On the relation between the magnitude and exponent of OTOCs

(Based on the paper with Alexei Kitaev [1812.00120])

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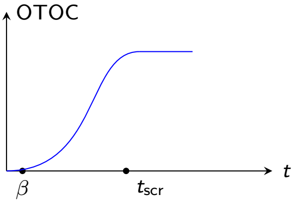
Outline

- Introduction: OTOC in SYK
- Kinetic equation approach: an algorithm that computes the Lyapunov exponent
- Ladder identity and branching time
- Applications of the ladder identity

Introduction

- OTOCs in large N systems with all-to-all interaction:



$$\text{Early time:} \quad \langle A(t)B(0)A(t)B(0)\rangle_{connected} \sim \frac{1}{N}\frac{e^{\lambda_L t}}{C}$$

- Convention $\beta = 2\pi$. MSS chaos bound $0 < \lambda_L \leqslant 1$.
- Main example: SYK $H_{SYK} = \sum_{jklm} J_{jklm}\chi_j\chi_k\chi_l\chi_m$

OTOC in SYK

- At strong coupling $J \gg 1$, SYK is almost maximally chaotic (Maldacena-Stanford, 2016)

$$\mathrm{OTOC}(t) \sim \frac{1}{N}\frac{e^{\lambda_L t}}{C}\,, \quad \lambda_L \sim 1 - \frac{c_1}{J}\,, \quad \frac{1}{C} \sim c_2 J$$

- Prefactor $\frac{1}{C}$ is big: pseudo-Goldstone mode (Schwarzian mode).

- The ratio $\frac{1-\lambda_L}{C}$ has a finite limit at $J \to \infty$.

- This talk is about an identity relating λ_L to C.
  [Corollary: although c_1, c_2 individually depends on the UV data, the product $c_1 c_2$ is universal.]

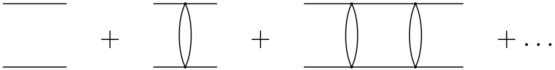
- Hope: improve the understanding of fast scrambling.

Four point function and kinetic equation

Average over fermion indices, in general depends on four times.

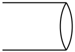
$$\text{OTOC}(\underbrace{t_1, t_2}_{future}, \underbrace{t_3, t_4}_{past}) = \frac{1}{N^2}\sum_{jk}\langle \chi_j(t_1)\chi_k(t_3)\chi_j(t_2)\chi_k(t_4)\rangle + G(t_{12})G(t_{34})$$

- At large N limit, four-point functions are dominated by ladder diagrams (Kitaev, Polchinski-Rosenhaus, Maldacena-Stanford ...)



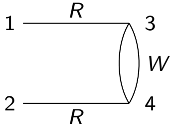
- Euclidean four-point function: Bethe-Salpeter equation

$$\mathcal{F} = \sum_n \mathcal{F}_n, \quad \mathcal{F}_n = \text{[kernel]} \cdot \mathcal{F}_{n-1} \Rightarrow \mathcal{F} = \mathcal{F}_0 + K\mathcal{F}$$



- OTOC: deform the contour to double Keldysh $\Rightarrow$ Kinetic equation

$$\text{OTOC} \approx K^R \cdot \text{OTOC}, \quad K^R(t_1, t_2, t_3, t_4):$$



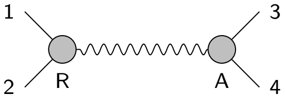
Single-mode ansatz

How does $\text{OTOC}(\underbrace{t_1, t_2}_{future}, \underbrace{t_3, t_4}_{past})$ look like?

- We are interested in time regime $t_1 \approx t_2 \gg t_3 \approx t_4$: large separation between the perturbations in the past and probes in the future. (But still before the scrambling time.)

- Single-mode ansatz ("scramblon") (Kitaev-Suh, 2017)

$$\text{OTOC}(t_1, t_2, t_3, t_4) \approx \frac{1}{N} \frac{e^{\lambda_L (t_1+t_2-t_3-t_4)/2}}{C} Y^R(t_{12}) Y^A(t_{34})$$



$Y^{R/A}$: vertex function

Eigenvalue $k_R(\alpha)$

The kinetic equation OTOC $\approx K^R$ OTOC means OTOC is an eigenvector of K^R with eigenvalue $k_R = 1$. In general finding OTOC is a complicated search. We can reduce it to a shooting problem:

- Guess a growing exponent α for scramblon $e^{\alpha t}$;
- Adjust vertex function such that the trial OTOC(α) is an eigenvector of K^R with general eigenvalue $k_R(\alpha)$;
- Find α_* such that $k_R(\alpha_*) = 1 \Rightarrow \lambda_L = \alpha_*$.

Prefactor

This algorithm is useful in finding λ_L (as well as vertex functions), but the prefactor is ambiguous

- Because the kinetic equation $\text{OTOC} \approx K^R\, \text{OTOC}$ is a homogeneous linear equation
- Origin of the ambiguity: we dropped the constant term F_0 in the exact equation $\text{OTOC} = K^R\, \text{OTOC} + F_0$
- We would like to get the correct prefactor with the data $(\lambda_L, Y^{R/A}, k_R(\alpha))$ we already have

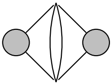
Ladder identity

$$\mathrm{OTOC}(t_1,t_2,t_3,t_4) \approx \frac{1}{N}\frac{e^{\lambda_L(t_1+t_2-t_3-t_4)/2}}{C}\,Y^R(t_{12})Y^A(t_{34})$$

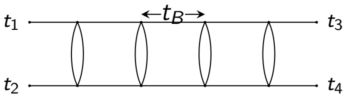
- Ladder identity:

$$\frac{2\cos\frac{\lambda_L\pi}{2}}{C}\cdot(-k_R'(\lambda_L))\cdot\left(Y^A,Y^R\right)=1\,.$$

- $\left(Y^A,Y^R\right)$: inner product of vertex functions:

$\left(Y^A,Y^R\right) =$  $= (q-1)J^2\int dt\, Y^A(t)\left(G^W(t)\right)^{q-2}Y^R(t)\,.$

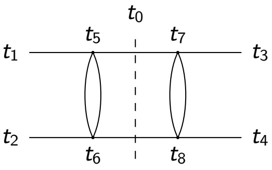
- Branching time $t_B := -k_R'(\lambda_L)$. Average separation between adjacent rungs.



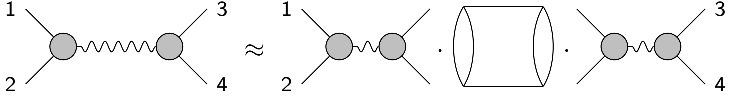
Idea of the derivation

Idea: cut a long ladder into pieces and find a consistency condition.

- Cut in the middle, find adjacent $\frac{t_5+t_6}{2} < t_0 < \frac{t_7+t_8}{2}$



- Cut-gluing consistency condition



- Left hand side contains one copy of C^{-1}, right hand side has two copies of C^{-1}.

Applications

The ladder identity:

$$\frac{2\cos\frac{\lambda_L \pi}{2}}{C} \cdot t_B \cdot \left(Y^A, Y^R\right) = 1\,.$$

Applications:

- Computational shortcuts: $C \Leftrightarrow \lambda_L$;
- In a 1D model, exact maximal chaos.

Computational shortcuts

- $C \Rightarrow \lambda_L$: for SYK at strong coupling, the prefactor C is determined by the coefficient of the Schwarzian action α_S. However, the Lyapunov exponent will receive small correction from the conformal matters (Maldacena-Stanford, 2016)

$$\delta\lambda_L \approx \frac{6\alpha_S}{Jk_R'(-1)\Delta(1-\Delta)(1-2\Delta)\tan(\pi\Delta)} .$$

- $\lambda_L \Rightarrow C$: for q-body interacting SYK model at $q \to \infty$ limit, the exact value of λ_L is easy to find using the kinetic equation (Maldacena-Stanford, 2016),

$$\frac{\lambda_L}{2\cos\frac{\pi\lambda_L}{2}} = \mathcal{J} .$$

But the prefactor is hard to compute. The ladder identity gives a result consistent with the computation using Liouville equation (Qi-Streicher, 2018).

$$\text{OTOC}(t_1, t_2; t_3, t_4) \approx \frac{1}{N\cos\frac{\lambda_L\pi}{2}} \frac{e^{\lambda_L(t_1+t_2-t_3-t_4)/2}}{\left(2\cosh\frac{\lambda_L t_{12}}{2}\right)\left(2\cosh\frac{\lambda_L t_{34}}{2}\right)} .$$

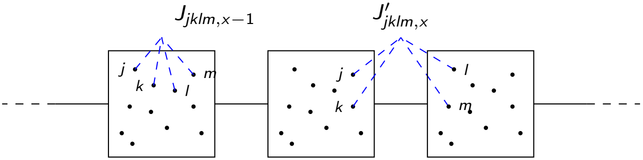
Maximal chaos in a 1D model

- Idea: regard λ_L and C as analytic functions of some parameter,

$$\frac{2\cos\frac{\lambda_L\pi}{2}}{C}\cdot t_B\cdot\left(Y^A, Y^R\right)=1\,.$$

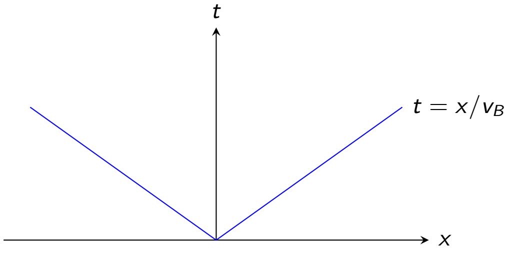
then the analytical properties of λ_L and C are locked by the ladder identity.

- A concrete example: SYK chain (YG-Qi-Stanford, 2016)



Operators at two different locations

Spatial propagation of chaos: measured by $\langle A(x,t)B(0,0)A(x,t)B(0,0)\rangle$



- Fourier transform:

$$\mathrm{OTOC}(x,t) = \int \frac{dp}{2\pi} e^{ipx}\, \mathrm{OTOC}(p,t)$$

- Each $\mathrm{OTOC}(p,t)$ can be solved using retarded kernel approach.

Fourier Transform

- The ladder identity holds for each OTOC(p, t):

$$C(p) = 2\cos\frac{\lambda_L(p)\pi}{2} \cdot t_B \cdot (Y^A, Y^R)\,,$$

- The dependence of t_B and (Y^A, Y^R) on p is not important (analytic and do not vanish in the domain of interest).

$$\mathrm{OTOC}(x,t) \sim \frac{1}{N} \underbrace{\int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{e^{\lambda_L(p)t + ipx}}{2\cos\frac{\pi\lambda_L(p)}{2}}}_{u(x,t)} \cdot \frac{Y^R Y^A}{t_B(Y^A, Y^R)}\,.$$

Butterfly wavefront

$$u(x,t) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{e^{\lambda_L(p)t + ipx}}{2\cos\frac{\lambda_L(p)}{2}}$$

- Butterfly wavefront: $u(x,t) \sim 1$.
- For large $x > 0$ and t, we can estimate the asymptotics by saddle point of the exponent.

$$\lambda_L'(p)t + ix = 0, \quad p = i|p|.$$

- The relevant saddle point $p_s = i|p_s|$ is purely imaginary: deform the integral contour to pass. Warning: pole contribution!

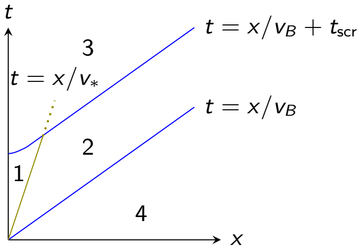
$$\cos\frac{\lambda_L(p_1)\pi}{2} = 0, \quad \lambda_L(p_1) = 1, \quad p_1 = i|p_1|\,.$$

Pole contribution: maximal chaos

- When $J > J_c$, the pole dominates the butterfly wavefront:

$$u_1(x,t) \approx \frac{e^{t-|p_1||x|}}{\pi i \lambda_L'(p_1)}, \quad v_1 = \frac{1}{|p_1|}$$

- Four regions with different OTOC behavior.



Summary and discussion

An identity relates C and λ_L;

$$\frac{2\cos\frac{\lambda_L \pi}{2}}{C} \cdot t_B \cdot \left(Y^A, Y^R\right) = 1\,, \quad t_B = (-k_R'(\lambda_L))$$

Some lessons from the applications:

- Computational shortcuts: in SYK model we find the correction $\delta\lambda_L \propto t_B^{-1}$. If we call $\lambda_L = 1$ the "coherent scrambling" (Schwarzian, gravity...), then the branching time t_B measures the "decoherence" effect.

- Maximal chaos: spatial locality provides a mechanism to enhance the Lyapunov exponent. Ladder identity: the enhancement cancels the correction exactly

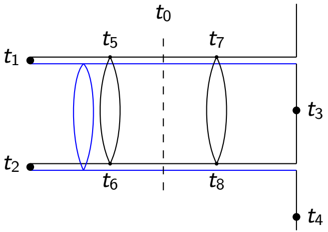
Thank you!

Derivation: $\cos\frac{\lambda_L \pi}{2}$ factor

- Naively, we would have a formula:

$$\text{OTOC} \approx \text{OTOC}_L \cdot \text{BOX} \cdot \text{OTOC}_R$$

- Subtlety: multiple choices on the double Keldysh contour

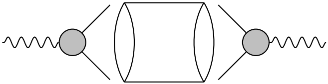


- Sum of two choices:

$$\text{OTOC} \approx \left(e^{i\frac{\lambda_L \pi}{2}} + e^{-i\frac{\lambda_L \pi}{2}}\right) \text{OTOC}_L \cdot \text{BOX} \cdot \text{OTOC}_R$$

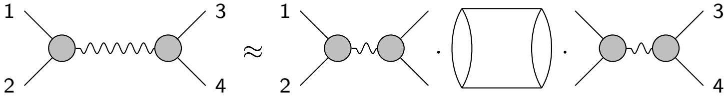
Derivation

- Gluing OTOC_L and OTOC_R through a box:



$$= t_B\left(Y^A, Y^R\right)$$

- Now inserting the single mode ansatz to the consistency condition $\mathrm{OTOC} \approx \left(e^{i\frac{\lambda_L \pi}{2}} + e^{-i\frac{\lambda_L \pi}{2}}\right) \mathrm{OTOC}_L \cdot \mathrm{BOX} \cdot \mathrm{OTOC}_R$



- Compare two sides (stripping off the vertex functions at two ends)

$$\frac{1}{C} = 2\cos\frac{\lambda_L \pi}{2} \cdot \frac{1}{C} \cdot t_B \cdot \left(Y^A, Y^R\right) \cdot \frac{1}{C}$$