Estimating the entropy of shallow circuit outputs is hard

arXiv:2002.12814

Alexandru Gheorghiu   Matty J. Hoban

ETH Zürich   California Institute of Technology 1891   Goldsmiths University of London

Entropy in (quantum) information theory

For some probability distribution $p : \{0,1\}^n \to [0,1]$

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Shannon entropy

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For some quantum state $\rho \in \mathcal{D}(\mathbb{C}^{\otimes n})$

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Von Neumann entropy

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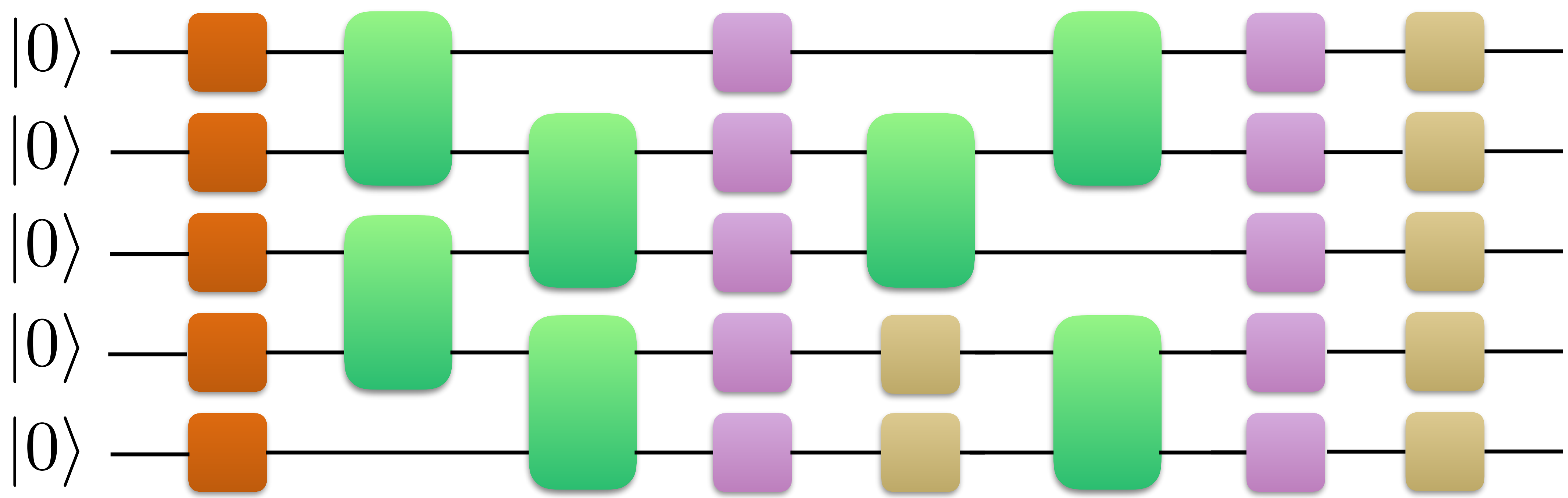
Von Neumann entropy

$$0 \leq S \leq n$$

Entropy estimation

Given description of …



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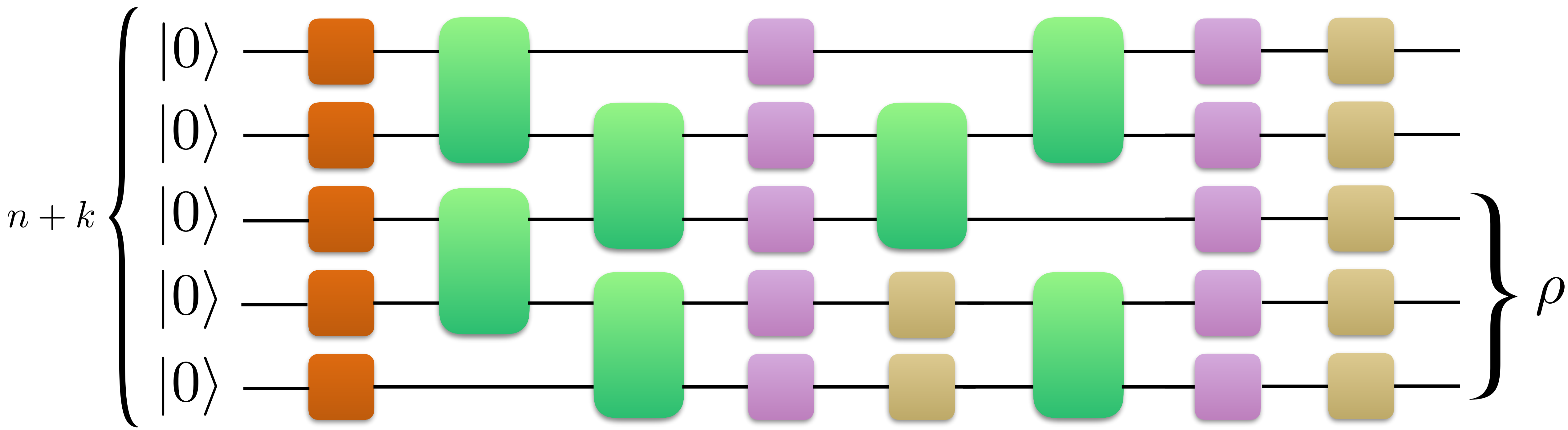

$$\rho \in \mathcal{D}(\mathbb{C}^{\otimes n})$$

Entropy estimation

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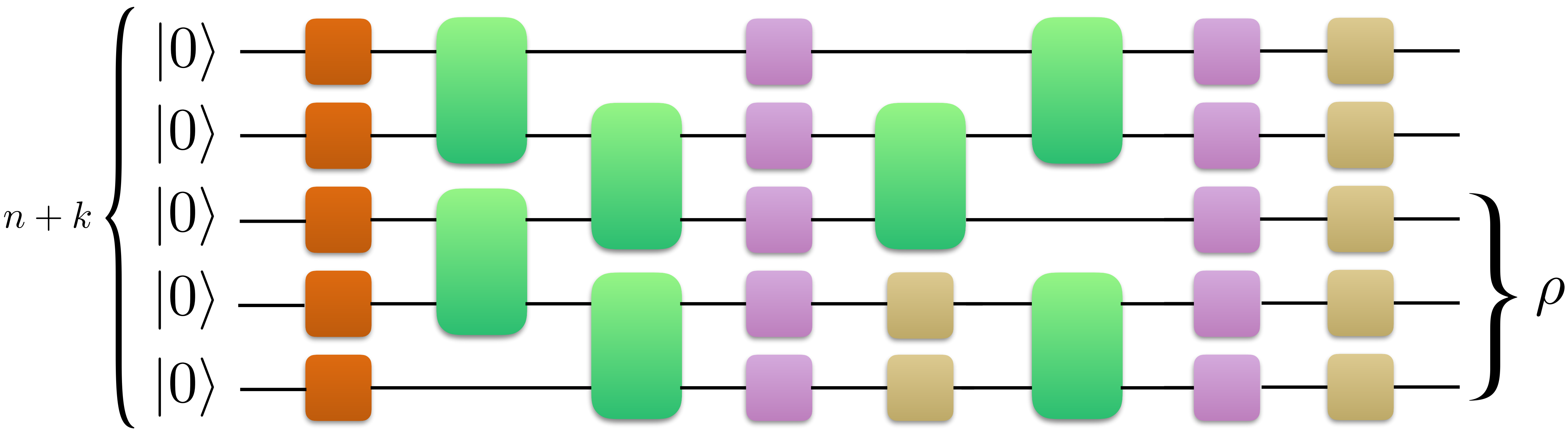

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(can similarly define classical case)

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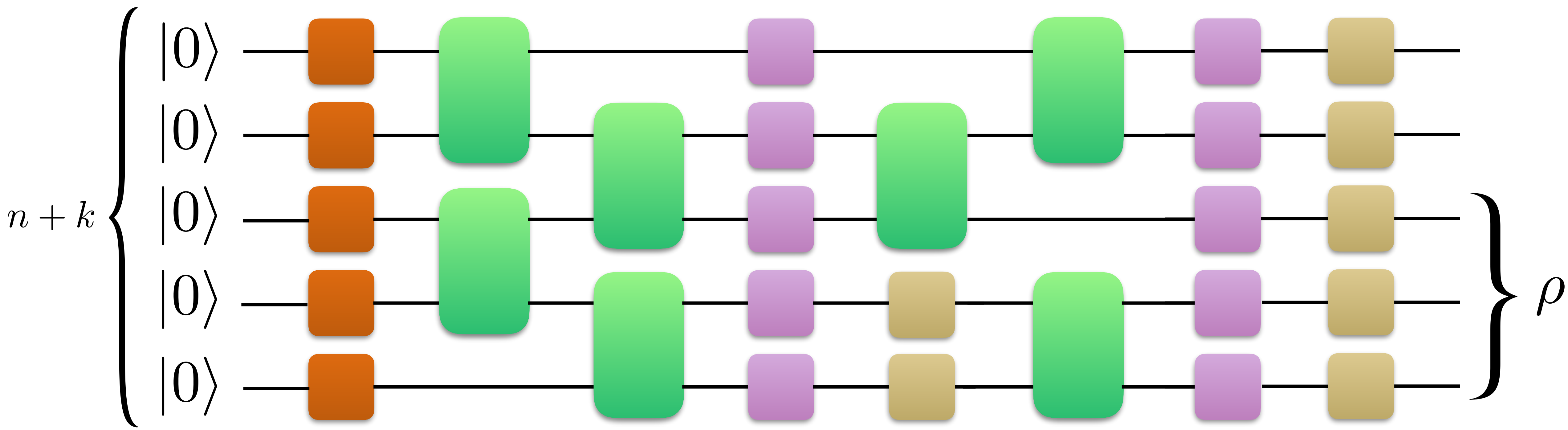


$$\rho \in \mathcal{D}(\mathbb{C}^{\otimes n})$$

(can similarly define classical case)

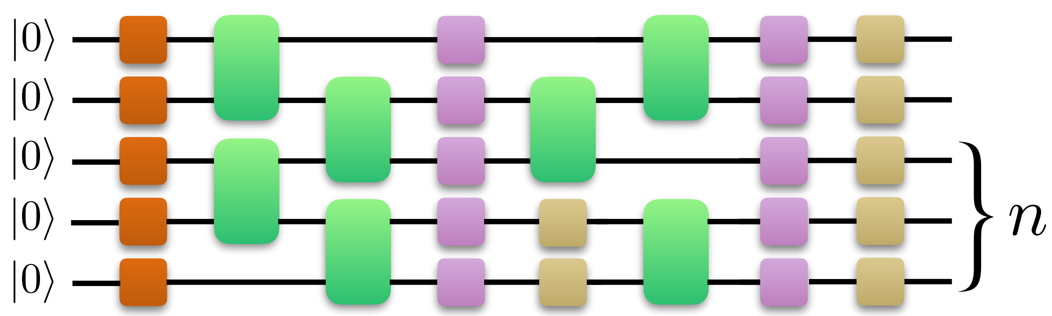
compute an entropy estimate $\hat{S}$

$$S(\rho) - 0.1 \leq \hat{S} \leq S(\rho) + 0.1$$

What is a hard problem?

Quantum Algorithm

What is a hard problem?



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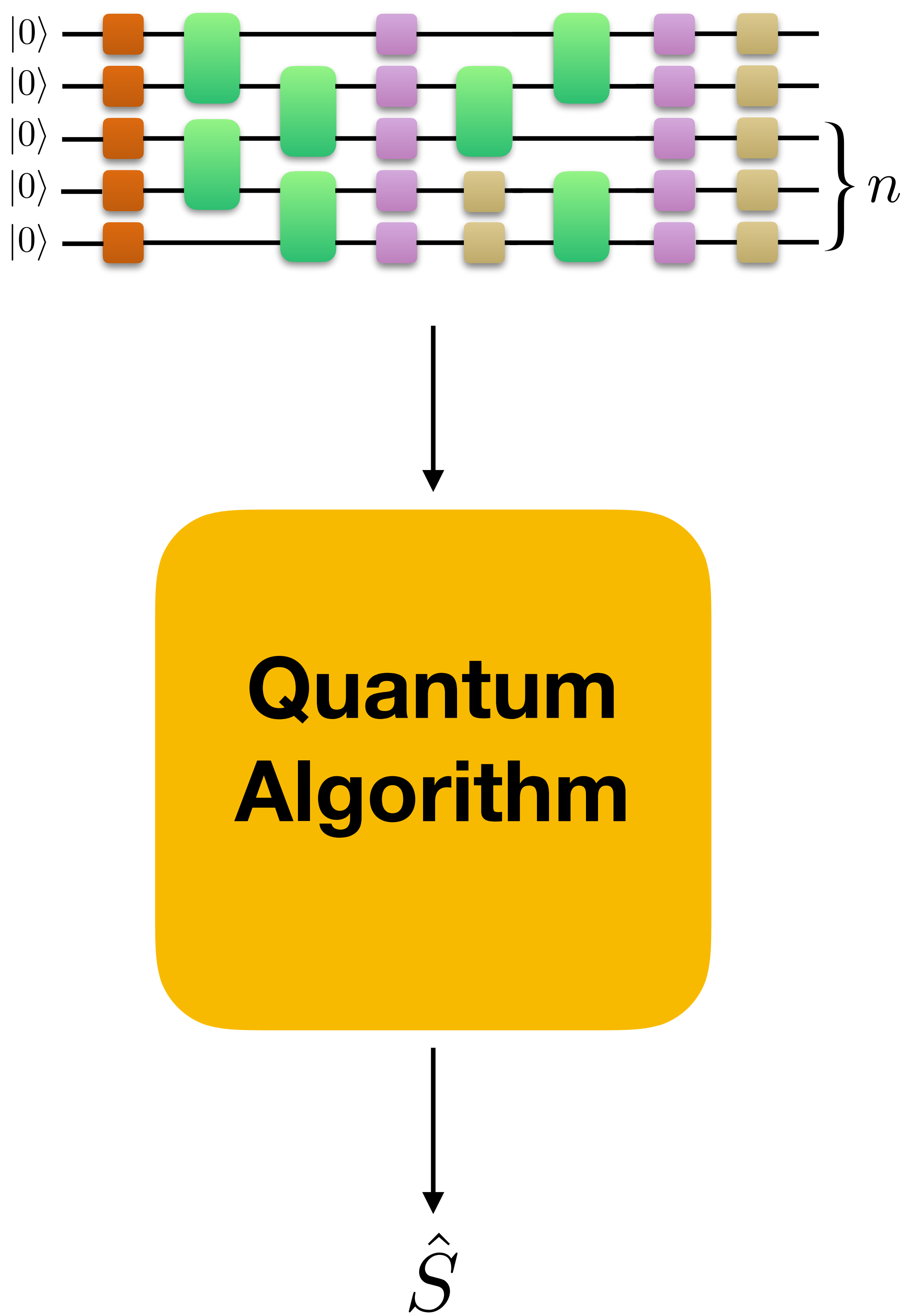

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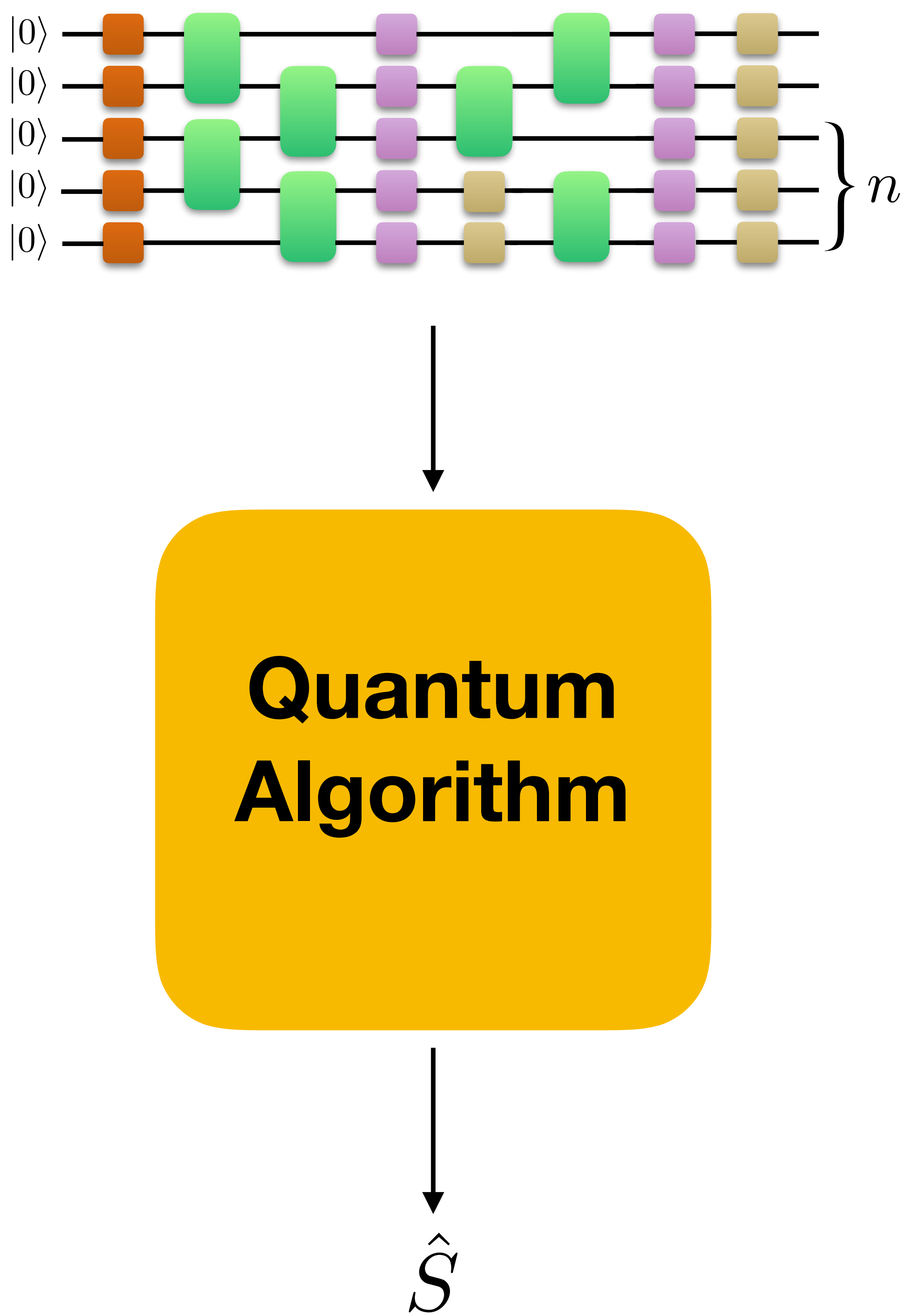

How much time does the *fastest* algorithm require as a function of n?

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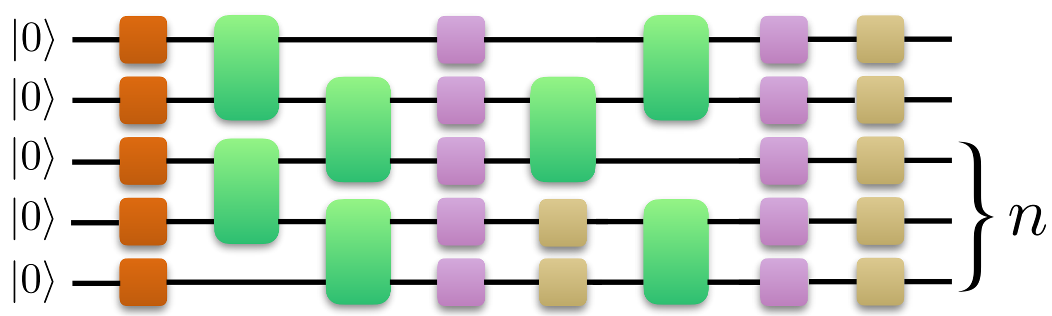

Quantum Algorithm

$\hat{S}$

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$poly(n)$ ⟶ **Efficient**

Otherwise ⟶ **Inefficient**

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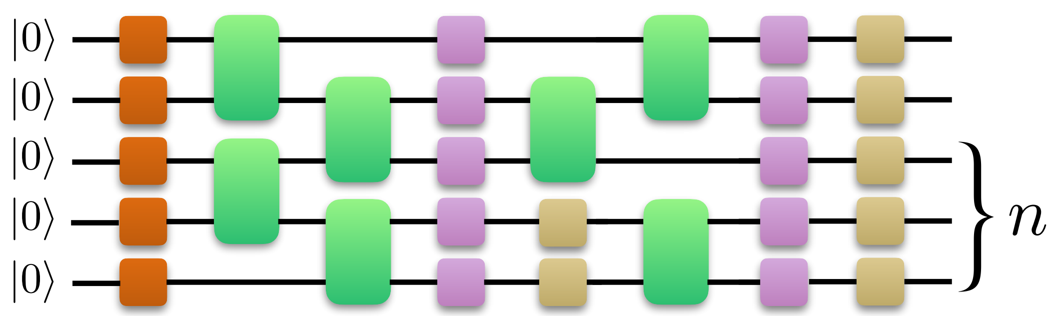

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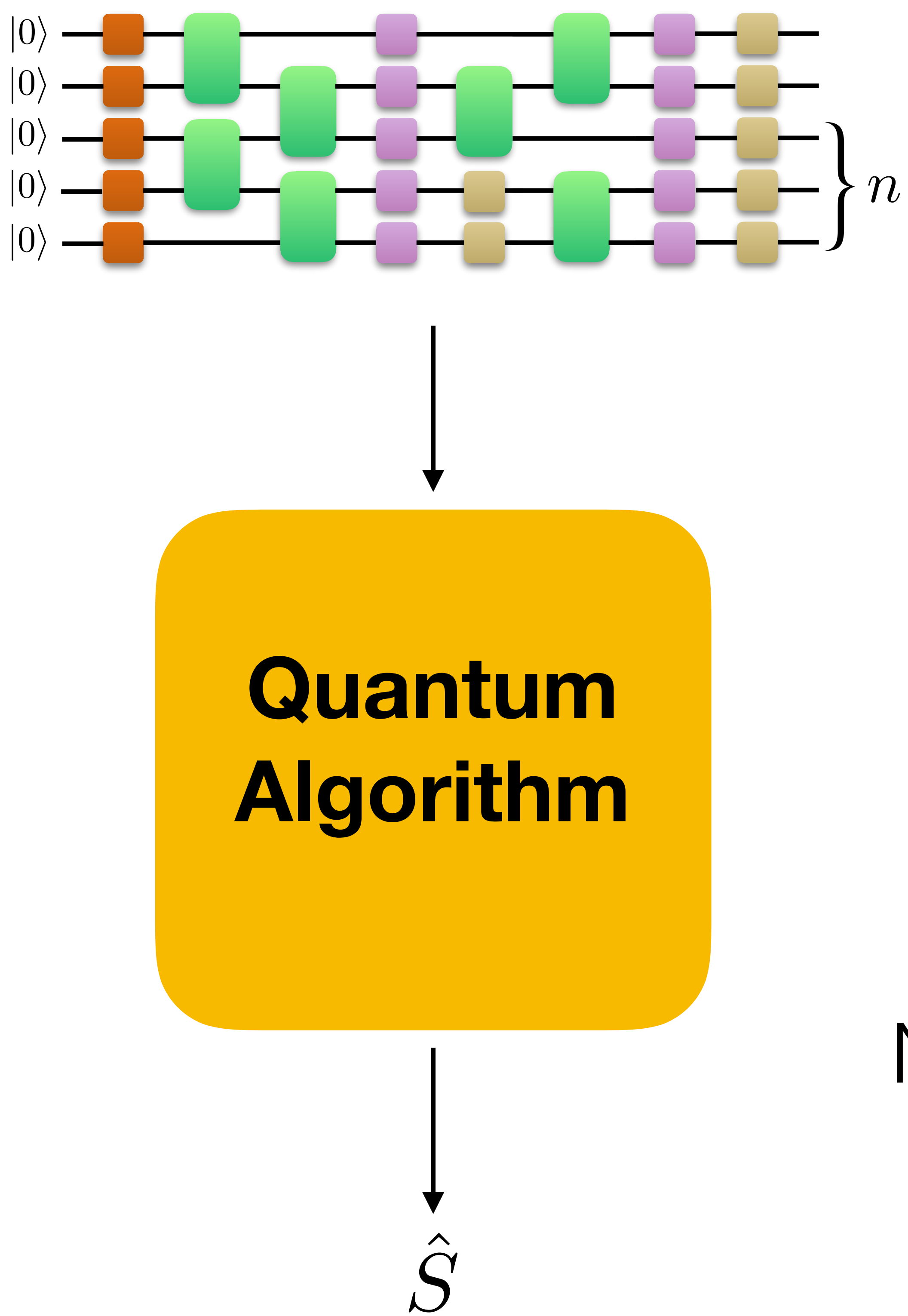

How much time does the *fastest* algorithm require as a function of n?

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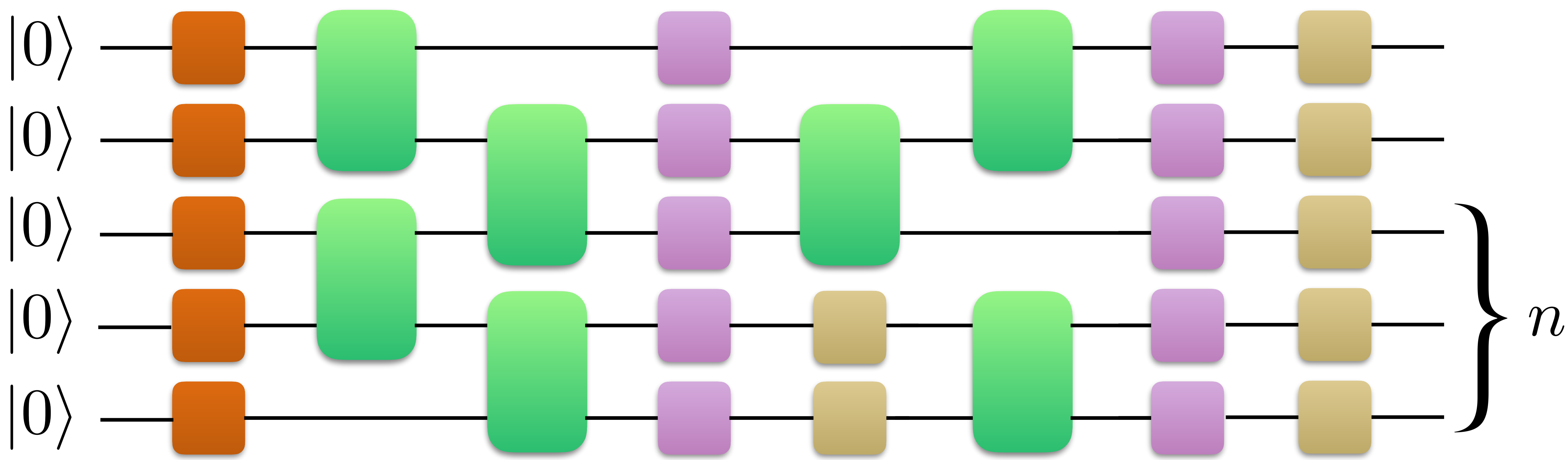
Otherwise ⟶ **Inefficient**

(problem is *hard*)

Naive approaches are inefficient

$2^{O(n)}$ time

Hardness of entropy estimation



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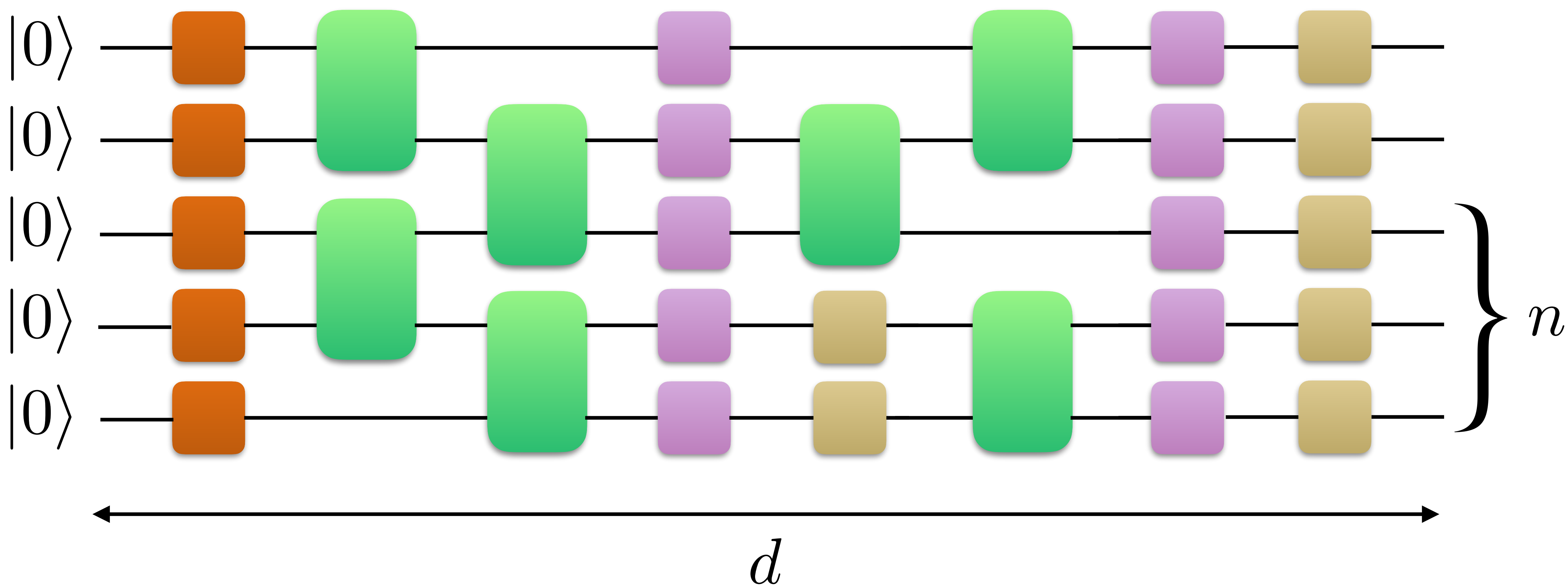

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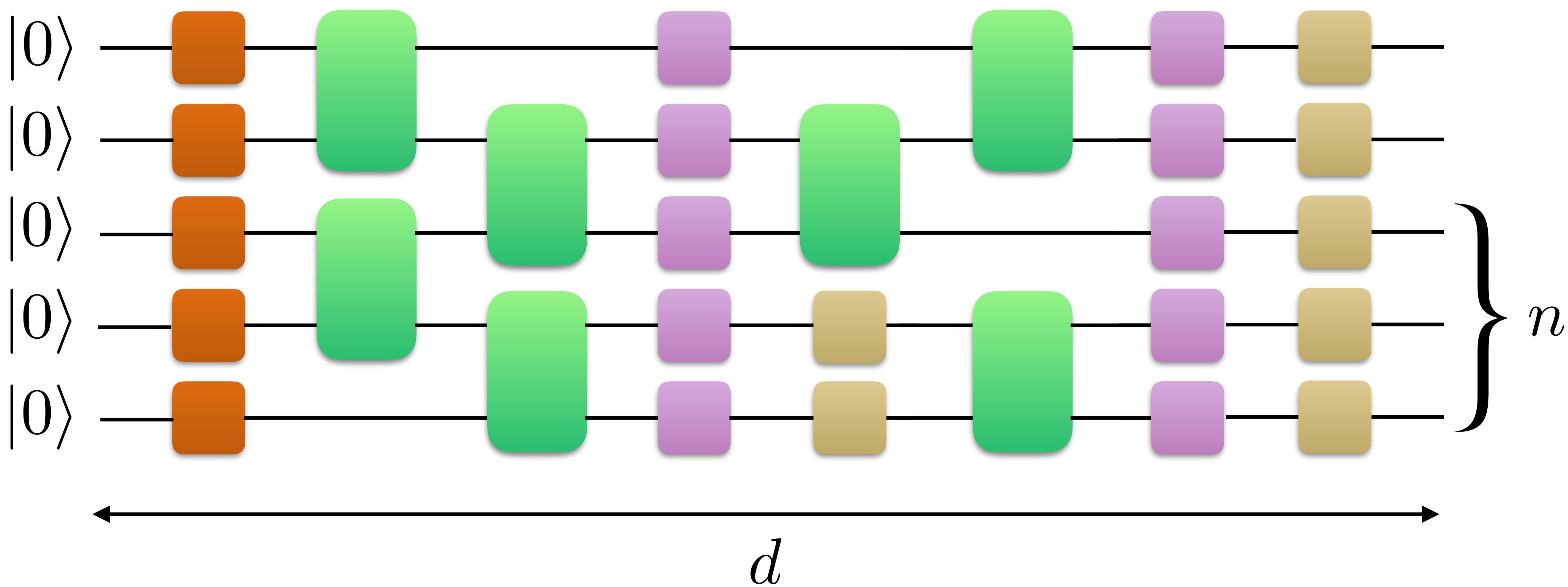
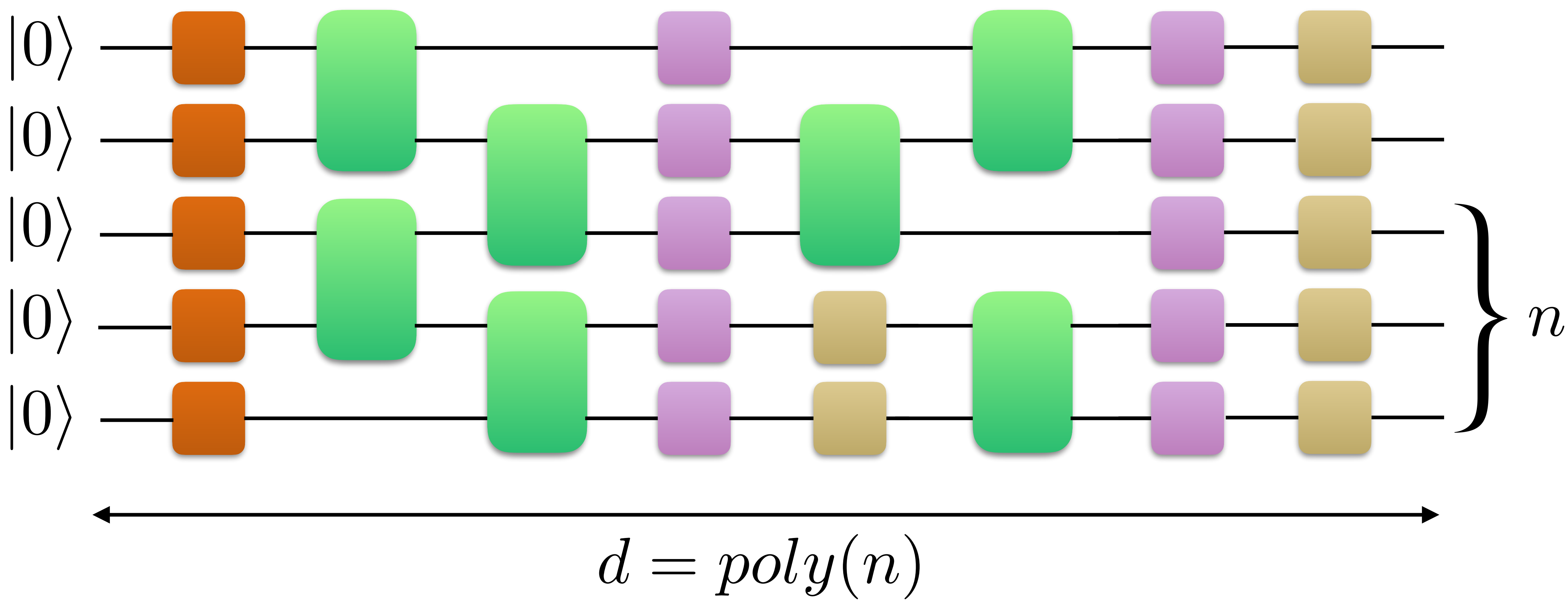


Assume circuit width is always poly(n)

Hardness of entropy estimation



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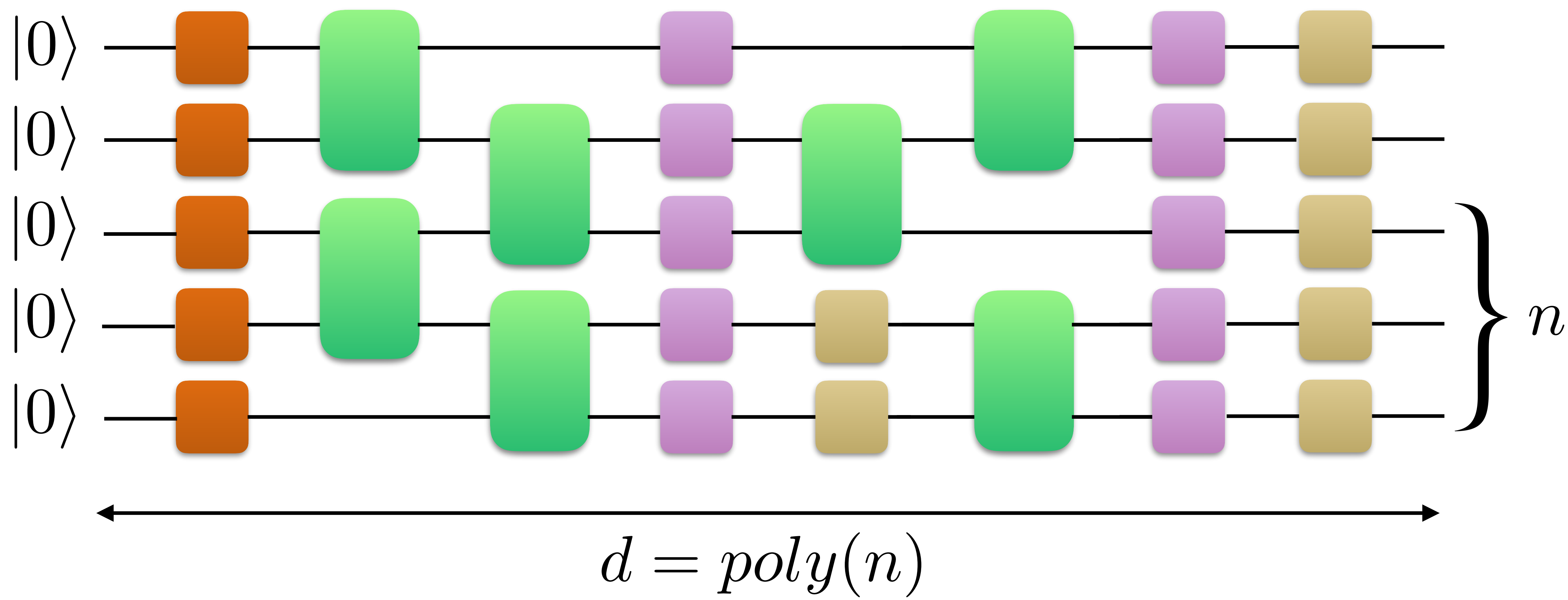
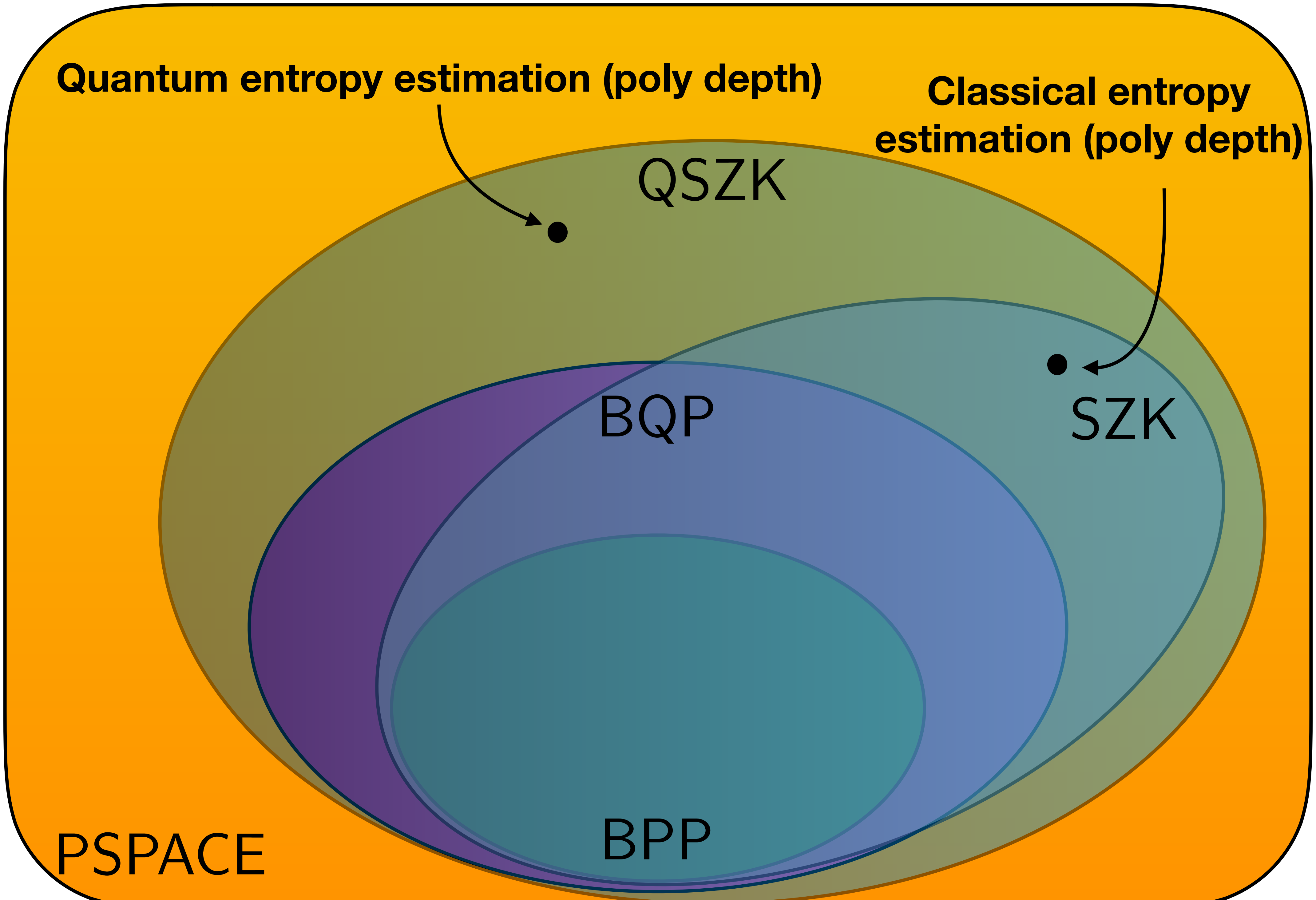


Hard based on plausible complexity-theoretic conjectures

[Goldreich, Vadhan '99]

[Ben-Aroya, Schwartz, Ta-Shma '10]

Hardness of entropy estimation



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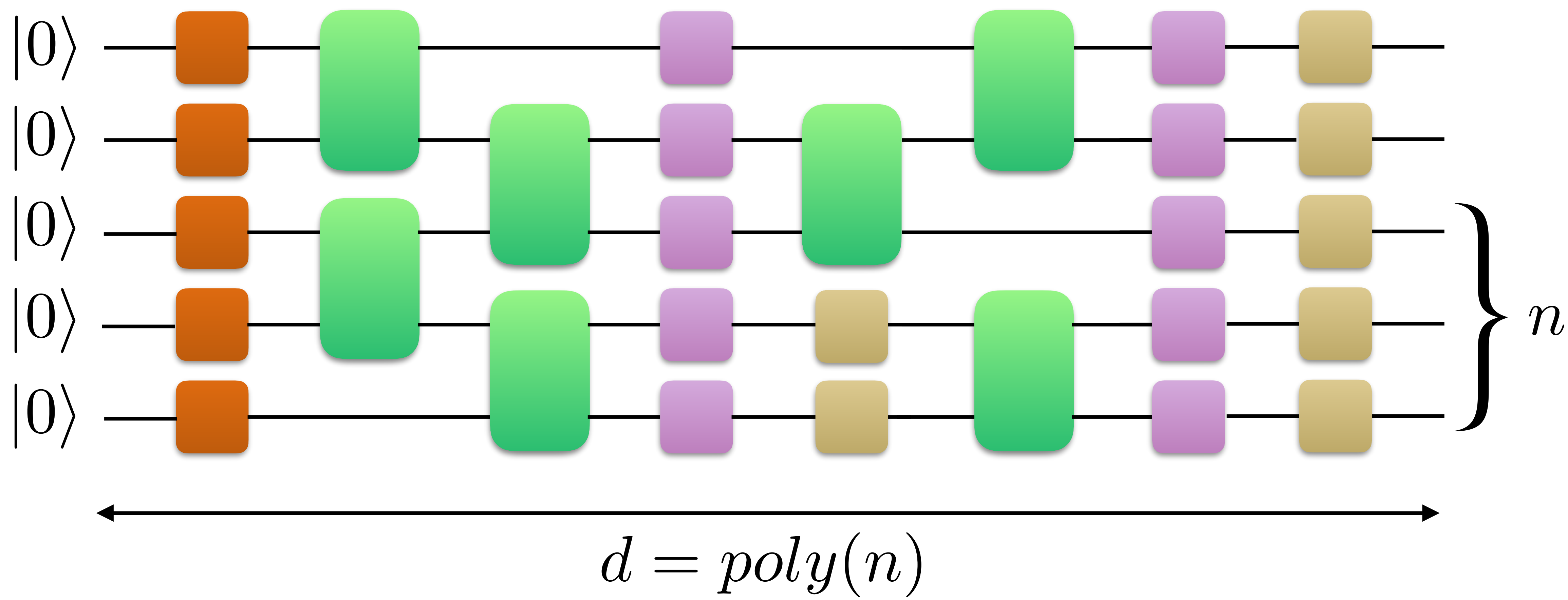


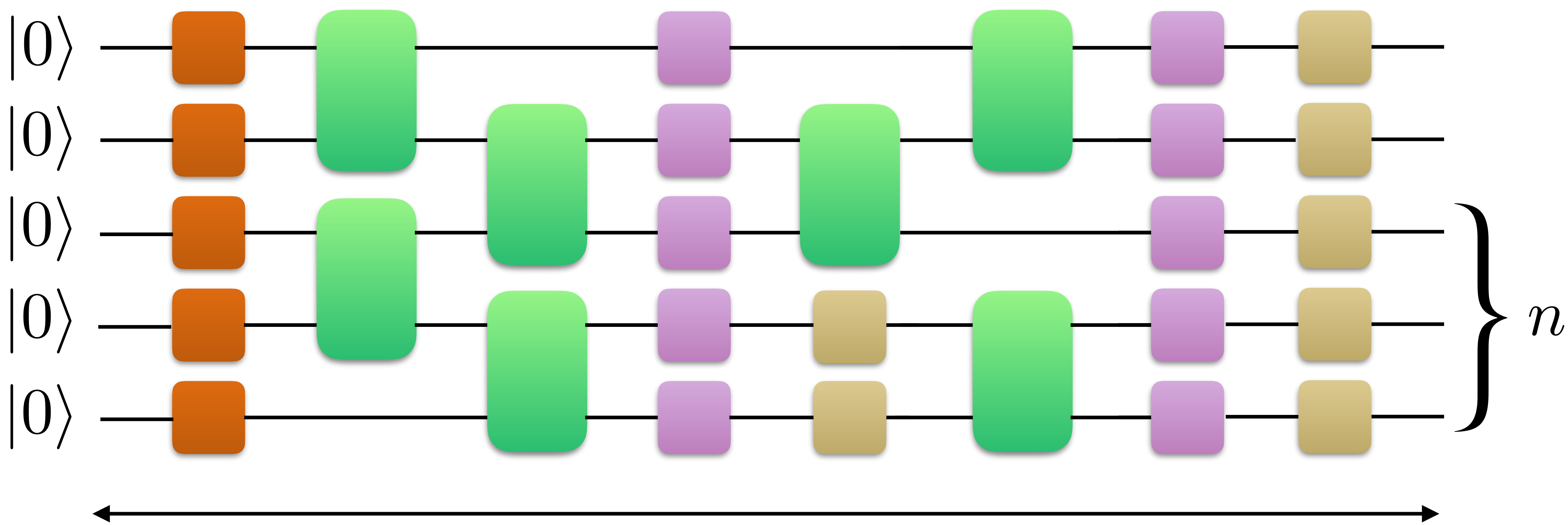
Hard based on plausible complexity-theoretic conjectures

(at least as hard as finding collisions for a function)

[Goldreich, Vadhan '99]

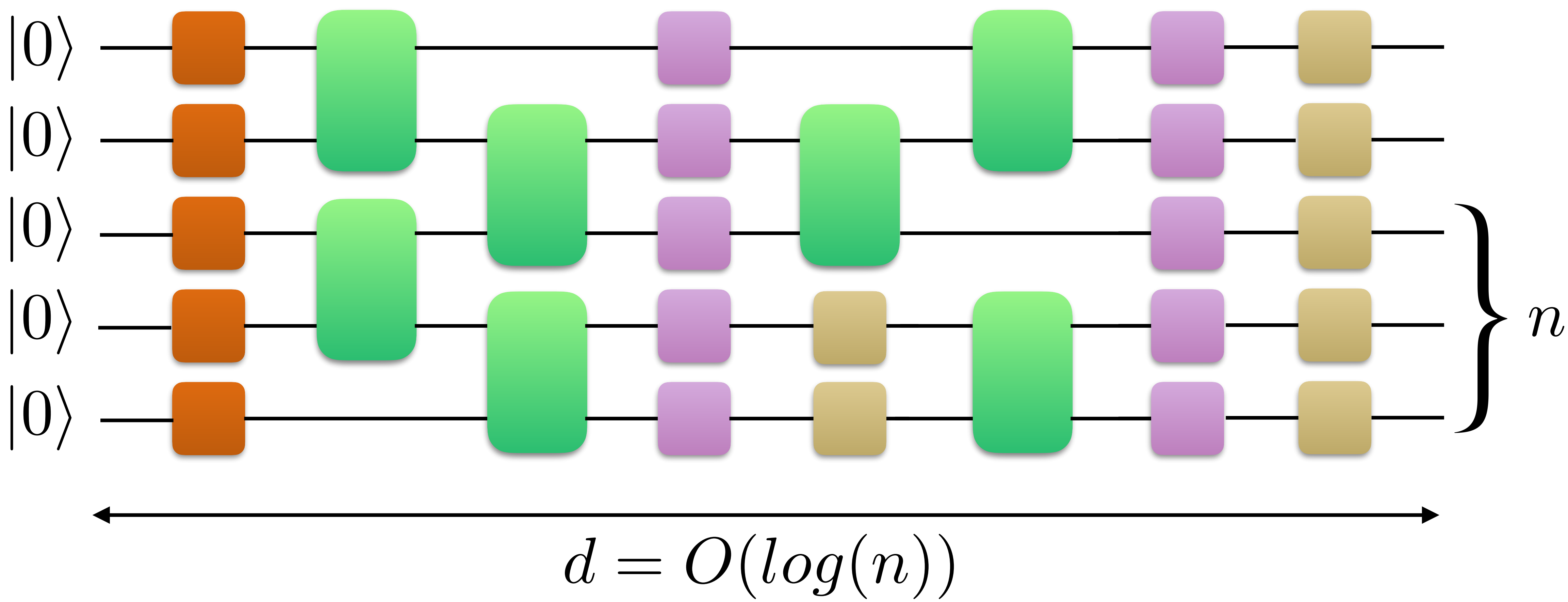
[Ben-Aroya, Schwartz, Ta-Shma '10]

Our results



$$d = O(log(n))$$

Our results



As hard as the *Learning With Errors (LWE)* problem

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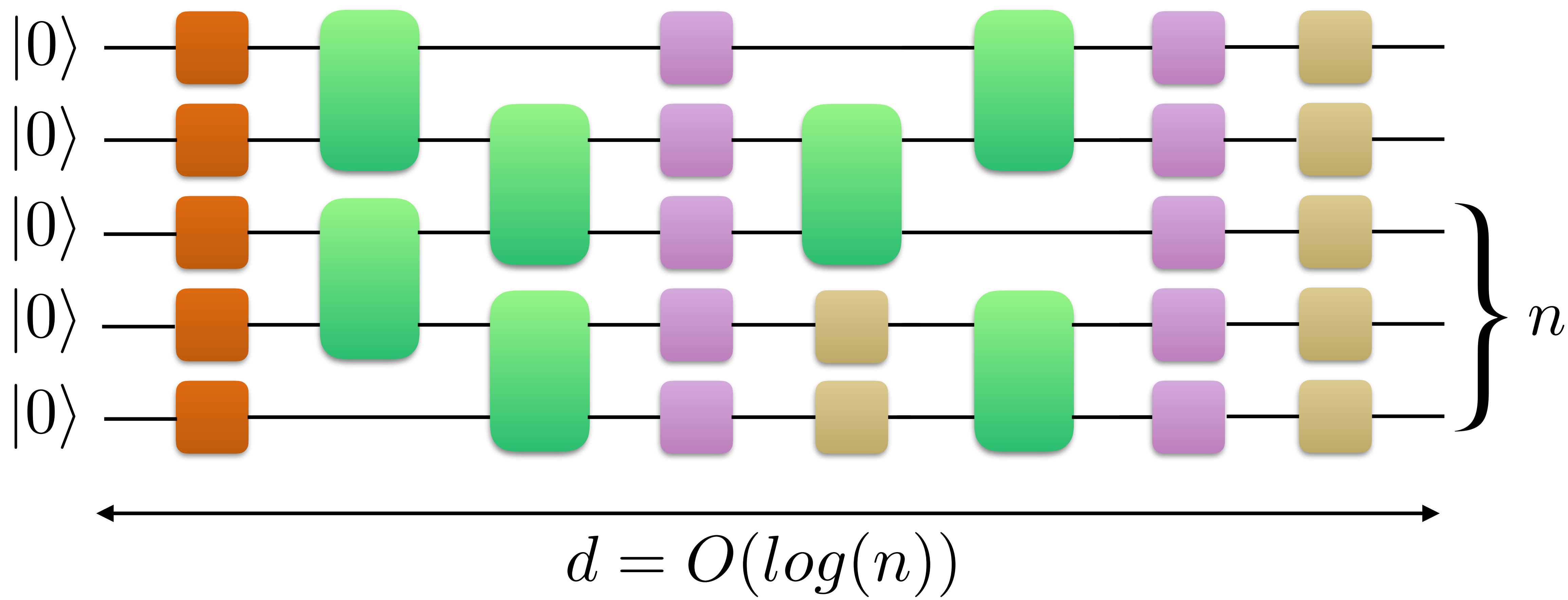

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LWE is a candidate problem for post-quantum cryptographic protocols

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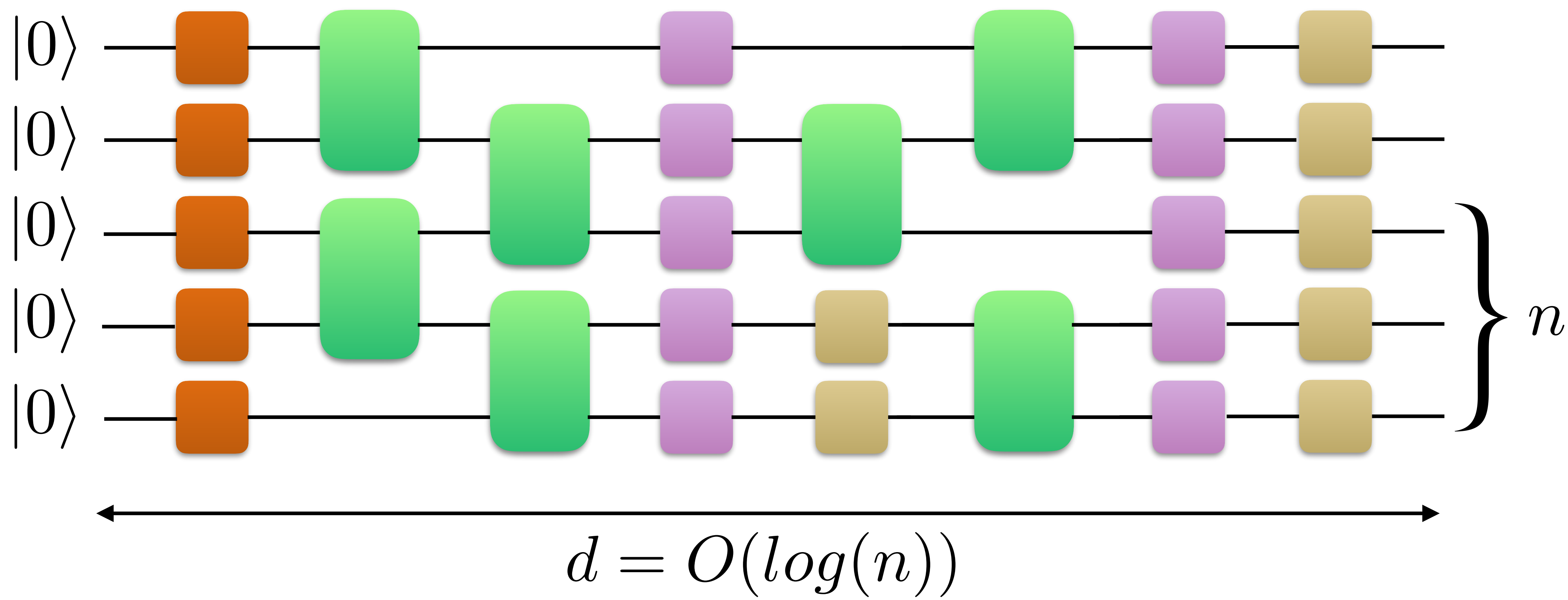
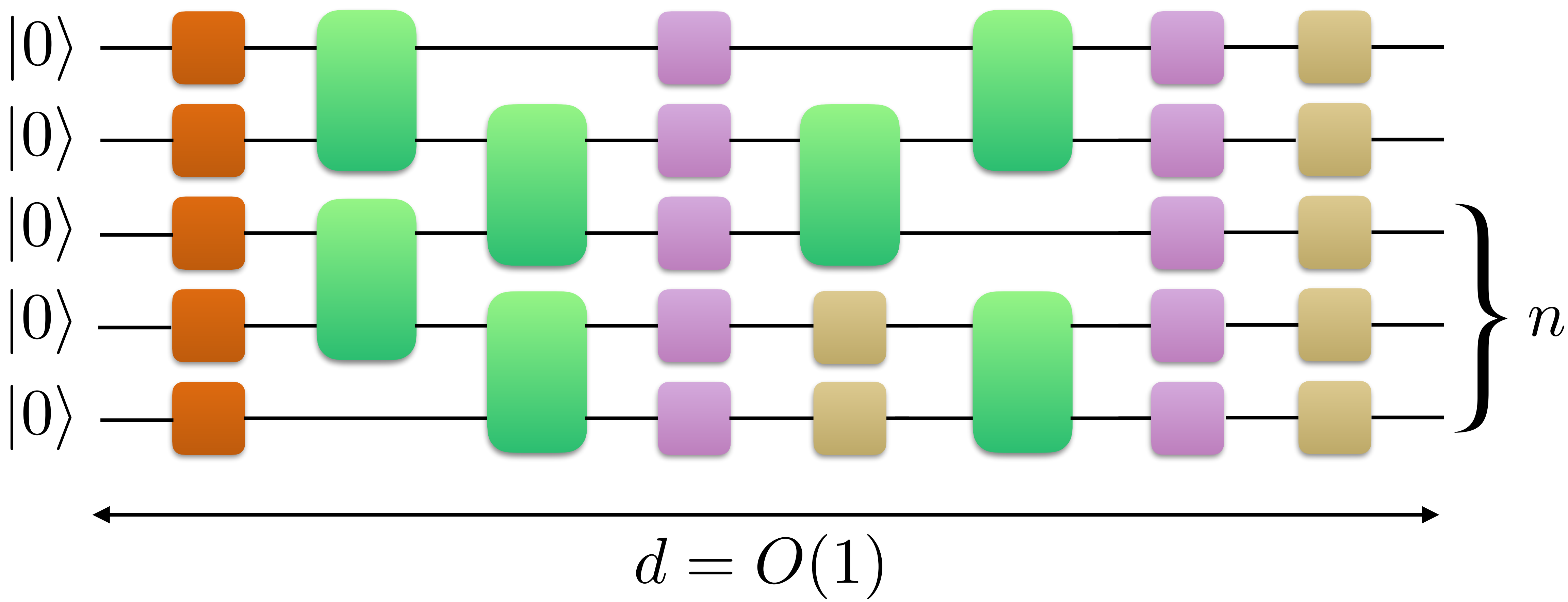


As hard as the *Learning With Errors (LWE)* problem

LWE is a candidate problem for post-quantum cryptographic protocols

Best known (quantum) algorithms require exponential time

Our results



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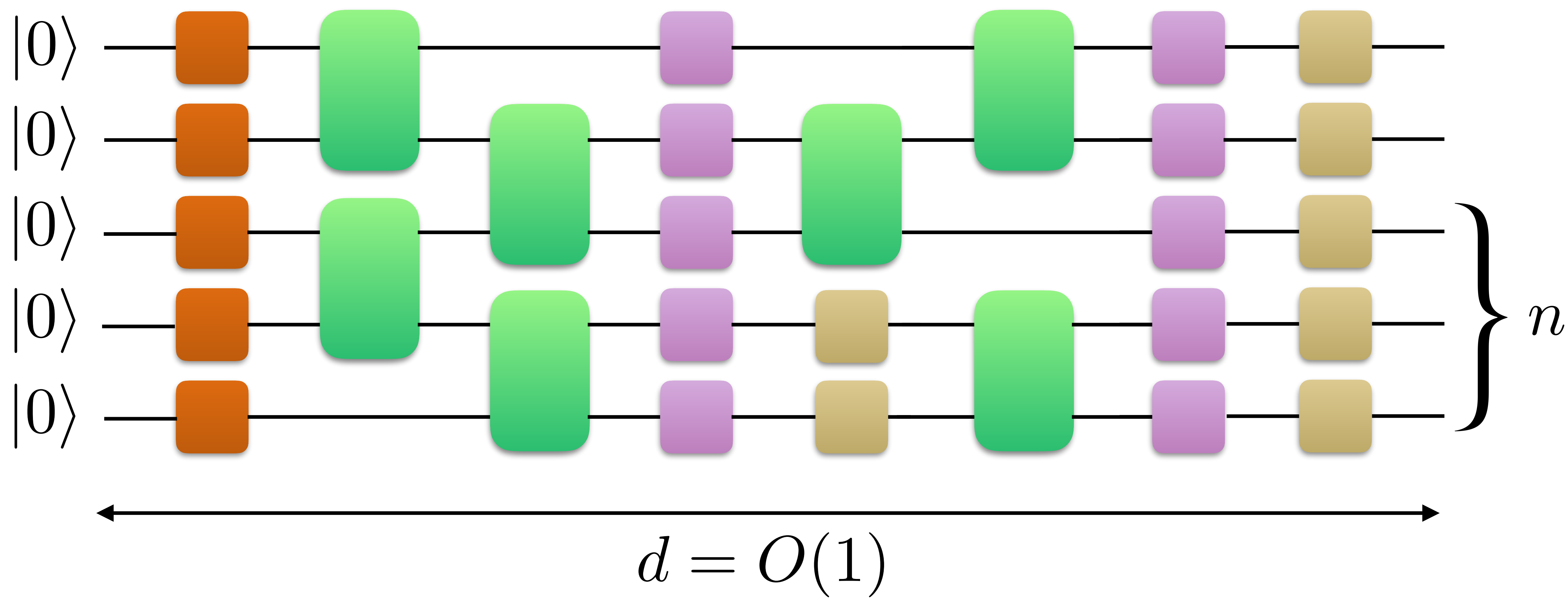

Classical circuit case is easy!

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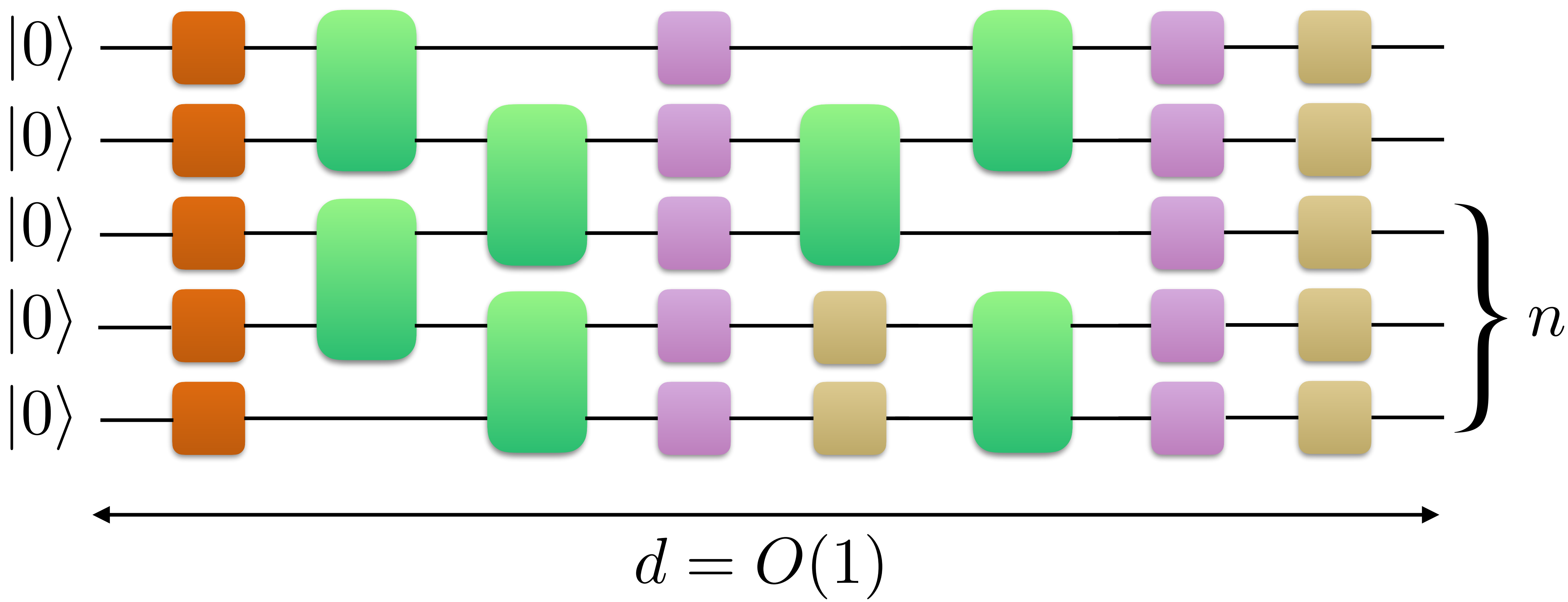

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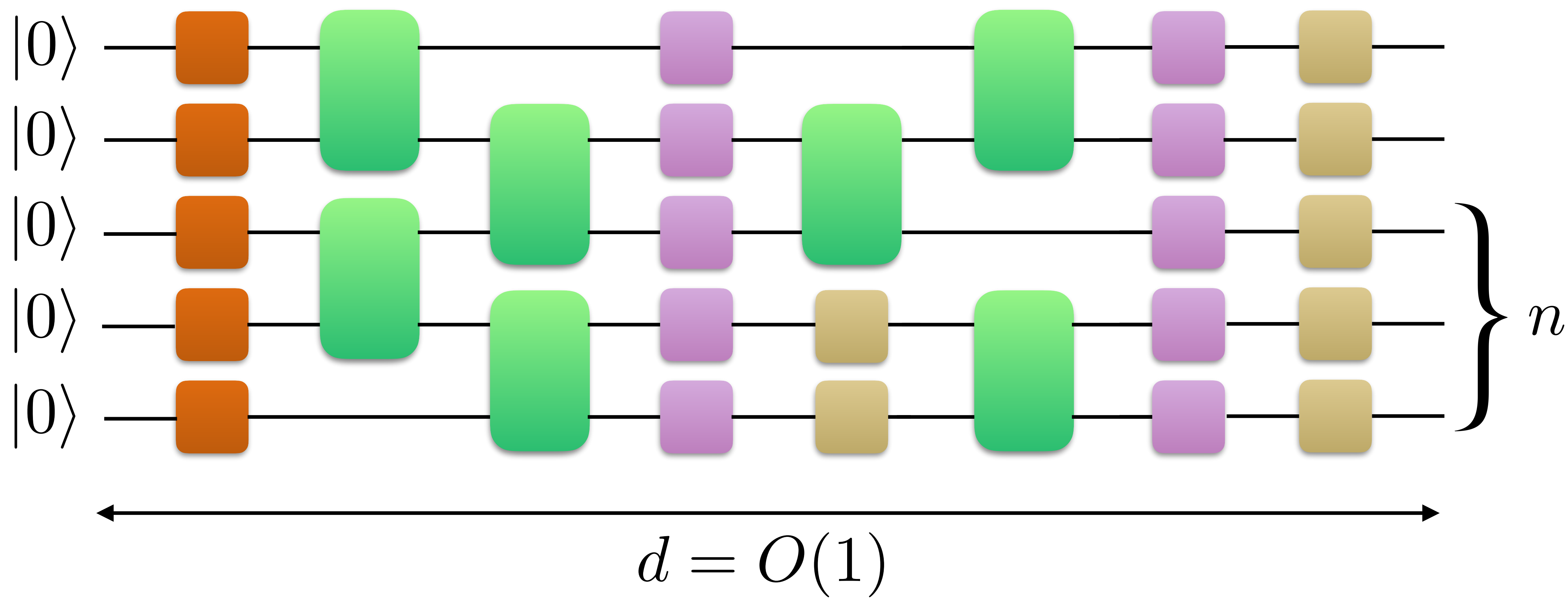
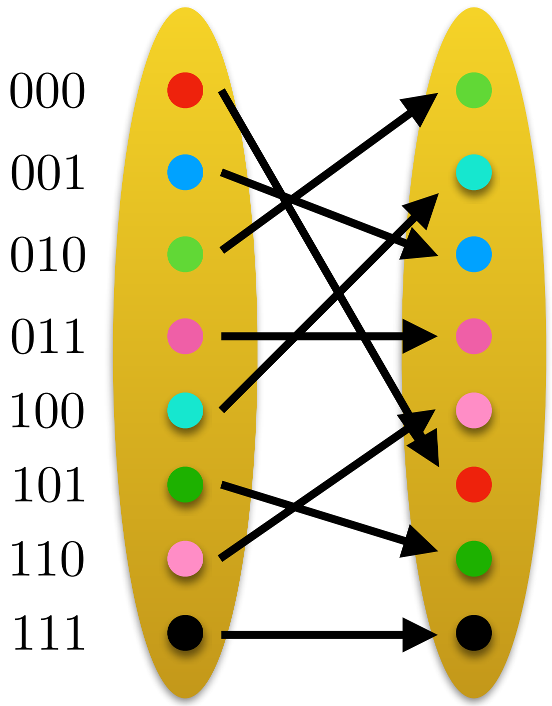


Classical circuit case is easy!

Quantum circuit case remains as hard as LWE

(requires arbitrary rotation gates)

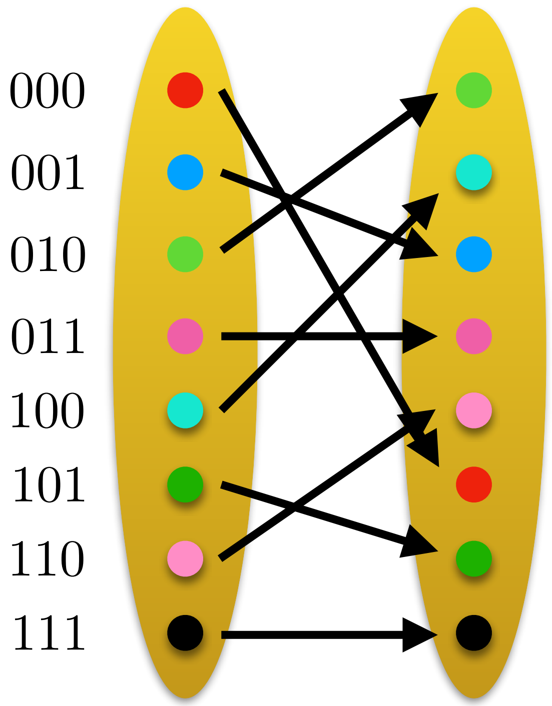
Idea



$$f : \{0,1\}^n \to \{0,1\}^n$$

1-to-1

Idea



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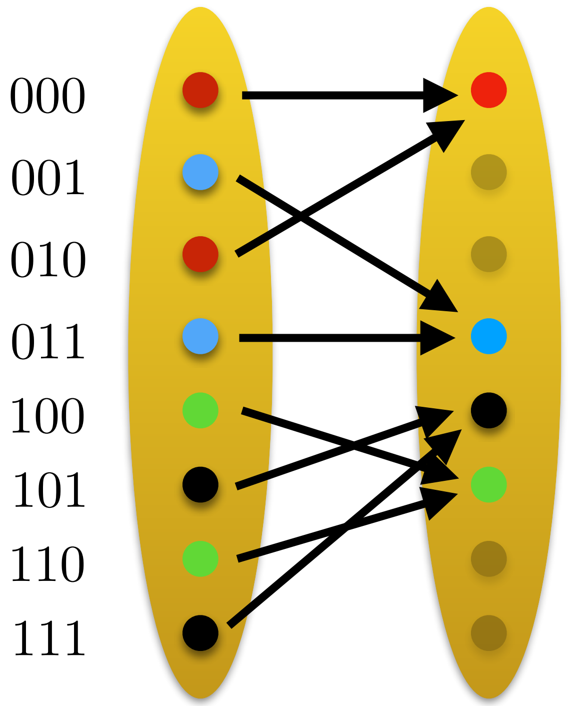
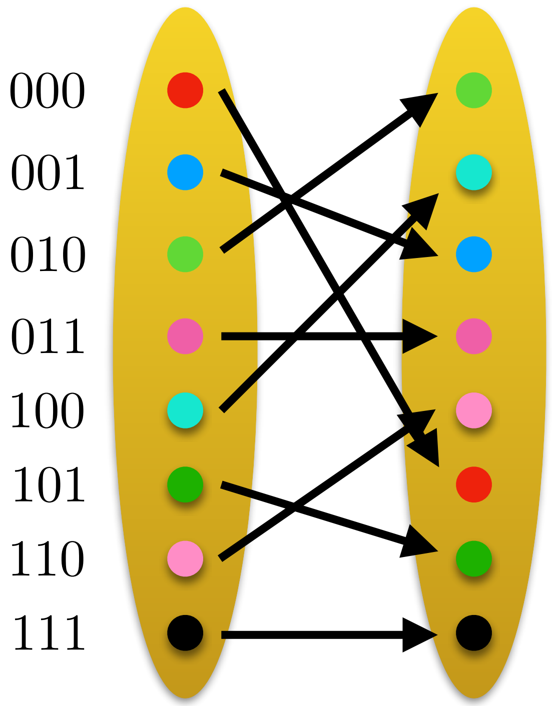


$$g : \{0,1\}^n \to \{0,1\}^n$$

2-to-1

Idea



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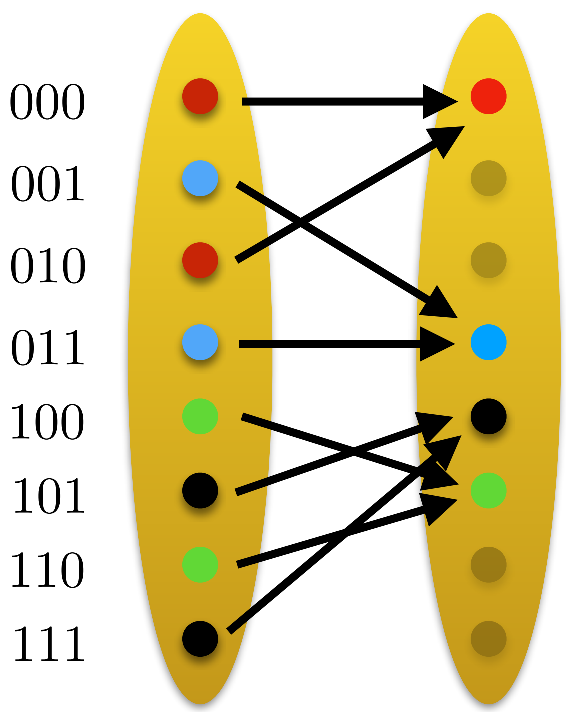
$$x \leftarrow_U \{0,1\}^n$$



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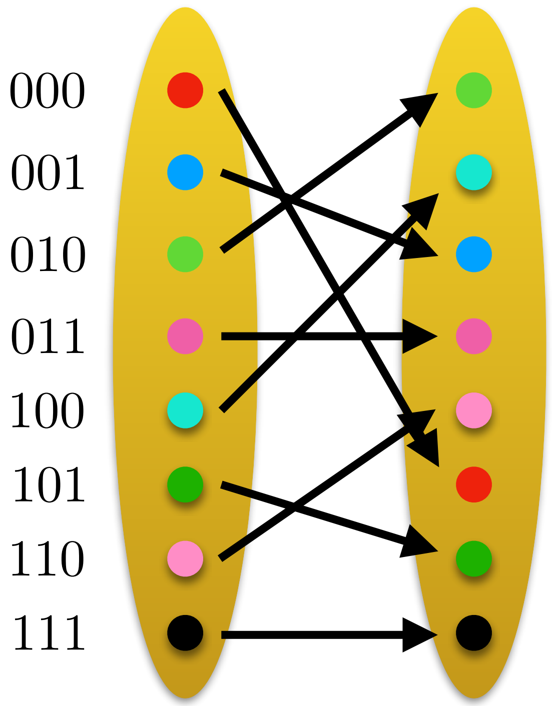

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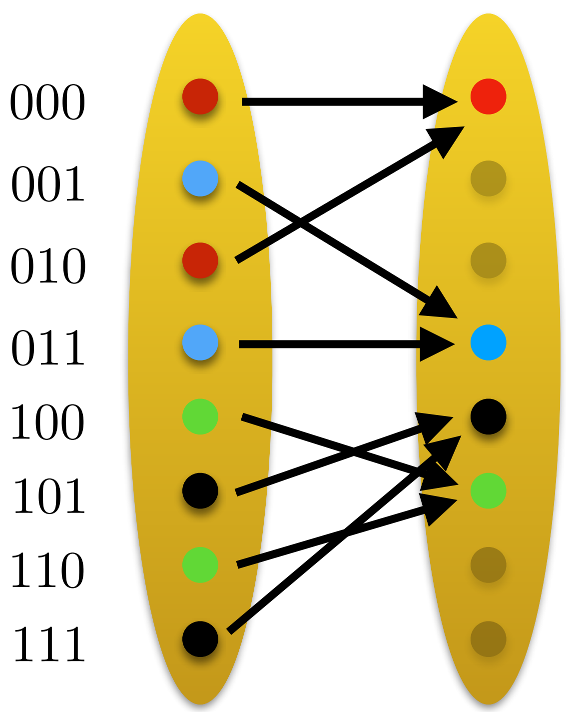
$$S(f(x)) = n$$



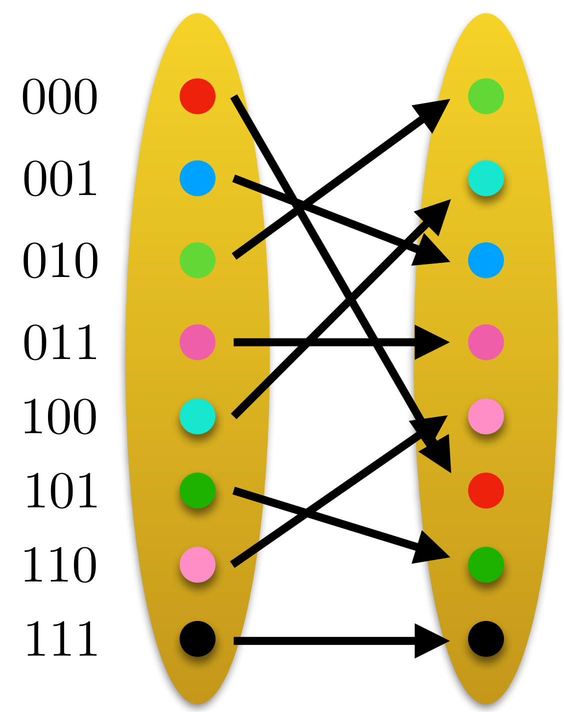
$$g : \{0,1\}^n \to \{0,1\}^n$$

2-to-1

$$x \leftarrow_U \{0,1\}^n$$

$$S(g(x)) = n - 1$$

Idea

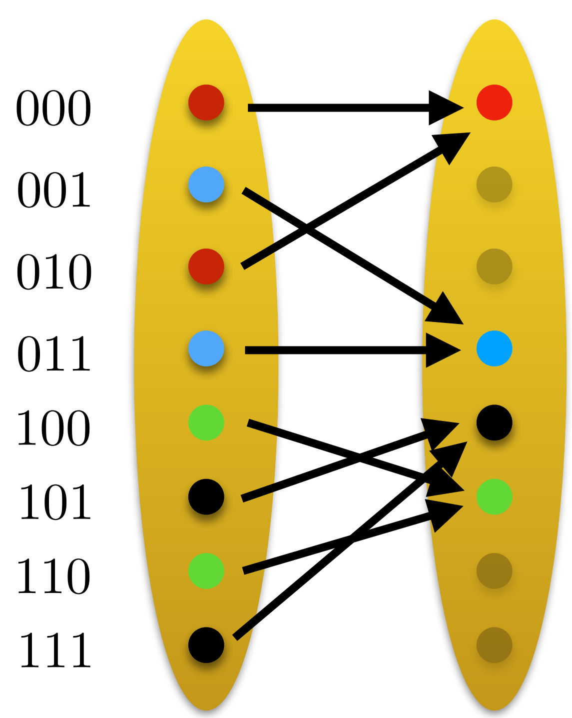


$$f : \{0,1\}^n \to \{0,1\}^n$$

1-to-1

$$\sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle$$

$$S(\rho_f) = n$$



$$g : \{0,1\}^n \to \{0,1\}^n$$

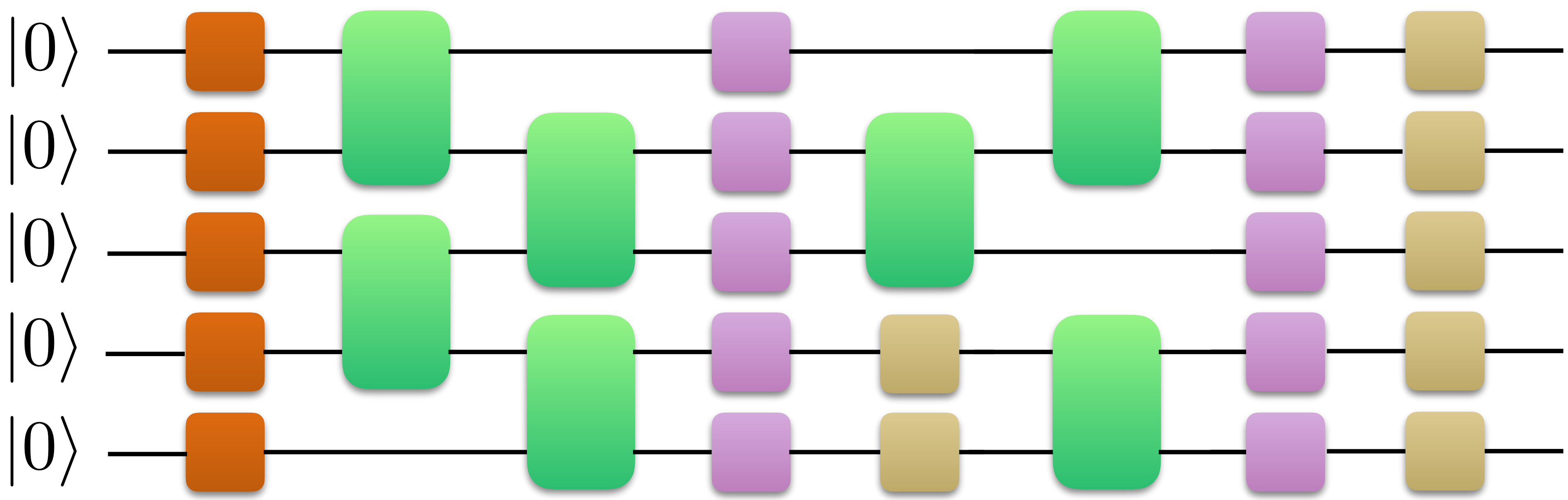
2-to-1

$$\sum_{x \in \{0,1\}^n} |x\rangle |g(x)\rangle$$

$$S(\rho_g) = n - 1$$

Idea

Given…



$$\sum_{x\in\{0,1\}^n} |x\rangle|h(x)\rangle$$

Idea

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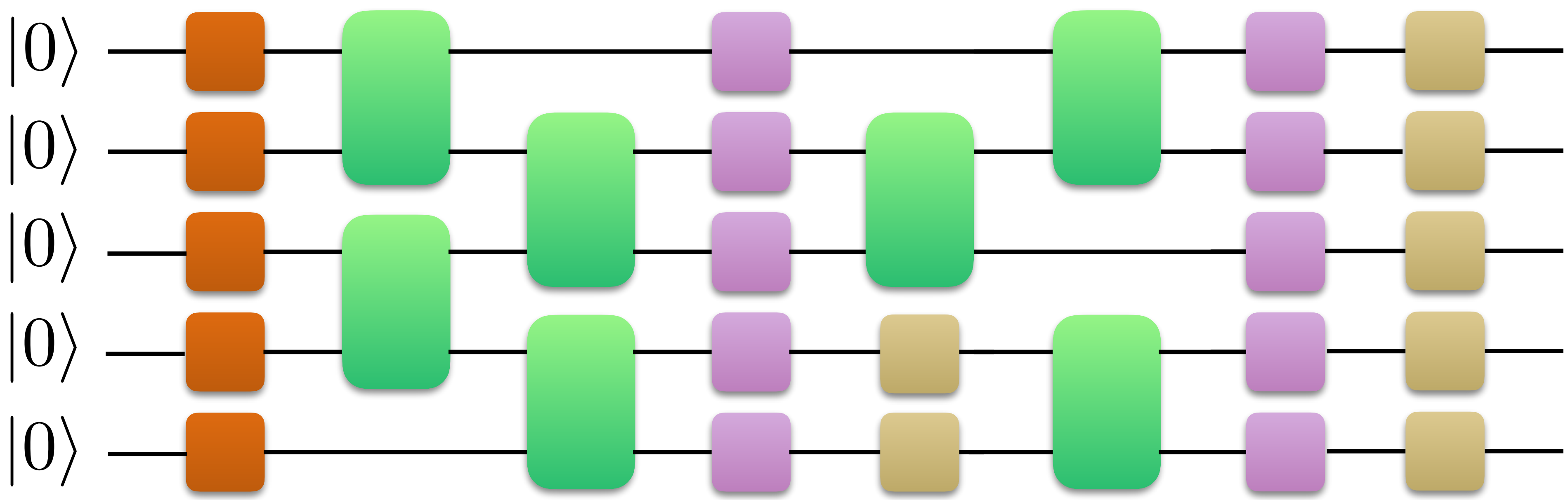

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is h a 1-to-1 or a 2-to-1 function?

Idea

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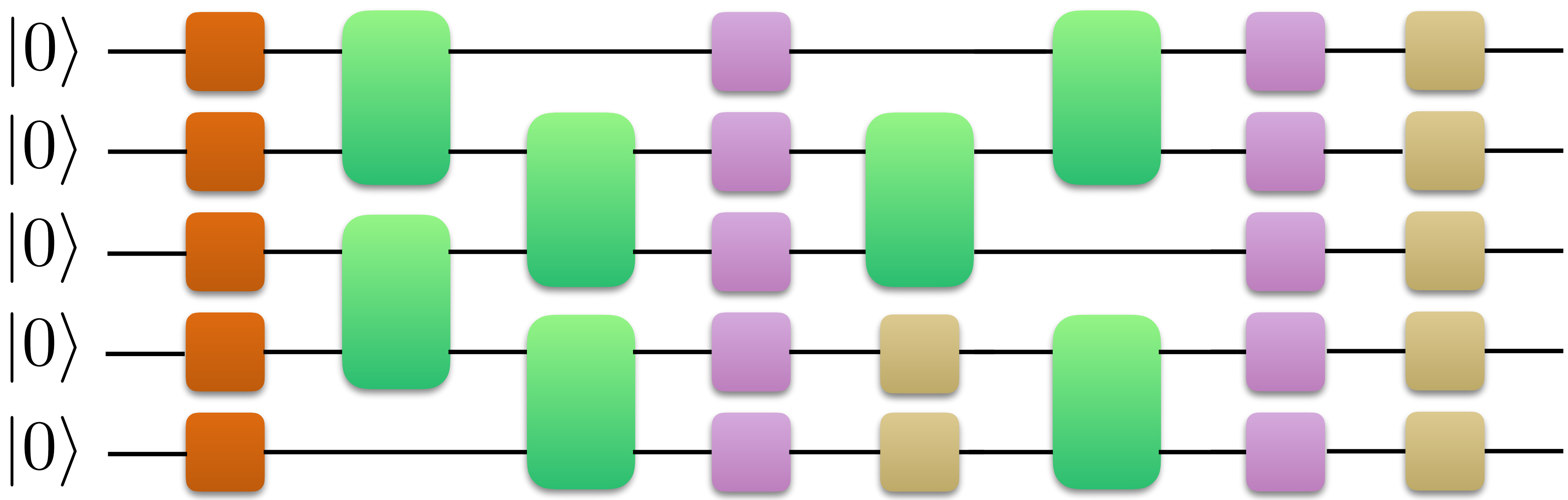

$$\sum_{x\in\{0,1\}^n} |x\rangle|h(x)\rangle$$

is h a 1-to-1 or a 2-to-1 function?

If we could estimate entropy, we could answer this question!

Idea

Can consider…

Idea

[Mahadev '18]

Can consider…

$$f(b,x) = Ax + b \cdot u + e \ (mod\ q)$$
$$g(b,x) = Ax + b \cdot (As + e') + e \ (mod\ q)$$

$$A \in \mathbb{Z}_q^{m \times n},\ x, s \in \mathbb{Z}_q^n,\ u, e, e' \in \mathbb{Z}_q^m$$

Idea

[Mahadev '18]

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such that, determining which is the 2-to-1 function is as hard as LWE

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Functions involve only linear-algebraic operations

Idea

[Mahadev '18]

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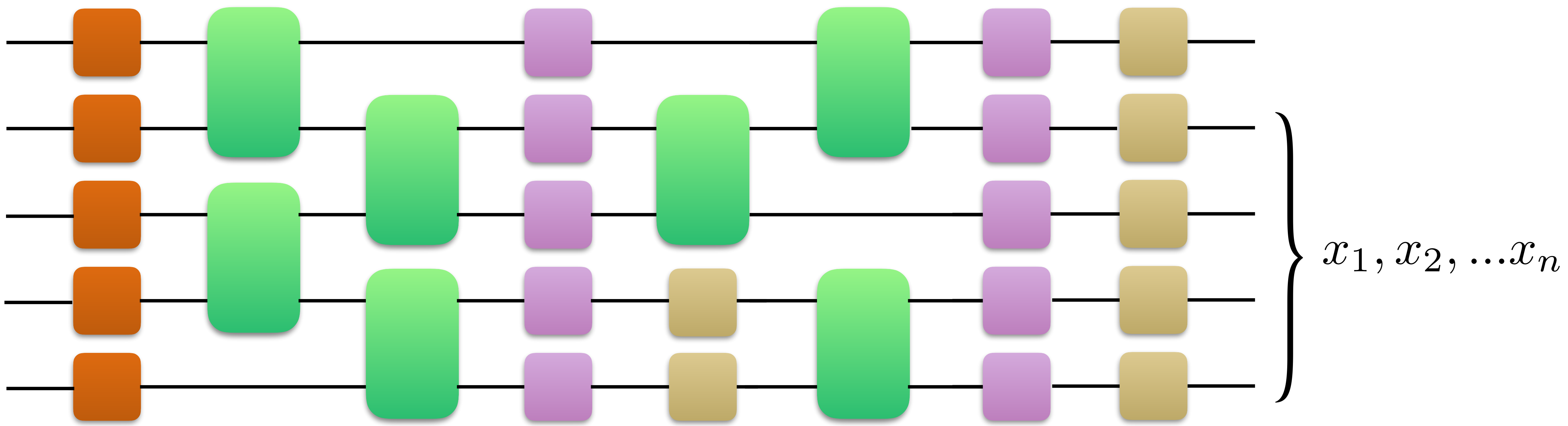
Functions involve only linear-algebraic operations

Can be performed in logarithmic depth!

Constant depth case

The classical case



Constant depth case

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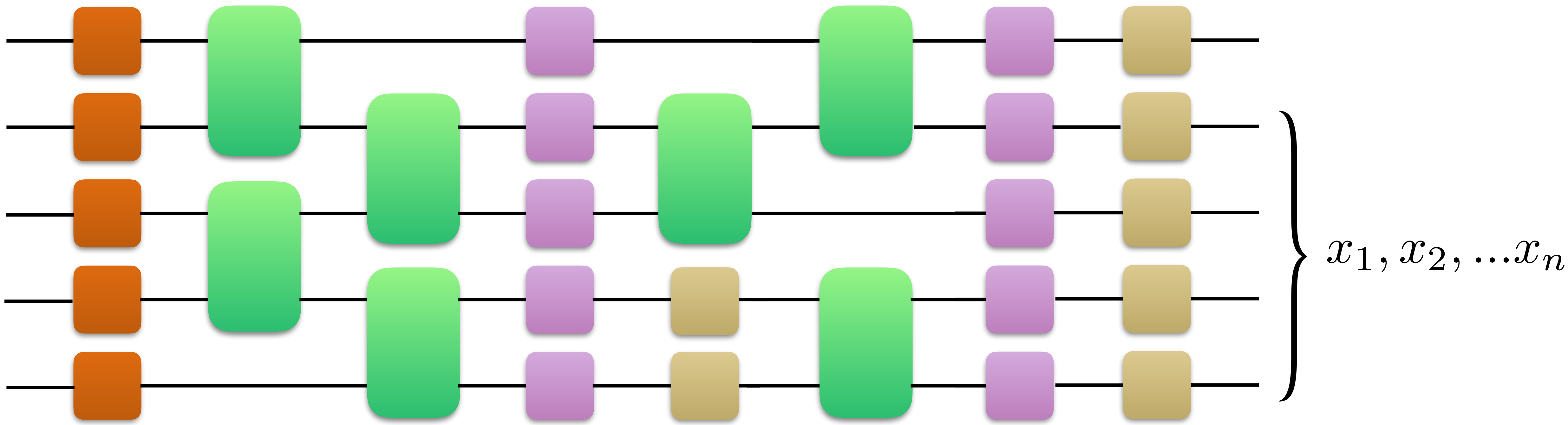
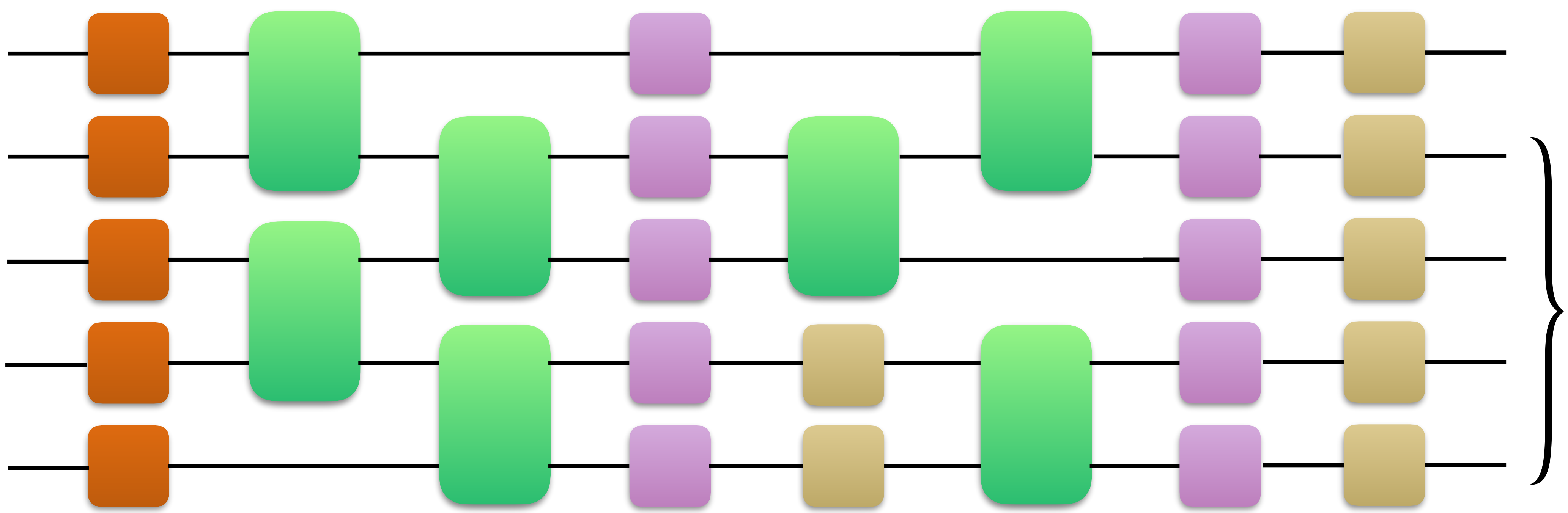


Use entropy chain rule

$$S(x_1, x_2, \ldots x_n) = \sum_i S(x_i | x_{i+1}, \ldots x_n)$$

Constant depth case

The classical case



$x_1, x_2, \ldots x_n$

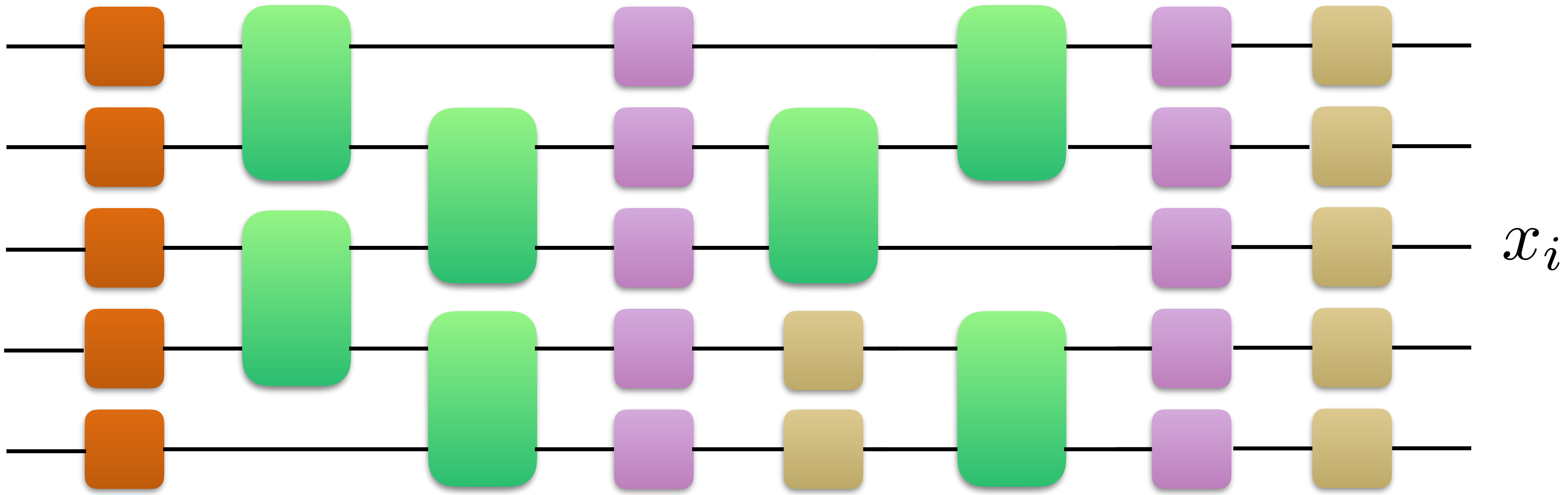
Use entropy chain rule

$$S(x_1, x_2, \ldots x_n) = \sum_i S(x_i | x_{i+1}, \ldots x_n)$$

$$S(x_i | x_j) = S(x_i) \; if \; x_i \; indep \; x_j$$

Constant depth case

The classical case



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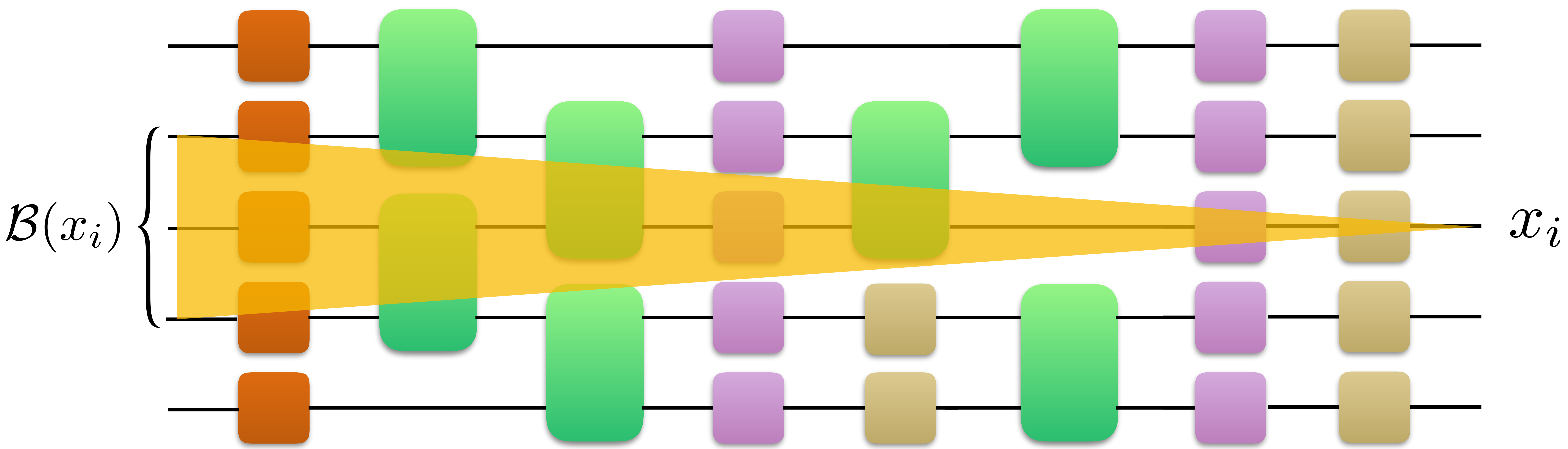

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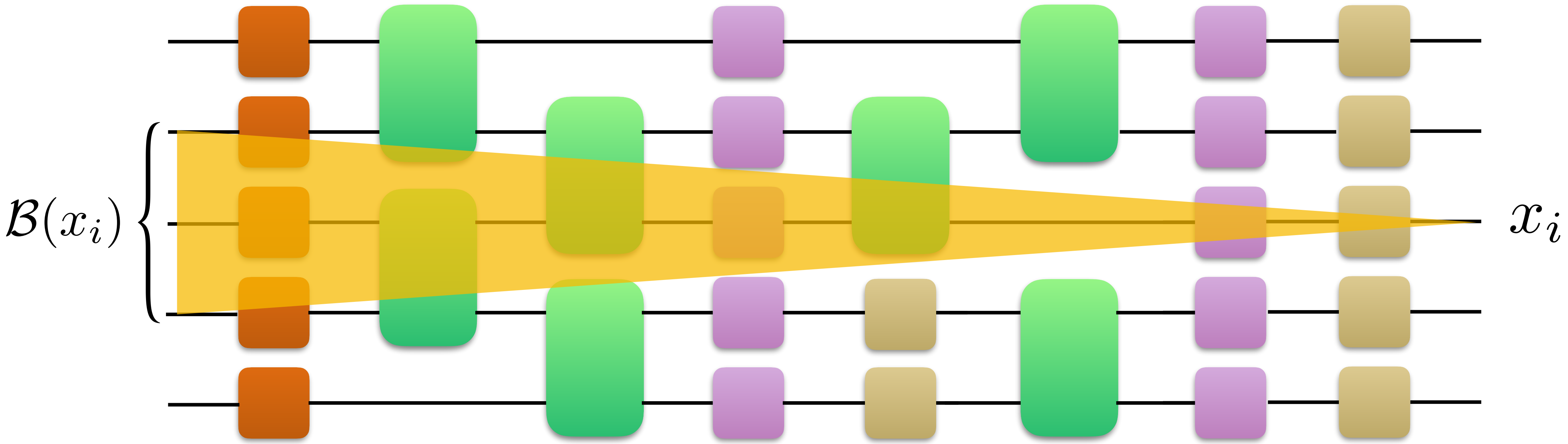
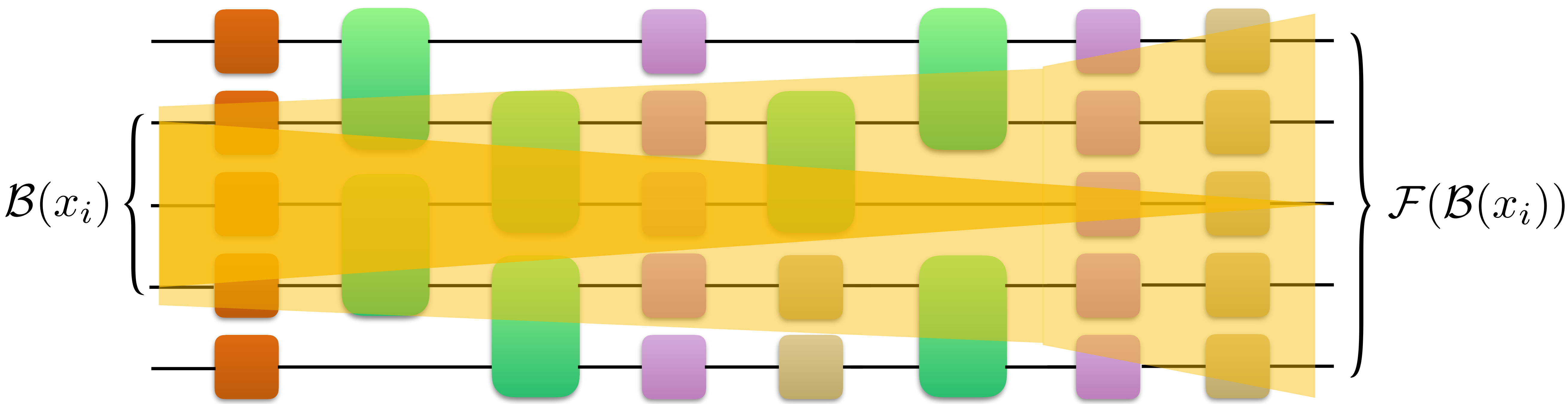


$$|\mathcal{B}(x_i)| \leq f_{in}^{d}$$

Constant depth case

The classical case



Constant depth case

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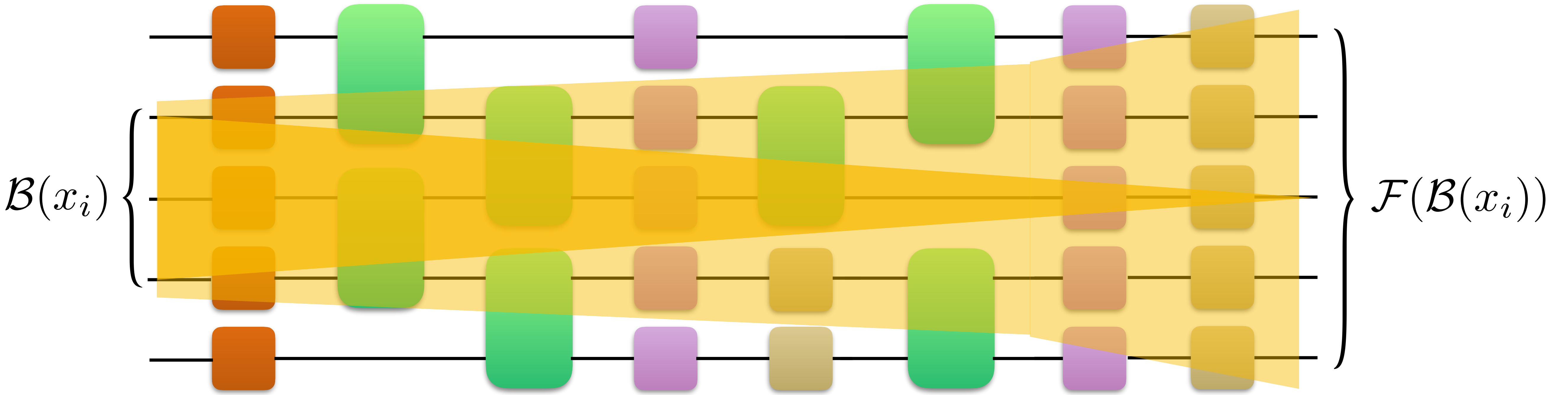
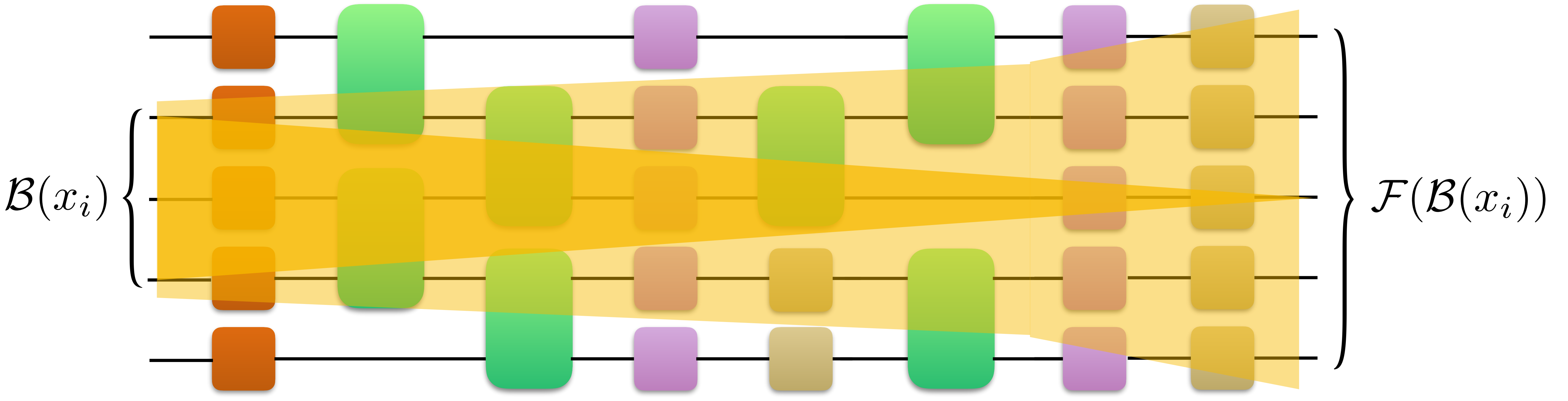


$$|\mathcal{F}(\mathcal{B}(x_i))| \leq (f_{in} f_{out})^d$$

$$|\mathcal{F}(\mathcal{B}(x_i))| = O(1)$$

Constant depth case

The classical case



$$|\mathcal{F}(\mathcal{B}(x_i))| \leq (f_{in} f_{out})^d$$

$$|\mathcal{F}(\mathcal{B}(x_i))| = O(1)$$

$$x_j \notin \mathcal{F}(\mathcal{B}(x_i)) \implies x_i \; indep \; x_j$$

Constant depth case

The classical case

$$S(x_1, x_2, ...x_n) = \sum_i S(x_i|x_{i+1}, ...x_n)$$

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$$S(x_1, x_2, ...x_n) = \sum_i S(x_i | \mathcal{F}(\mathcal{B}(x_i)) \setminus \{x_i\})$$

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$$S(x_i | \mathcal{F}(\mathcal{B}(x_i)) \setminus \{x_i\}) = S(\mathcal{F}(\mathcal{B}(x_i))) - S(\mathcal{F}(\mathcal{B}(x_i)) \setminus \{x_i\})$$

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Completely determined by $\mathcal{B}(\mathcal{F}(\mathcal{B}(x_i)))$

Constant depth case

The classical case

$$S(x_1, x_2, \ldots x_n) = \sum_i S(x_i | x_{i+1}, \ldots x_n)$$

$$S(x_1, x_2, \ldots x_n) = \sum_i S(x_i | \mathcal{F}(\mathcal{B}(x_i)) \setminus \{x_i\})$$

$$S(x_i | \mathcal{F}(\mathcal{B}(x_i)) \setminus \{x_i\}) = S(\mathcal{F}(\mathcal{B}(x_i))) - S(\mathcal{F}(\mathcal{B}(x_i)) \setminus \{x_i\})$$

Completely determined by $\mathcal{B}(\mathcal{F}(\mathcal{B}(x_i)))$

Can be computed in O(1) time

Constant depth case

The classical case

$$S(x_1, x_2, ...x_n) = \sum_i S(x_i | x_{i+1}, ...x_n)$$

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Completely determined by $\mathcal{B}(\mathcal{F}(\mathcal{B}(x_i)))$

Can be computed in O(1) time

Whole sum can be computed in O(n) time!

Constant depth case

In the quantum case this argument breaks down!

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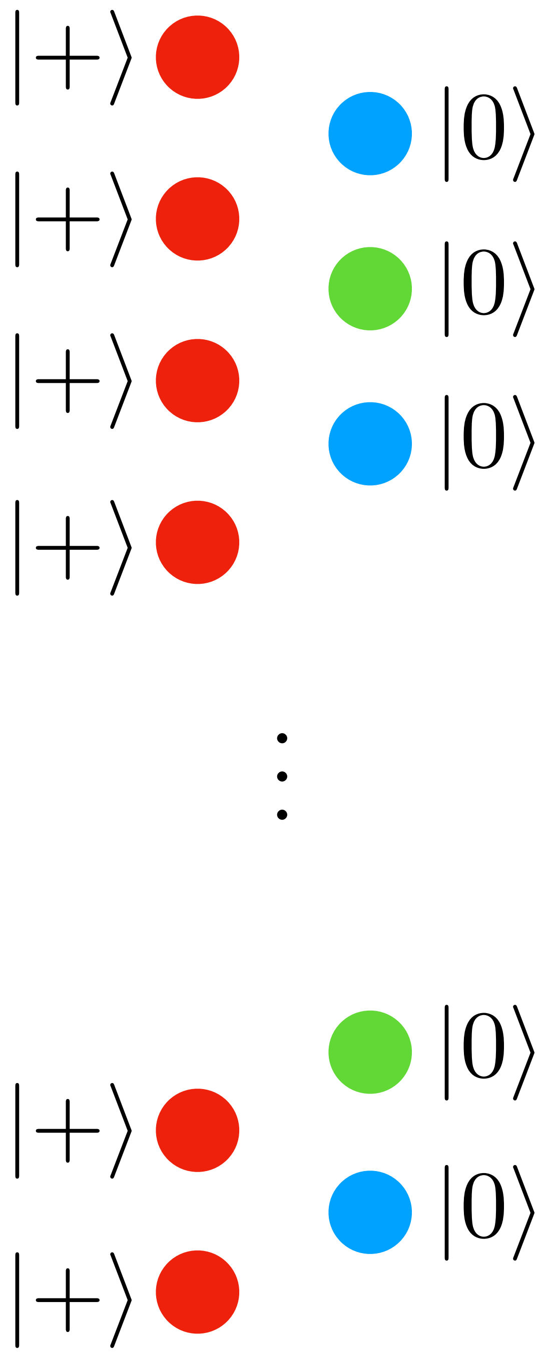

Constant depth case

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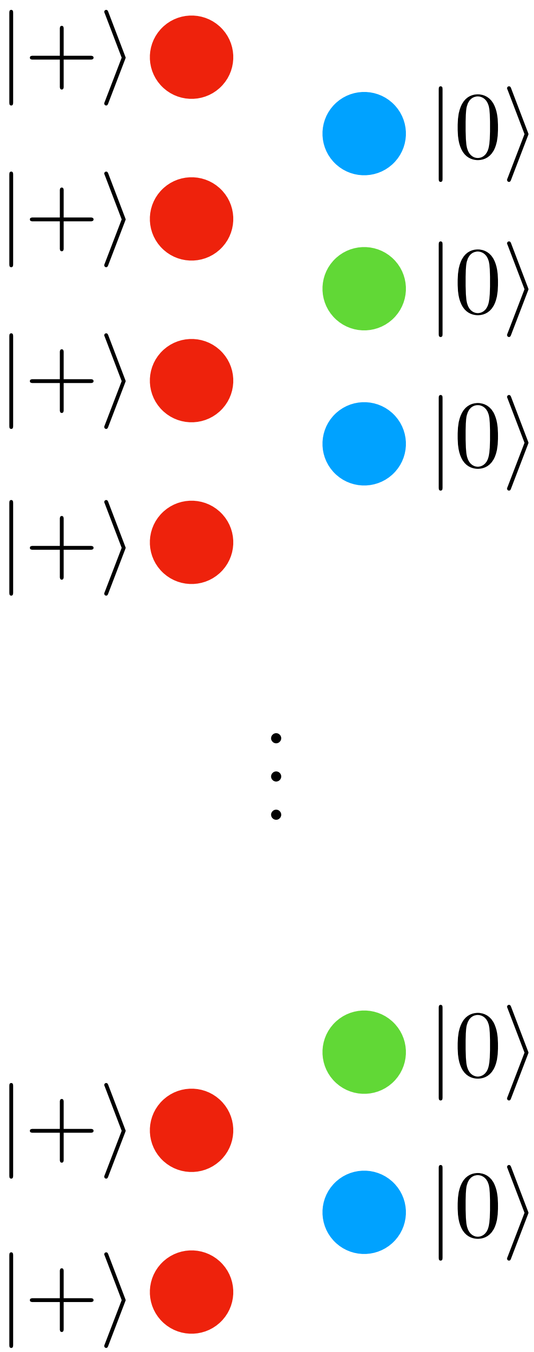

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Constant depth case

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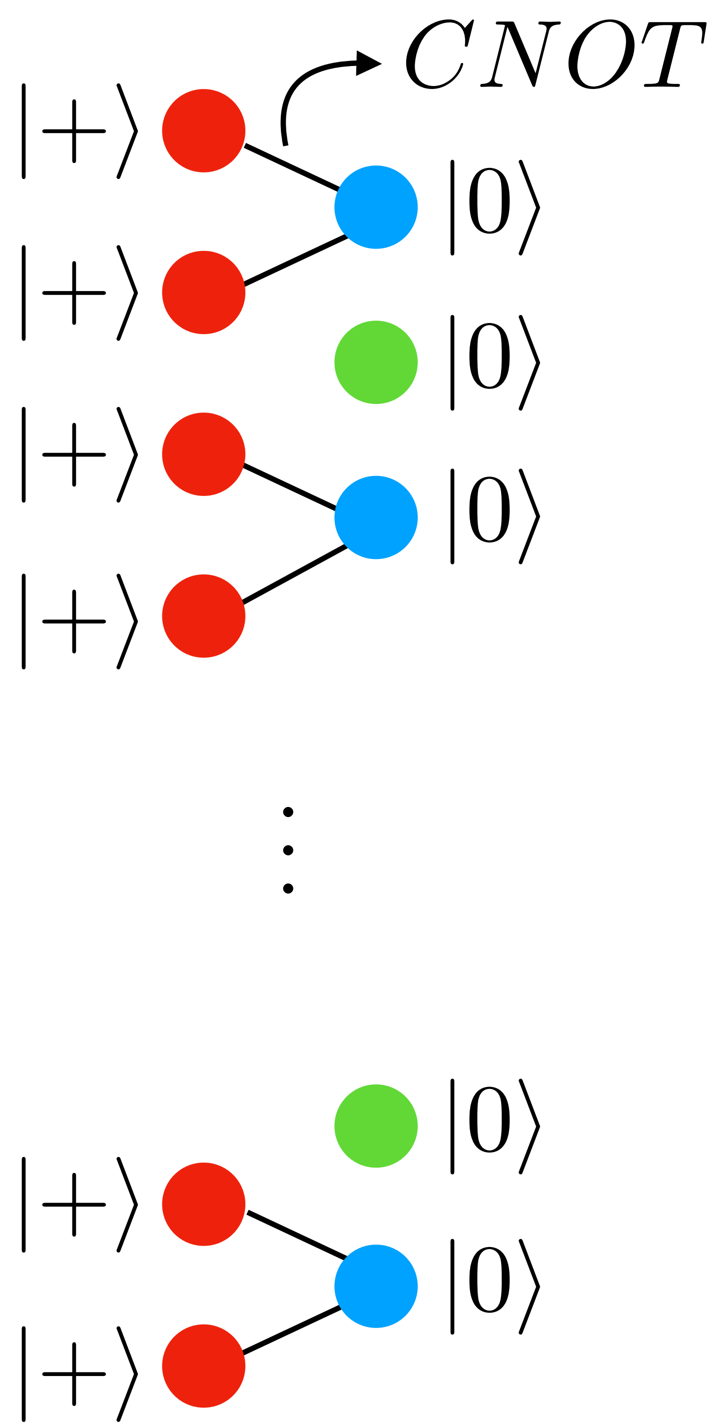

$$CNOT = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

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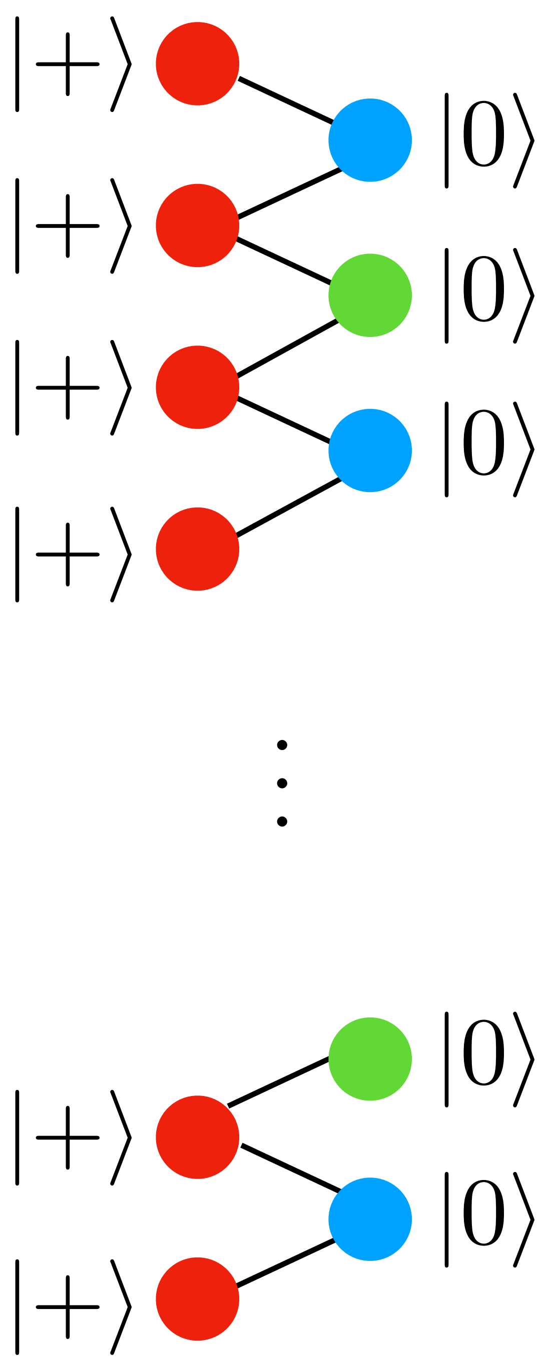

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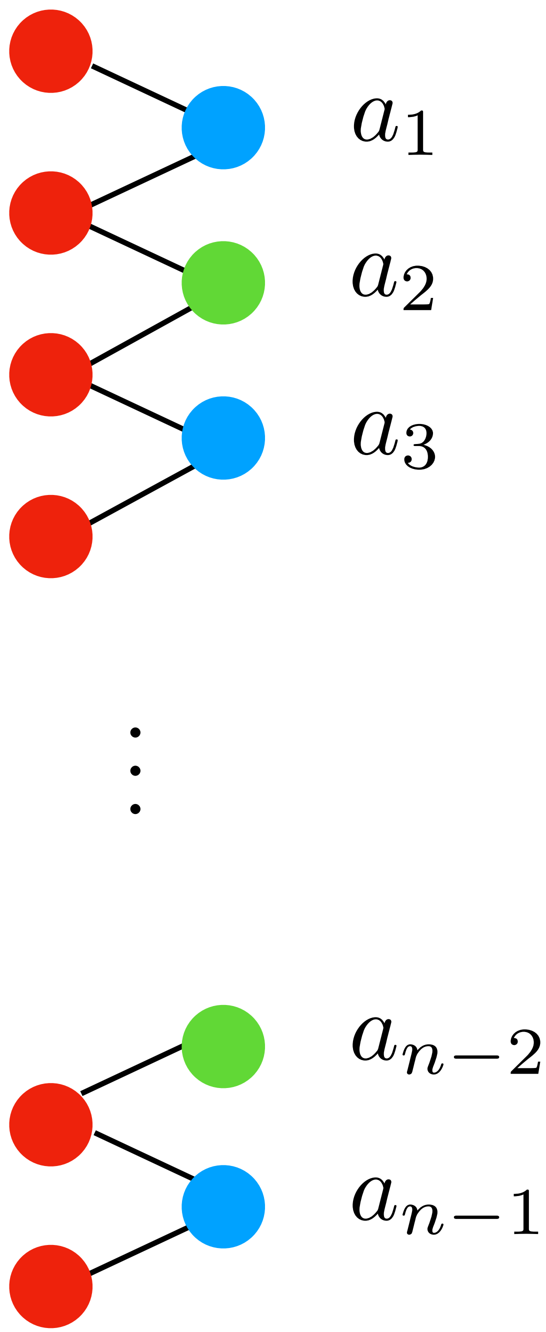

Measure non-red qubits

Constant depth case

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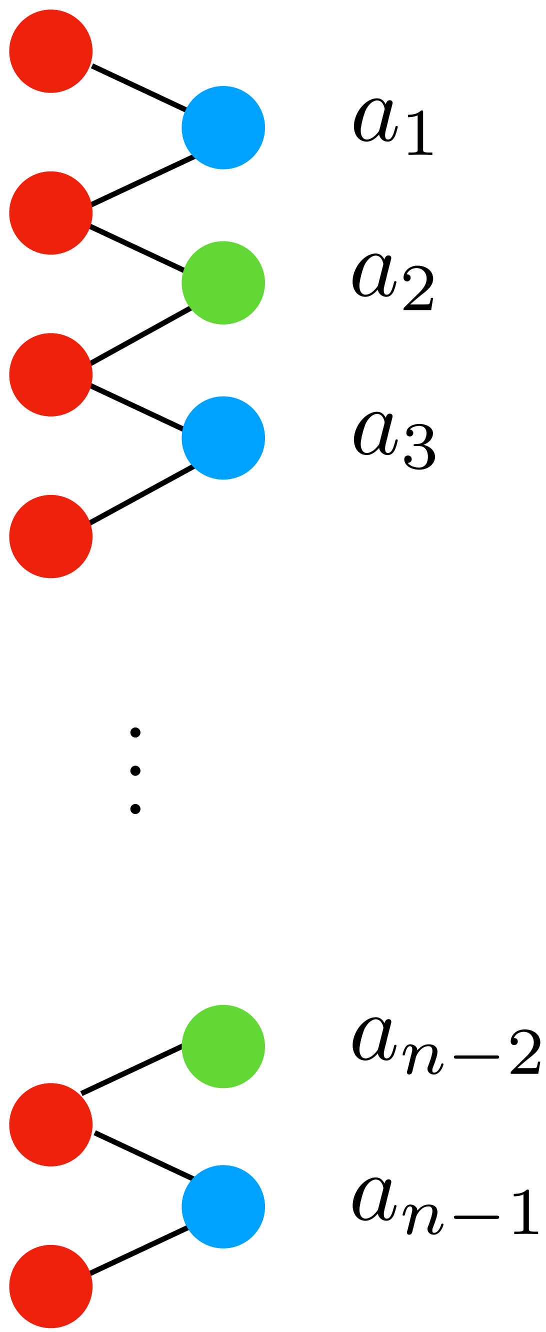

Measure non-red qubits

$$\frac{|z\rangle + |\bar{z}\rangle}{\sqrt{2}}$$

$$z_i \oplus z_{i+1} = a_i$$
$$\bar{z}_i \oplus \bar{z}_{i+1} = a_i$$

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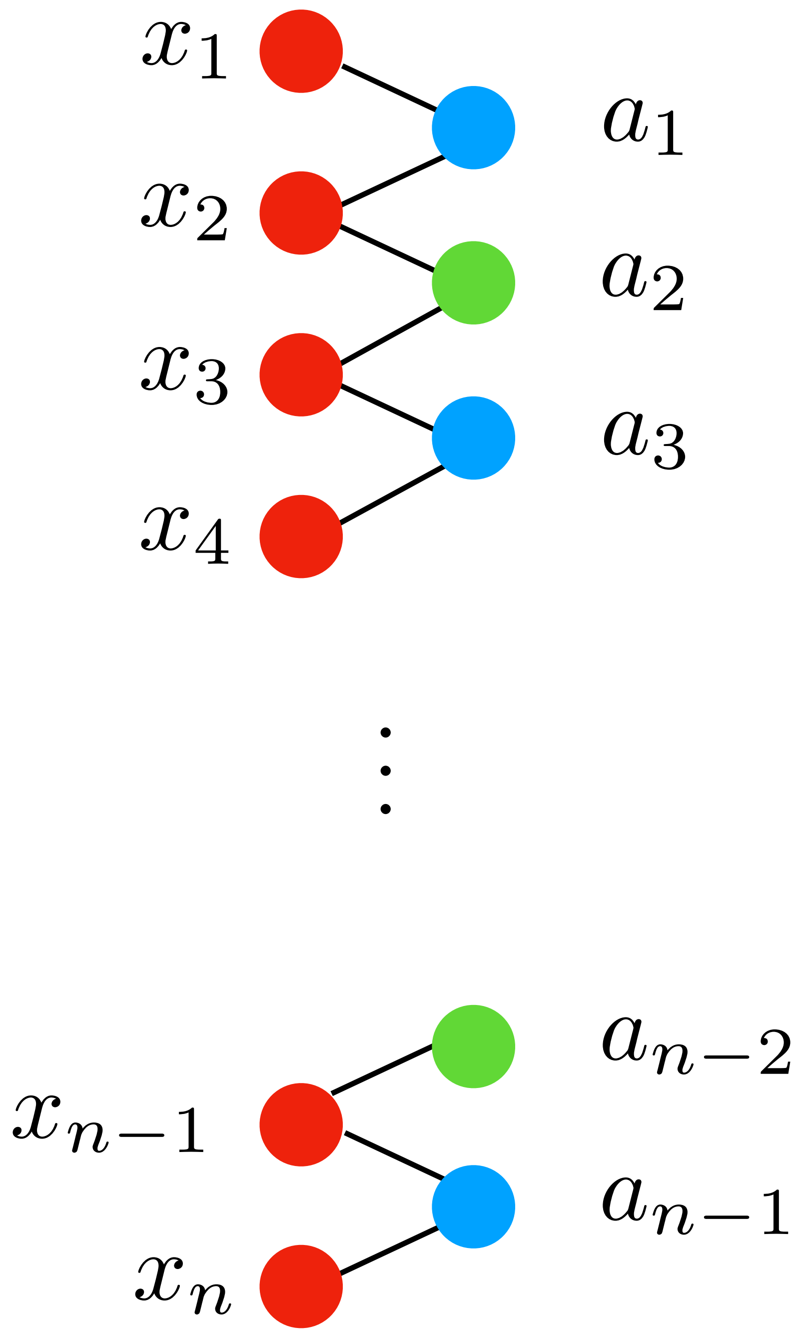

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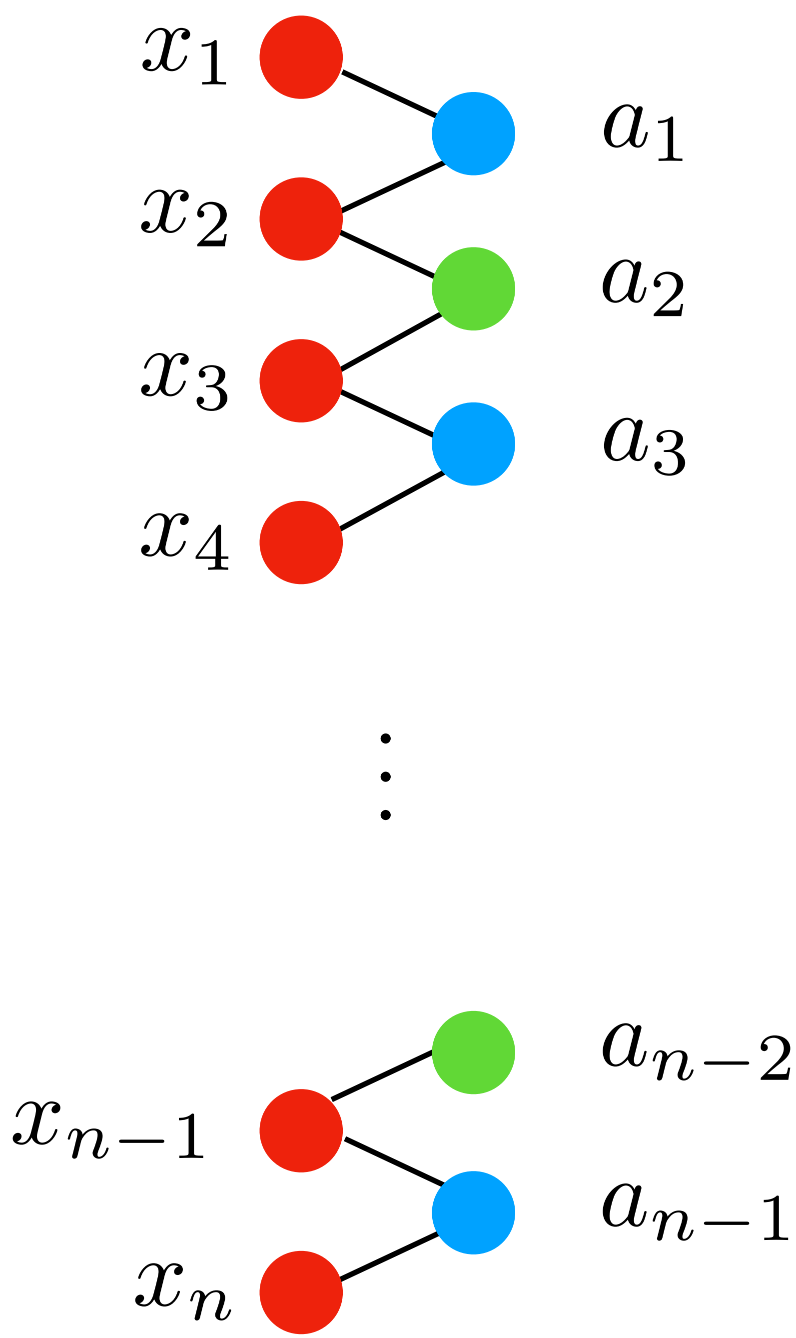

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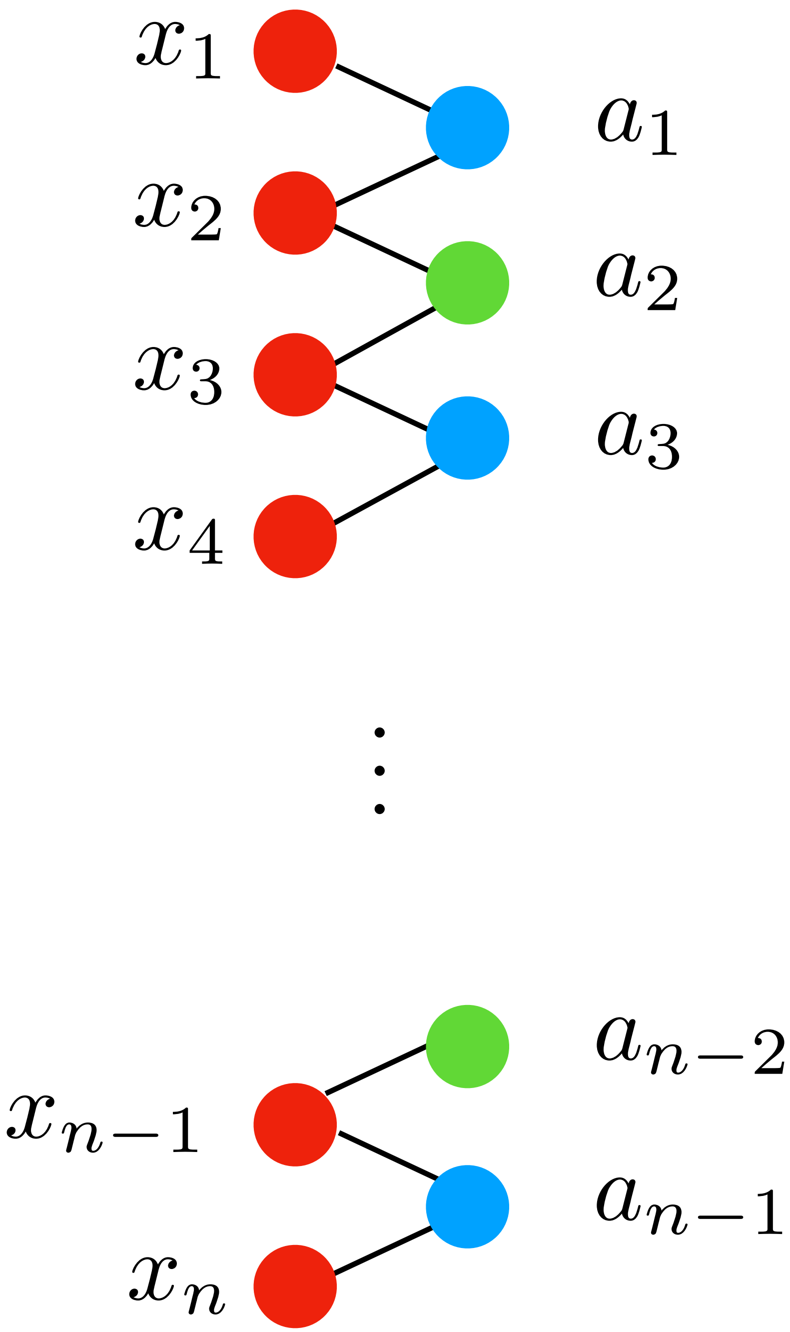

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Measure red qubits

$$p(x_1|a_1, ..., a_{n-1}) = 1/2$$
$$p(x_1|a_1, ..., a_{n-1}, {\color{red}x_n}) \in \{0, 1\}$$

Constant depth case

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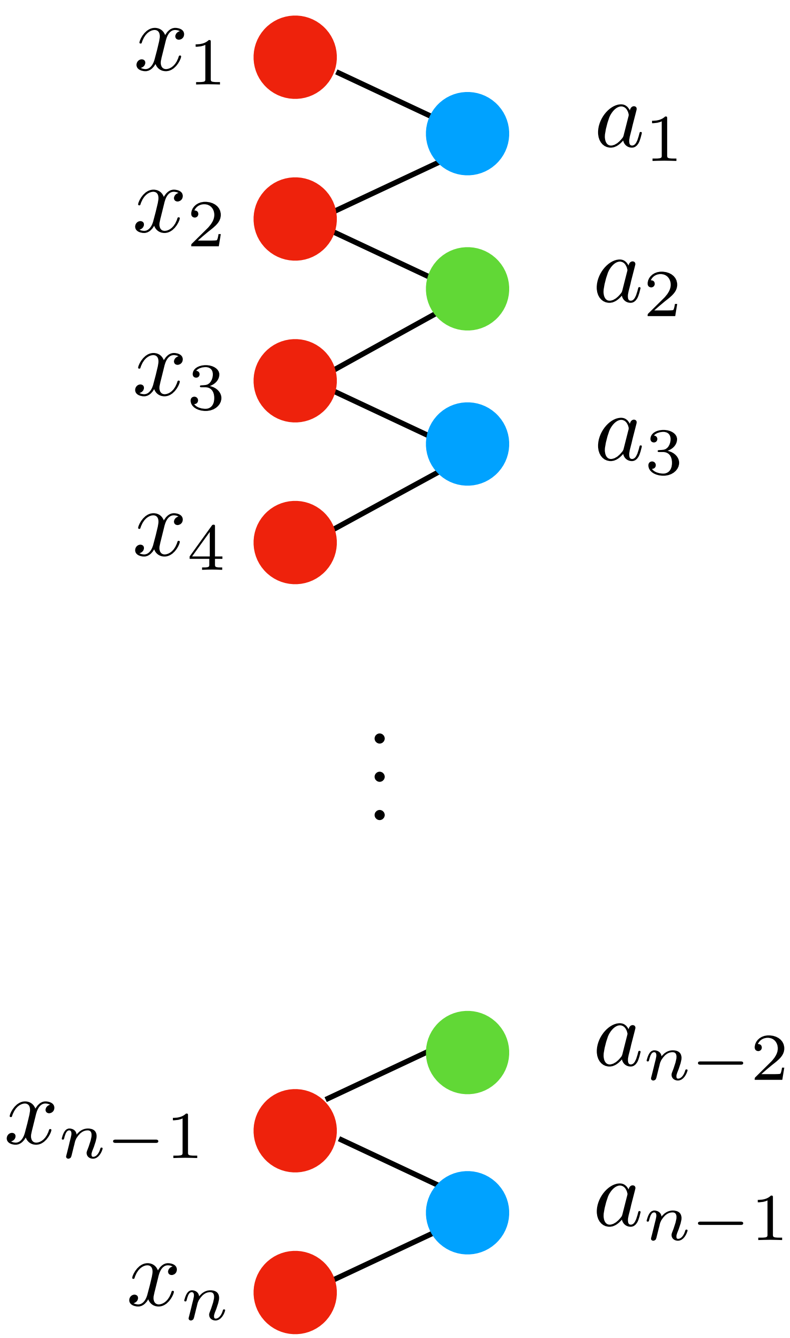

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$$\frac{|z\rangle + |\bar{z}\rangle}{\sqrt{2}}$$

$$z_i \oplus z_{i+1} = a_i$$
$$\bar{z}_i \oplus \bar{z}_{i+1} = a_i$$

Measure red qubits

$$S(x_1|a_1, ..., a_{n-1}) = 1$$
$$S(x_1|a_1, ..., a_{n-1}, \textcolor{red}{x_n}) = 0$$

Constant depth case

But why is the quantum case hard?

Constant depth case

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$$\frac{|z\rangle + |\bar{z}\rangle}{\sqrt{2}}$$

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Perform Z rotations on qubits i and j $\quad R_Z(\theta)_i \quad R_Z(\phi)_j$

Constant depth case

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Perform Z rotations on qubits i and j $\quad R_Z(\theta)_i \quad R_Z(\phi)_j$

$$\frac{1}{\sqrt{2}}(|z\rangle + e^{\theta+\phi}|\bar{z}\rangle)$$

Up to a global phase

Constant depth case

Leverage this fact to encode

$$f(b, x) = Ax + b \cdot u + e \ (mod \ q)$$

$$g(b, x) = Ax + b \cdot (As + e') + e \ (mod \ q)$$

in phases of noisy GHZ states

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in phases of noisy GHZ states

$$\frac{1}{\sqrt{2}}\left(|z\rangle + e^{\frac{2\pi i}{q}(\langle a_i, x\rangle + b \cdot u_i + e_i)}|\bar{z}\rangle\right)$$

Constant depth case

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Constant depth case

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$$f(b,x) = Ax + b\cdot u + e\ (mod\ q)$$
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in phases of noisy GHZ states

$$\sum_x |x\rangle|\widetilde{f(x)}\rangle$$

$$\sum_x |x\rangle|\widetilde{g(x)}\rangle$$

Conclusion

Classical and quantum entropy estimation are hard for log-depth circuits!

For constant depth, classical is easy, quantum is hard

Quantum requires arbitrary rotation gates.
Possible with fixed gate set?

Connections to cryptography

Potential connections to quantum gravity (AdS/CFT)

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Classical and quantum entropy estimation are hard for log-depth circuits!

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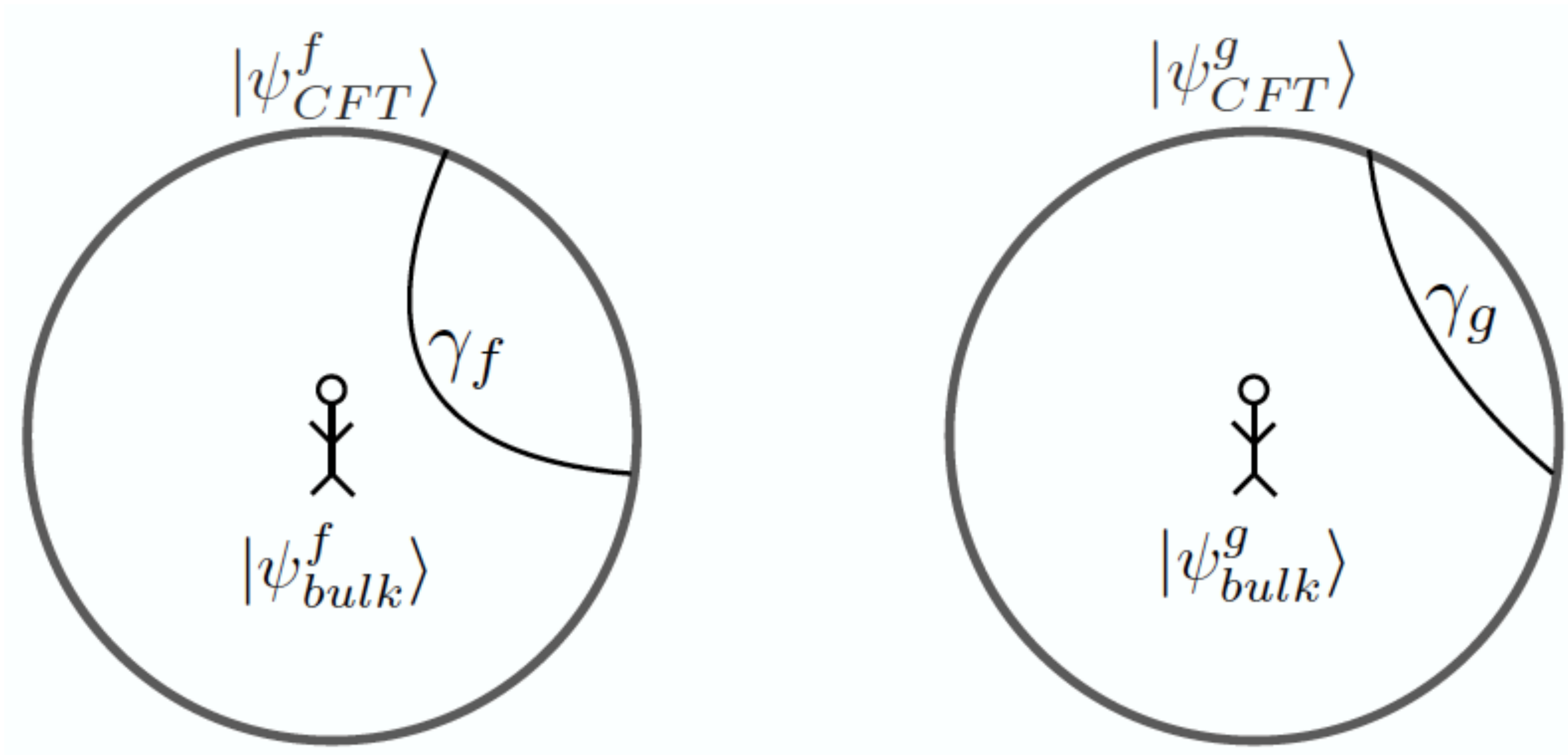
Thanks!

AdS/CFT


$|\psi^f_{CFT}\rangle$
γ_f
$|\psi^f_{bulk}\rangle$
$|\psi^g_{CFT}\rangle$
γ_g
$|\psi^g_{bulk}\rangle$