Averaged null energy condition in curved spacetime from AdS/CFT

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based on joint works with
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Introduction

“Energy” and “Causality” for a physically sensible system

• Energy should be positive (at least bounded below) to have a stable ground state.

In general relativity, energy conditions are defined in terms of stress-energy tensor $T_{\mu\nu}$ and specification of observers $\xi^b$.

• Propagation of physical fields should be “causally well behaved.”

Causality is governed locally by light cone and globally by achronal null geodesics.
Purpose

• We examine the compatibility of energy conditions and global causality in the context of AdS/CFT duality

• -- derive a certain type of energy condition “ANEC” on curved boundary spacetime.
Plan

• Null Energy Condition and its role in GR: review

• Averaged NEC (ANEC) from AdS/CFT
Null Energy Condition
and
Its usages in GR
Physically sensible conditions for Stress-Energy tensor

RHS of the Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$$

Example: FLRW cosmology

$$T_{ab} = \rho u_a u_b + P h_{ab}$$

Weak Energy Conditions (WEC)

For any timelike vector $\xi^a$

$$T_{ab} \xi^a \xi^b \geq 0$$

- holds for classical matter system

Ex. FLRW case

$$\rho \geq 0, \quad \rho + P \geq 0$$

Dominant Energy Conditions (DEC)

For any future dir. timelike vector $\xi^a$

$$-T^a\!_b \xi^b$$ also is a future dir. timelike or null vector

- Conservation theorem

Ex. FLRW case

$$\rho \geq |P|$$
Focusing conditions for geodesic congruences

- **Strong Energy Conditions (SEC)**
  For any timelike vector $\xi^a$ \( \left( T_{ab} - \frac{1}{2} T g_{ab} \right) \xi^a \xi^b \geq 0 \)

- **Timelike Focusing** \( R_{ab} \xi^a \xi^b \geq 0 \)

  Ex. FLRW case \( \rho + 3P \geq 0 \) \( \rho + P \geq 0 \)

- **Null Energy Conditions (NEC)**
  For any null vector $k^a$ \( T_{ab} k^a k^b \geq 0 \)

- **Null Focusing** \( R_{ab} k^a k^b \geq 0 \) Null-limit of WEC
Importance of Null Energy Condition

Null Energy Condition (NEC) plays an important role in GR

\[ T_{kk} := T_{ab}k^a k^b \geq 0 \quad k^a : \text{null vector} \]

NEC is the positivity of energy density measured by (locally fastest) observers with the speed of light.
Null Energy Condition (NEC) plays an important role in GR.

\[ T_{kk} := T_{ab} k^a k^b \geq 0 \quad k^a : \text{null vector} \]

NEC is the positivity of energy density measured by (locally fastest) observers with the speed of light. It governs the focusing of null geodesic congruence, from which one can learn a lot about the spacetime geometry under consideration.

It is used in an essential way in proofs of theorems in GR: 

- Singularity theorems
- Area theorem
- Topology censorship etc.
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Focusing theorem

Surface orthogonal null

\[ k^\alpha = \left( \frac{d}{d\lambda} \right)^\alpha \]

Classical expansion

\[ \theta = \nabla_a k^a = \frac{1}{A} \frac{dA}{d\lambda} \]

Shear

\[ \sigma_{ab} \]

Ex: light-rays emanating from 2-sphere in flat spacetime

\[ A = 4\pi r^2 \]

Out-going

\[ r \propto \lambda \]

\[ \theta_+ = \frac{2}{r} \]

In-going

\[ \theta_- = -\frac{2}{r} \]
Focusing theorem

Surface orthogonal null

\[ k^a = \left( \frac{d}{d\lambda} \right)^a \]

Classical expansion

\[ \theta = \nabla_a k^a = \frac{1}{A} \frac{dA}{d\lambda} \]

Shear

\[ \sigma_{ab} \]

Ex: light-rays emanating from 2-sphere in \textit{flat} spacetime

\[ \theta_+ = \frac{2}{r} > 0 \]
\[ \theta_- = -\frac{2}{r} < 0 \]
Focusing theorem

Surface orthogonal null

\[ k^a = \left( \frac{d}{d\lambda} \right)^a \]

Classical expansion

\[ \theta = \nabla_a k^a = \frac{1}{A} \frac{dA}{d\lambda} \]

Shear

\[ \sigma_{ab} \]

Ex: light-rays emanating from 2-sphere in curved spacetime
e.g. inside BH

\[ 0 < -\theta < 0 \]

both (out-, in-going) expansions can be negative

\[ \theta_+ < 0 \quad \theta_- < 0 \]
Focusing theorem

Surface orthogonal null
\[ k^a = \left( \frac{d}{d\lambda} \right)^a \]

Classical expansion
\[ \theta = \nabla_a k^a = \frac{1}{\mathcal{A}} \frac{d\mathcal{A}}{d\lambda} \]

Shear
\[ \sigma_{ab} \]

In curved spacetime, the expansion obeys Raychaudhuri equation
\[ \frac{d\theta}{d\lambda} = -\frac{1}{D-2} \theta^2 - \sigma^2 - T_{kk} \]

In going

Out-going

Strong gravity

Out-going

In-going

Strong gravity
Focusing theorem

Surface orthogonal null

\[ k^a = \left( \frac{d}{d\lambda} \right)^a \]

Raychaudhuri equation

Classical expansion

\[ \theta = \nabla_a k^a = \frac{1}{A} \frac{dA}{d\lambda} \]

Shear

\[ \sigma_{ab} \]

Raychaudhuri equation

\[ \frac{d\theta}{d\lambda} = -\frac{1}{D-2} \theta^2 - \sigma^2 \bigcirc T_{kk} \]

if initially \( \theta_0 < 0 \)

\[ \theta \to -\infty \]

\[ A \to 0 \]

Conjugate point wrt initial surface or antipodal point

\[ A \]
Focusing theorem

Focusing $\theta \to -\infty$ of null geodesic congruence generically occurs under NEC and Null generic condition/initial convergence $\theta_0 < 0$
Area theorem and BH 2nd Law

Causal nature of the BH event horizon does not allow focusing $\theta \to -\infty$ of the null generators toward future.

If NEC holds, the expansion of BH horizon must be non-negative

$$\theta_+ = \frac{1}{A} \frac{dA}{d\lambda} \geq 0$$

Area must be non-decreasing.
Area theorem and BH 2\textsuperscript{nd} Law

Causal nature of the BH event horizon does not allow focusing $\theta \rightarrow -\infty$ of the null generators toward future.

Classical Focusing w. NEC
BH must have its own Entropy

\[ S_{BH} = \frac{A}{4G\hbar} \quad \delta S_{BH} \geq 0 \]

Bekenstein 72

Generalized Entropy

\[ S_{\text{gen}} = S_{BH} + S_{\text{out}} \]

Bekenstein 73

Generalized 2\textsuperscript{nd} Law

\[ \delta S_{\text{gen}} \geq 0 \]

e.g. Page 76, Unruh-Wald 82, Zurek-Thorne 85, Frolov-Page 93
Null Energy Condition

Null Energy Condition (NEC) plays an important role in GR

\[ T_{kk} := T_{ab} k^a k^b \geq 0 \]

- holds for classical matter systems

It governs the focusing of null geodesic congruence, from which one can learn a lot about the spacetime geometry under consideration.

It is used in an essential way in the proof of

*Singularity theorems*

*Area theorem*

*Topology censorship*

*No-Bulk-shortcut theorem*
No-bulk-shortcut and AdS/CFT

- We are concerned with the compatibility of bulk-boundary causal structures and AdS/CFT duality.

- Consider two boundary points $p, q$ on a boundary achronal null geodesic $\gamma$.

Achronal null $\gamma$ is the fastest null geodesic: No two points on $\gamma$ can be connected by a boundary timelike curve.

Any null geodesic is achronal in Minkowski space but not always so in curved spacetimes.
EHT Black Hole Shadow

not achronal null geodesics

Credit: EHT Collaboration
No-bulk-shortcut and AdS/CFT

- We are concerned with the compatibility of bulk-boundary causal structures and AdS/CFT duality.
- Consider two boundary points \( p, q \) on a boundary achronal null geodesic.
- Suppose there is a bulk-shortcut (timelike) curve \( \xi \). Then \( p \) can be connected with a boundary point \( r \) strictly past to \( q \) by a bulk causal curve (dashed-line).
- Then bulk propagator could carry information faster than boundary propagator. This is no sensible causal property for bulk-boundary duality.
Basic questions

(i) Given a boundary metric, what kind of effects on the boundary theory does a sensible bulk causal property (i.e., no bulk-shortcut) give rise to?

(ii) What kind of boundary metric admits bulk-shortcut?

• Key to answering these questions is the Averaged Null Energy Condition (ANEC)
No-bulk-shortcut Theorem

No acausal propagation \( \quad ( \text{Gao-Wald 00} ) \)

Bulk NEC $\rightarrow$ No shortcut through the bulk

If this is the case, bulk propagators could carry information faster than the corresponding boundary propagators.

Causal curve in the bulk

Achronal null geodesic on the boundary
No-bulk-shortcut Theorem

No acausal propagation \( \text{(Gao-Wald 00)} \)

Bulk NEC $\rightarrow$ No shortcut through the bulk

Causal curve in the bulk

This can be the case when there exists a \underline{naked} singularity in the bulk

\text{Al-Maeda-Mefford 19}

Naked singularity

Achronal null geodesic on the boundary
Violation of NEC and Non-local conditions

NEC (any local energy conditions) can be violated by quantum fields.

\[ T_{kk} \geq 0 \quad \langle T_{ab} \rangle k^a k^b \geq 0 \]

- by e.g. Hawking-radiation

Averaged Null Energy Conditions (ANEC)

\[ \int \langle T_{ab} \rangle k^a k^b d\lambda \geq 0 \]

Non-local: defined along any complete null geodesic

e.g. Wald-Yurtsever 91

ANEC can be violated by Quantum Fields.

\[ \int \langle T_{ab} \rangle k^a k^b d\lambda \not\geq 0 \]

e.g. Visser 96

Achronal averaged Null Energy Conditions (AANEC)

e.g. Graham and Olum 07
• However, quantum field effects violate NEC and ANEC.

• Counter-examples of ANEC:
  A local violation of NEC can be enhanced by conformal transformations. This can occur since ANEC itself is not conformally invariant.

  Urban-Olum 10
  Al-Maeda-Mefford 19
Basic questions

(i) Given a curved boundary metric, what kind of effects on the boundary theory does No-bulk-shortcut property give rise to?

(ii) What kind of boundary metric admits bulk-shortcut?

• Key to answering these questions is the Averaged Null Energy Condition (ANEC)
Purpose

• Taking the assertion of No-bulk-shortcut theorem as our guiding principle, we discuss the compatibility of bulk-boundary causality and AdS/CFT duality.

• We derive ANEC with a weight function on boundary curved spacetime.

• In odd-dimensions, it is conformally invariant (CANEC)
ANEC from AdS/CFT
ANECD (CANEC) from AdS/CFT

- \((d+1)\)-asymptotic AdS metric and Fefferman-Graham expansion:

\[
\begin{align*}
    ds^2 &= dz^2 + \hat{g}_{\mu\nu}(z, x) \frac{dx^\mu dx^\nu}{z^2} \\
    \hat{g}_{\mu\nu}(z, x) &= \sum_{n=0}^{\infty} g(n)_{\mu\nu}(x) z^n + z^d \ln z^2 h_{\mu\nu}(x)
\end{align*}
\]

Only when \(d\) is even

- Boundary stress-energy tensor: de Haro-Solodukhin-Skenderis 01

\[
\langle T_{\mu\nu} \rangle = \frac{d!}{16\pi G_{d+1}} g(d)_{\mu\nu} + X_{\mu\nu}
\]

\(X_{\mu\nu}\) : gravitational anomaly

(\(\rightarrow\) vanishes in odd-dimensions)
In Minkowski space, all null curves are complete and achronal (fastest null curve).

In curved space, achronal null curves are limited: e.g. horizon generators

The above formulas involve the curved background metric, curvature, source terms and become more complicated in higher dimensions.
ANEC in curved (closed) universe from AdS/CFT

Iizuka-Al-Maeda 19, 20

4-dim vacuum AdS-bulk

\[ R_{ab} - \frac{1}{2} R g_{ab} = -\Lambda g_{ab} \]

( in the bulk, NEC is satisfied \rightarrow \text{No-bulk-shortcut} )

3-dim. boundary metric of static compact universe \[ g^{(0)\mu\nu} \]

\[ ds^2 = -2 \, dU \, dV + \sin^2 \left( \frac{V-U}{\sqrt{2}} \right) d\varphi^2 \]
ANEC in curved (closed) universe from AdS/CFT

4-dim vacuum AdS-bulk

\[ R_{ab} - \frac{1}{2} R g_{ab} = -\Lambda g_{ab} \]

( in the bulk, NEC is satisfied \( \rightarrow \) No-bulk-shortcut )

3-dim. boundary metric of static compact universe \( g^{(0)\mu\nu} \)

\[ ds^2_3 = -2\, dU\, dV + \sin^2\left(\frac{V-U}{\sqrt{2}}\right) d\varphi^2 \]

Consider a boundary achronal null geodesic w/ tangent

\[ l^\mu = (\partial_V)^\mu \quad (0 \leq \lambda \leq \sqrt{2\pi}) \]

Null-null component of boundary Ricci tensor is

\[ \hat{R}_{\mu\nu} l^\mu l^\nu = \frac{1}{2} \]
Consider a bulk causal curve \( \xi \) that connects two boundary points \( p, r \) with tangent vector:

\[
\hat{g}_{ab} K^a K^b \leq 0
\]

Since \( \xi \) is a causal curve

From No-shortcut theorem, the value of \( U \) at \( r \) must be that

\[
U(\sqrt{2\pi}) \geq 0
\]

Boundary achronal null geodesic

\[
\gamma : p \to q
\]

Consider a bulk causal curve \( \xi \) that connects two boundary points \( p, r \) with tangent vector:

\[
K^a = \begin{pmatrix}
\frac{d z}{d \lambda}, & \frac{d U}{d \lambda}, & \frac{d V}{d \lambda}, & \frac{d \varphi}{d \lambda}
\end{pmatrix}
\]

\[
z = \varepsilon z_1 + \varepsilon^2 z_2 + \cdots
\]

\[
K^U = \varepsilon^2 \frac{d u_2}{d \lambda} + \varepsilon^3 \frac{d u_3}{d \lambda} + \cdots
\]
Consider a bulk causal curve that connects two boundary points, with tangent vector:

**Boundary achronal null geodesic**

\[ \gamma : p \rightarrow q \]

From No-shortcut theorem, the value of \( U \) at \( r \) must be that

\[ U(\sqrt{2\pi}) \geq 0 \]

Consider a bulk causal curve \( \xi \) that connects two boundary points \( p, r \) with tangent vector:

\[ 2^{\text{nd}}-\text{order of } \epsilon \text{ yields } z_1 = \frac{1}{2}z_1 \rightarrow z_1 = \sin \frac{\lambda}{\sqrt{2}} \]

thus \( z_1 \) is just like a boundary Jacobi field \( \eta \)

\[ K^a = \left( \frac{dz}{d\lambda}, \frac{dU}{d\lambda}, \frac{dV}{d\lambda}, \frac{d\varphi}{d\lambda} \right) \]

\[ z = \epsilon z_1 + \epsilon^2 z_2 + \cdots \]

\[ K^U = \epsilon^2 \frac{du_2}{d\lambda} + \epsilon^3 \frac{du_3}{d\lambda} + \cdots \]
• From $\hat{g}_{ab} K^a K^b \leq 0$ and 3rd-order expansion, it follows:

$$ U(\sqrt{2}\pi) \geq \frac{\varepsilon^3}{2} \int_0^{\sqrt{2}\pi} z_1^3 \frac{g(3)_{\mu\nu}}{l^\mu l^\nu} d\lambda \geq 0 $$

• Holographic renormalized stress-energy formula yields $g(3)_{\mu\nu}$ corresponds to $\langle T_{\mu\nu} \rangle$

• Boundary Raychaudhuri equation implies that $z_1$ can be viewed as the boundary Jacobi field $\eta$

$$ \ddot{z}_1 = -\hat{R}_{\mu\nu} l^\mu l^\nu z_1 \quad \quad \quad \ddot{\eta} = -\hat{R}_{\mu\nu} l^\mu l^\nu \eta $$

Thus we obtain:

$$ \int_{\lambda_-}^{\lambda_+} \eta^3 \langle T_{\mu\nu} \rangle l^\mu l^\nu d\lambda \geq 0 $$

• This is invariant under conformal transformation:

$$ \tilde{g}(0)_{\mu\nu} = \Omega^2 g(0)_{\mu\nu} \quad \frac{d\tilde{\lambda}}{d\lambda} = \Omega^2 \quad \tilde{\eta} = \Omega \eta \quad \langle \tilde{T}_{\mu\nu} \rangle = \Omega^{-1} \langle T_{\mu\nu} \rangle $$
• One can derive this conformally invariant ANEC (CANEC) in the case of more generic 3-dim. or 5-dim Boundary spacetime w/ positive spatial curvature

\[ \int_{\lambda_-}^{\lambda_+} \eta^3 \langle T_{\mu\nu} \rangle l^\mu l^\nu d\lambda \geq 0 \quad \int_{\lambda_-}^{\lambda_+} \eta^5 \langle T_{\mu\nu} \rangle l^\mu l^\nu d\lambda \geq 0 \]

3-dim. bndry case \quad 5-dim. bndry case

• These inequalities are given in terms only of the boundary quantities.

• These inequalities can be used to restrict the extent of the negative value of the null energy \( \langle T_{\mu\nu} \rangle l^\mu l^\nu \)

• The conf. transf. trick used for ANEC cannot invalidate CANEC.
• So far, we have focused on the odd-dimensional boundary case.

• In even-dimensions, one has to deal with conformal anomalies, which make relevant formulas more complicated.

• Boundary stress-energy tensor: de Haro-Solodukhin-Skenderis 01

\[
\langle T_{\mu\nu} \rangle = \frac{d!}{16\pi G_{d+1}} g(d)_{\mu\nu} + X_{\mu\nu}
\]

\(X_{\mu\nu}: \text{gravitational anomaly}\)
• Using a similar holographic method, one can derive a lower bound for weighted ANEC in even-dimensions.

• $d=2$ case: We can obtain the ANEC.

• $d=4$ case:

$$
4\pi G \int_{v_-}^{v_+} \eta^4 \langle T_{\mu\nu} \rangle \, l^\mu l^\nu \geq \frac{1}{12} \int_{v_-}^{v_+} \eta^4 \left[ \theta_+^2 \left( R_{uv} + \frac{R_{\theta\theta}}{r^2} \right) - \frac{\theta_+^2}{2} \cdot \mu(r) \right. \\
\left. + \frac{1}{2} \theta_+ \theta_- R_{vv} + \left( \frac{3}{2r^2} - \frac{R_{\theta\theta}}{r^2} - 2R_{uv} \right) R_{vv} \right]
$$

$\theta_{\pm}$: Expansion of boundary null geodesics

$\mu(r) := \frac{1}{r^2} + \theta_+ \theta_-$: Quasi-local mass density

$R_{\mu\nu}$: Boundary Ricci curvature
• Using a similar holographic method, one can derive a lower bound for weighted ANEC in even-dimensions.

• $d=2$ case: We can obtain the ANEC.

• $d=4$ case:

\[
4\pi G \int_{v_+}^{v_+} \eta^4 \langle T_{\mu\nu} \rangle \, l^\mu l^\nu \geq \frac{1}{12} \int_{v_-}^{v_+} \eta^4 \left[ \theta_+^2 \left( R_{uv} + \frac{R_{\theta\theta}}{r^2} \right) - \frac{\theta_+^2}{2} \cdot \mu(r) \right. \\
+ \frac{1}{2} \theta_+ \theta_- R_{vv} + \left( \frac{3}{2r^2} - \frac{R_{\theta\theta}}{r^2} - 2R_{uv} \right) R_{vv} \left. \right]
\]

Einstein-Static cylinder case

\[
\geq \frac{1}{8} \int_{v_-}^{v_+} \eta^4 > 0
\]
• Schwarzchild-AdS bulk and boundary ANEC

\[ ds^2 = -\left( r^2 + 1 - \frac{M}{r^2} \right) dt^2 + \left( r^2 + 1 - \frac{M}{r^2} \right)^{-1} dr^2 + r^2 (d\rho^2 + \sin^2 \rho d\Omega^2) \]

• The boundary weighted ANEC implies that the bulk mass parameter must be non-negative:

\[ M \geq 0 \]

• -- suggests No-bulk-shortcut condition to be connected with the positive mass theorem in asymptotically AdS spacetime.
• A higher curved boundary spacetime in which ANEC is violated. [Iizuka-Al-Maeda 19, 20]

• C.f. conformal symmetry implies ANEC for CFT in dS and AdS and a similar lower bound for ANEC in closed universe (Lorentizan cylinder) is obtained. [Rosso 20]

• For some closed universe the minimum of the ANEC can be negative. [Fischetti-Hickling-Wiseman 16]
We have pursued the question of what is a sensible causal interaction between the bulk and conformal boundary in the context of AdS/CFT duality with Gao-Wald’s No-bulk-shortcut property as our guiding principle for the appropriate choice of boundary geometry.

We have derived, in the case of generic 3-dim. or 5-dim boundaries w/ compact positive spatial curvature, a conformally invariant ANEC (CANEC), which is expressed in terms only of the boundary quantities.

No-bulk-shortcut ⇒ CANEC

In the even-dimensional case, one has to deal with conformal anomalies. We have derived ANEC in 2-dimensions and a lower bound for ANEC with a certain weight function in 4-dimension.