begin{cases} \frac{c}{6} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}}, & \frac{1}{2} \leq x \leq 1 \\ 0, & 0 \leq x \leq \frac{1}{2} \end{cases} \qquad x = \frac{\ell_1 \ell_2}{(\ell_1 + d)(\ell_2 + d)}.$$

[Takayanagi-Umemoto(17), Nguyen-Devakul-Halbasch-Zaletel-Swingle (17)]

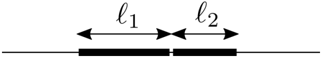
- E_W should be compared with

$$\begin{aligned}\mathscr{E} &= \lim_{n_e \to 1} \ln \mathrm{Tr}\,(\rho^{T_2})^{n_e} \\ &= \lim_{n_e \to 1} \ln \langle \sigma_{n_e}(w_1, \bar{w}_1)\bar{\sigma}_{n_e}(w_2, \bar{w}_2)\bar{\sigma}_{n_e}(w_3, \bar{w}_3)\sigma_{n_e}(w_4, \bar{w}_4)\rangle_{\mathbb{C}}.\end{aligned}$$

where σ_n is the twist operator with dimension $h_n = (c/24)(n - 1/n)$. The replica limit from even integer to $n_e \to 1$. [Calabrese-Cardy-Tonni (12)]

Adjacent limit

- In the limit of adjacent intervals $d \to 0$:



$$E_W \to \frac{c}{6} \ln \left[\frac{4}{\epsilon} \frac{\ell_1 \ell_2}{(\ell_1 + \ell_2)} \right].$$

- Agrees with CFT result (universal) [Calabrese-Cardy-Tonni (12)]:

$$\begin{aligned} \mathscr{E} &= \lim_{n_e=1} \ln \left\langle \sigma_{n_e}(w_1, \bar{w}_1) \bar{\sigma}^2_{n_e}(w_2, \bar{w}_2) \sigma_{n_e}(w_4, \bar{w}_4) \right\rangle_{\mathbb{C}} \\ &= \frac{c}{4} \ln \left[\frac{\ell_1 \ell_2}{\ell_1 + \ell_2} \right] + const. \end{aligned}$$

if the constant is properly chosen, $\mathscr{E} = (3/2) E_W$.

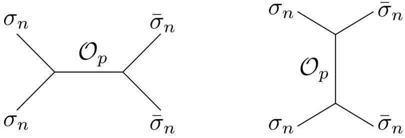
Disjoint intervals

- We need four pt correlation function for the disjoint case:

$$\mathscr{E} = \lim_{n_e \to 1} \ln \langle \sigma_{n_e}(w_1, \bar{w}_1) \bar{\sigma}_{n_e}(w_2, \bar{w}_2) \bar{\sigma}_{n_e}(w_3, \bar{w}_3) \sigma_{n_e}(w_4, \bar{w}_4) \rangle_{\mathbb{C}}.$$

- Will focus on the dominant conformal block

$$\langle \sigma_{n_e}(\infty) \bar{\sigma}_{n_e}(1) \bar{\sigma}_{n_e}(x, \bar{x}) \sigma_{n_e}(0) \rangle \sim \mathcal{F}(h_p, h_i, x) \bar{\mathcal{F}}(h_p, h_i, \bar{x})$$



- Intermediate state is either identity operator of double twist operator σ_n^2 with dimension

$$h_{\sigma_n^2} = \begin{cases} \frac{c}{24}\left(\frac{n}{1} - \frac{1}{n}\right) & \to 0 \quad n : \text{odd} \\ \frac{c}{12}\left(\frac{n}{2} - \frac{2}{n}\right) & \to -\frac{c}{8} \quad n : \text{even} \end{cases}$$

Series expansion

- Dominant conformal block with double twist operator $(y = 1 - x)$: [Headkrick (10), Kulaxizi-Parnachev-Policastro (14)]

$$\mathcal{F}(h_p, y) = y^{h_p}\left[1 + \frac{h_p}{2}y + \frac{h_p(h_p+1)^2}{4(2h_p+1)}y^2 \right.$$
$$\left. + \frac{h_p^2(1-h_p)^2}{2(2h_p+1)[c(2h_p+1)+2h_p(8h_p-5)]}y^2 + \cdots\right].$$

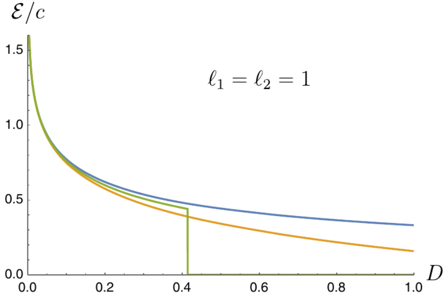
- Taking the large c limit and then replica limit $h_p \to -c/8$,

$$\mathcal{F}(h_p, y) = y^{-\frac{c}{8}}\left[1 - \frac{cy}{16} + \frac{c^2y^2}{512} - \frac{c^3y^3}{24576} + \cdots\right].$$

Matches with the cross-ratio expansion of $(3/2)E_W$ in large c.

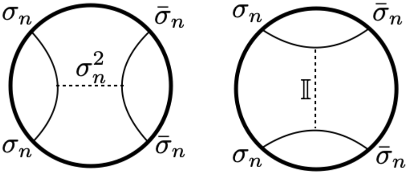
Monodromy method

[Hartman (13), Faulkner (13), Kulaxizi-Parnachev-Policastro (14)]



Geodesic Witten diagram calculation

[Hirai-Tamaoka-Yokoya (18), Prudenziati (19)]



- $$\begin{aligned}&\langle\sigma_n(x_1)\bar{\sigma}_n(x_2)\bar{\sigma}_n(x_3)\sigma_n(x_4)\rangle\\&\sim\int_{\gamma_{ij}}\int_{\gamma_{kl}}G_{b\partial}G_{b\partial}G_{bb}^{\Delta_{\sigma_n^2}}G_{b\partial}G_{b\partial}\\&\sim(|x_{12}||x_{34}|)^{-2\Delta_n}\int_{\gamma_{12}}\int_{\gamma_{34}}d\lambda d\lambda' e^{-\Delta_{\sigma_n^2}\sigma(y,y')}\end{aligned}$$

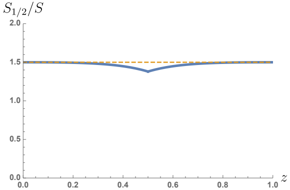
  where $\sigma(y,y')$ is the distance between bulk points y and y'.
- In the large c limit, the integral localizes at the minimal entanglement wedge:

  $$\langle\sigma_n(x_1)\bar{\sigma}_n(x_2)\bar{\sigma}_n(x_3)\sigma_n(x_4)\rangle\sim e^{-\Delta_{\sigma_n^2}\sigma_{min}}$$

  We then take the replica limit $\Delta_{\sigma_n^2}\to -c/4$.

A few more comments

- Check in other configurations; e.g., bipartite at finite temperatures, thermofield double, etc.
- Back reaction for other configurations



- Bit thread picture; cross section = maximal number of distillable Bell pairs? [Agón-de Boer-Pedraza(18)] Negativity gives upper bound of distillable entanglement.
- Applications; operator negativity.