2 Mar. 2021
YIT P workshop
Recent Progress in Theoretical Physics

Averaged null energy condition in curved spacetime from AdS/CFT

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based on joint works with
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2008.07942 1911.02654

Introduction

"Energy" and "Causality" for a physically sensible system

- Energy should be positive (at least bounded below) to have a stable ground state.

  In general relativity, energy conditions are defined in terms of *stress-energy tensor* $T_{\mu\nu}$ and specification of observers ξ^b .

- Propagation of physical fields should be "causally well behaved."

  Causality is governed locally by light cone and globally by *achronal null* geodesics.

Purpose

- We examine the compatibility of energy conditions and global causality in the context of AdS/CFT duality

- -- derive a certain type of energy condition "ANEC" on curved boundary spacetime.

Plan

- Null Energy Condition and its role in GR: review
- Averaged NEC (ANEC) from AdS/CFT

Null Energy Condition
and
Its usages in GR

Physically sensible conditions for Stress-Energy tensor

RHS of the Einstein equations $R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$

Example: FLRW cosmology $T_{ab} = \rho u_a u_b + P h_{ab}$

Weak Energy Conditions (WEC)

For any timelike vector ξ^a $\Rightarrow$ $T_{ab}\xi^a\xi^b \geqslant 0$

- holds for classical matter system Ex. FLRW case $\rho \geqslant 0$ $\rho + P \geqslant 0$

Dominant Energy Conditions (DEC)

For any future dir. timelike vector ξ^a

$-T^a{}_b\xi^b$ also is a future dir. timelike or null vector

- Conservation theorem Ex. FLRW case $\rho \geqslant |P|$

Focusing conditions for geodesic congruences

Strong Energy Conditions (SEC)

For any timelike vector ξ^a ➡ $\left(T_{ab} - \frac{1}{2}T g_{ab}\right)\xi^a\xi^b \geqslant 0$

- Timelike Focusing $R_{ab}\xi^a\xi^b \geqslant 0$

Ex. FLRW case $\rho + 3P \geqslant 0 \quad \rho + P \geqslant 0$

Null Energy Conditions (NEC)

For any null vector k^a ➡ $T_{ab}k^a k^b \geqslant 0$

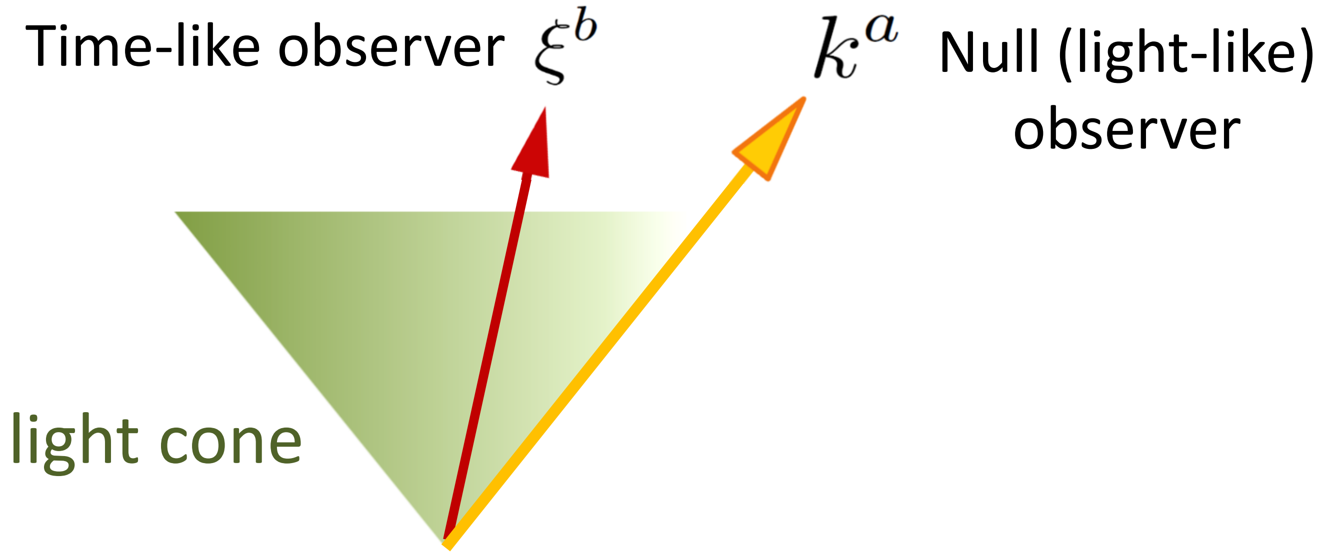
- Null Focusing $R_{ab}k^a k^b \geqslant 0$ Null-limit of WEC

Importance of Null Energy Condition

Null Energy Condition (NEC) plays an important role in GR

$$T_{kk} := T_{ab}k^a k^b \geqslant 0 \qquad k^a \text{ : null vector}$$

NEC is the positivity of energy density measured by (locally fastest) observers with the speed of light.



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It governs the focusing of null geodesic congruence, from which one can learn a lot about the spacetime geometry under consideration.

It is used in an essential way in proofs of theorems in GR:

Singularity theorems

Area theorem

Topology censorship *etc.*

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Focusing theorem

Surface orthogonal null

$$k^a = \left(\frac{d}{d\lambda}\right)^a$$

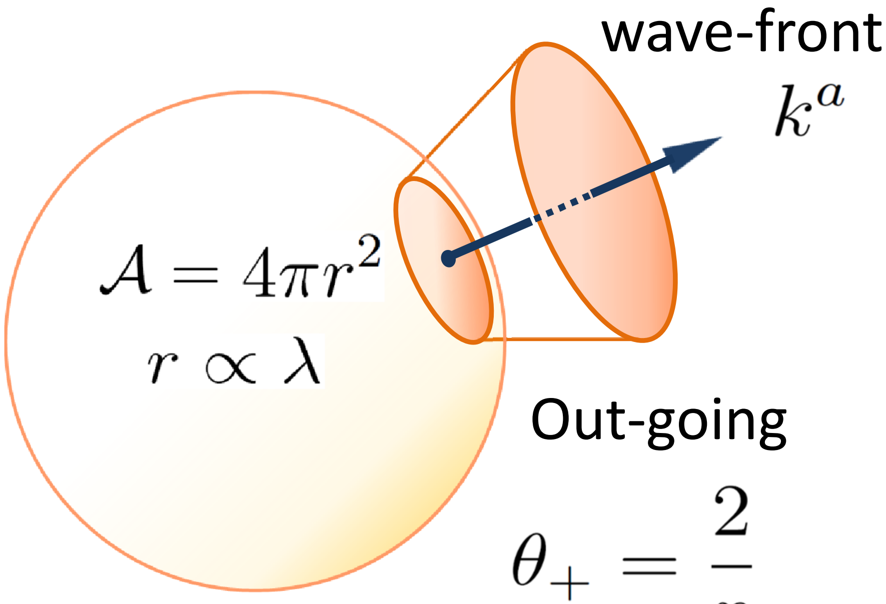
Classical expansion

$$\theta = \nabla_a k^a = \frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\lambda}$$

Shear

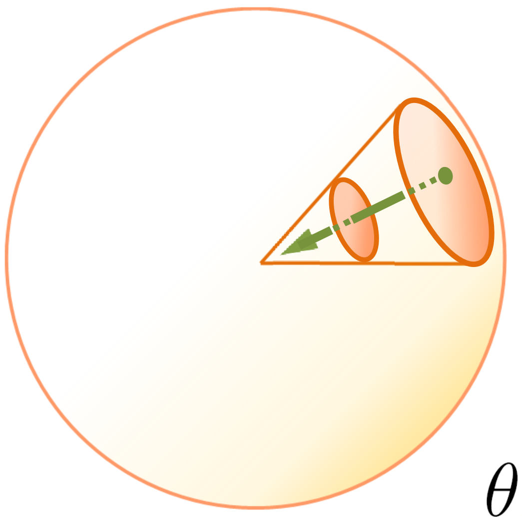
$$\sigma_{ab}$$

Ex: light-rays emanating from 2-sphere in flat spacetime



Out-going

$$\theta_+ = \frac{2}{r}$$



In-going

$$\theta_- = -\frac{2}{r}$$

Focusing theorem

Surface orthogonal null

$$k^a = \left(\frac{d}{d\lambda}\right)^a$$

Classical expansion

$$\theta = \nabla_a k^a = \frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\lambda}$$

Shear

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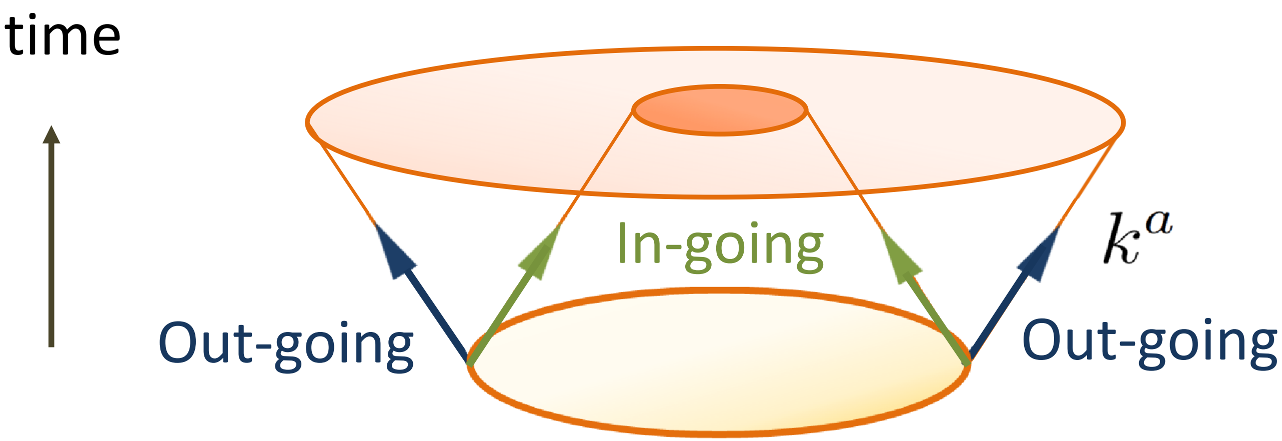


$$\theta_+ = \frac{2}{r} > 0$$

$$\theta_- = -\frac{2}{r} < 0$$

Focusing theorem

Surface orthogonal null

$$k^a = \left(\frac{d}{d\lambda}\right)^a$$

Classical expansion

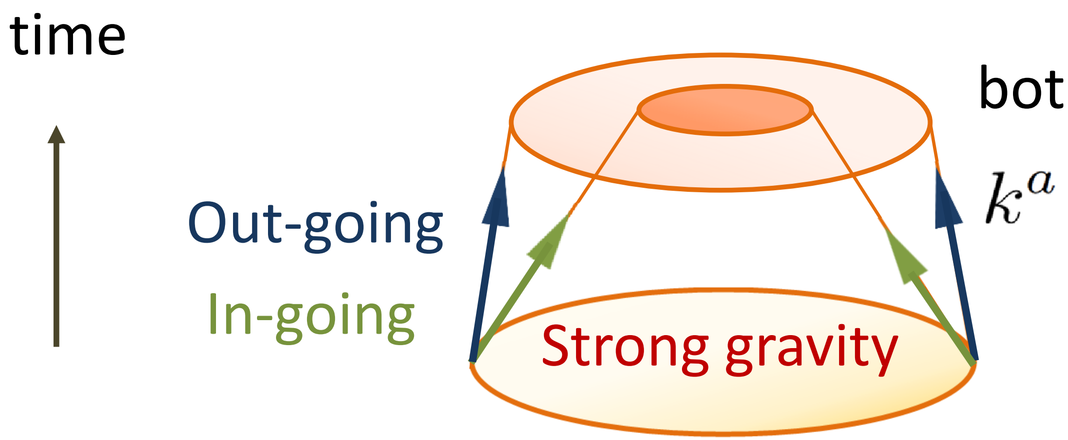
$$\theta = \nabla_a k^a = \frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\lambda}$$

Shear

$$\sigma_{ab}$$

Ex: light-rays emanating from 2-sphere in curved spacetime

e.g. inside BH



both (out-, in-going) expansions
can be negative

$$\theta_+ < 0 \qquad \theta_- < 0$$

Focusing theorem

Surface orthogonal null

$$k^a = \left(\frac{d}{d\lambda}\right)^a$$

Classical expansion

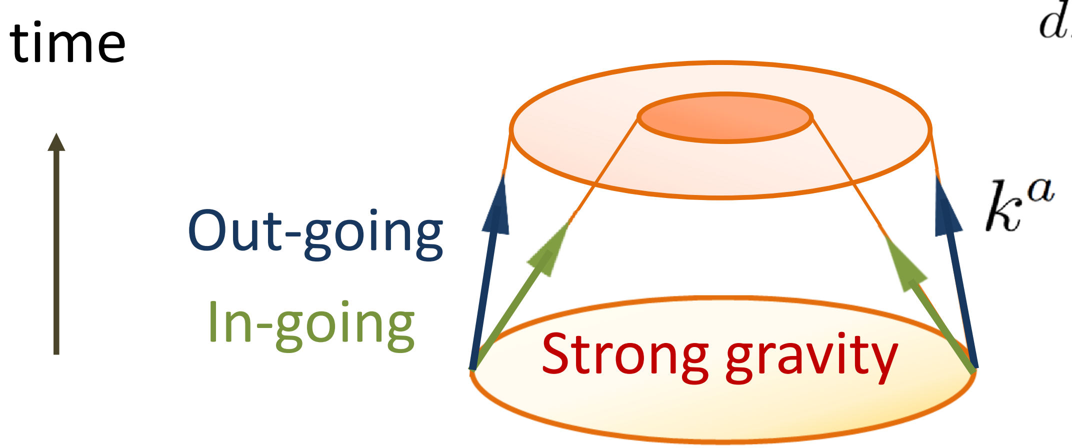
$$\theta = \nabla_a k^a = \frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\lambda}$$

Shear

$$\sigma_{ab}$$

In curved spacetime, the expansion obeys Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 - \sigma^2 - T_{kk}$$



Focusing theorem

Surface orthogonal null

$$k^a = \left(\frac{d}{d\lambda}\right)^a$$

Classical expansion

$$\theta = \nabla_a k^a = \frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\lambda}$$

Shear

$$\sigma_{ab}$$

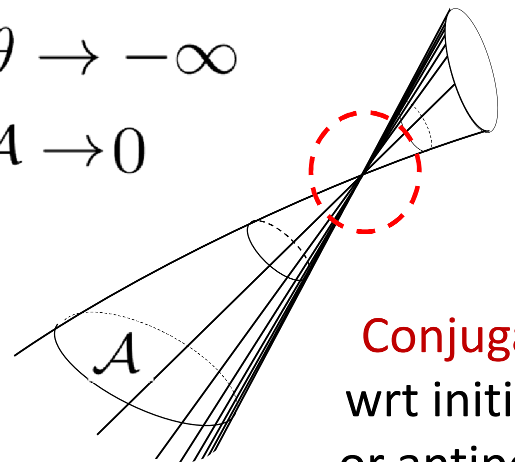
Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 - \sigma^2 - T_{kk}$$

$$T_{kk} \geqslant 0 \Rightarrow \boxed{\frac{d\theta}{d\lambda} \leqslant 0} \Rightarrow$$

if initially $\theta_0 < 0$

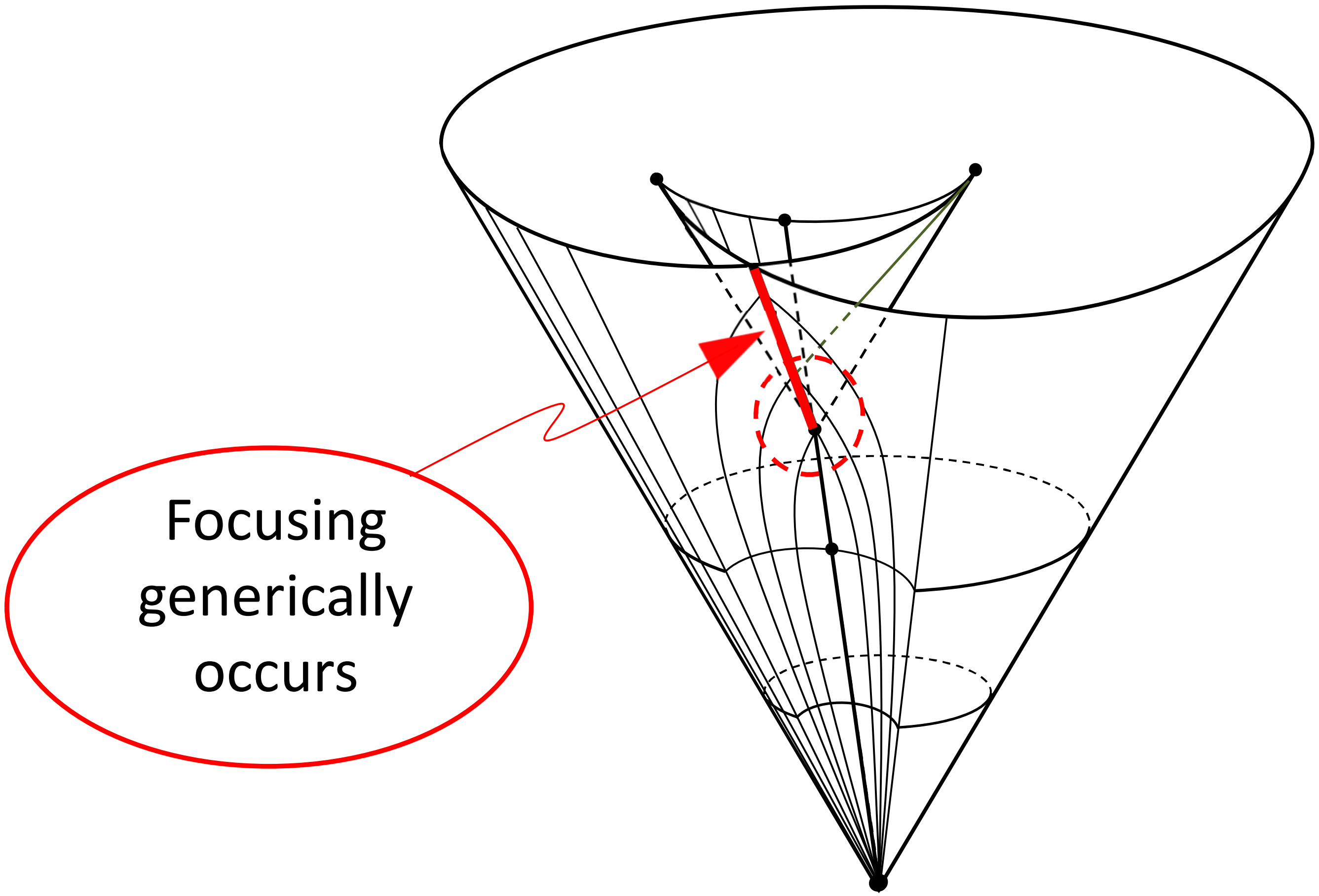
$$\left\{ \begin{array}{l} \theta \to -\infty \\ \mathcal{A} \to 0 \end{array} \right.$$



Conjugate point
wrt initial surface
or antipodal point

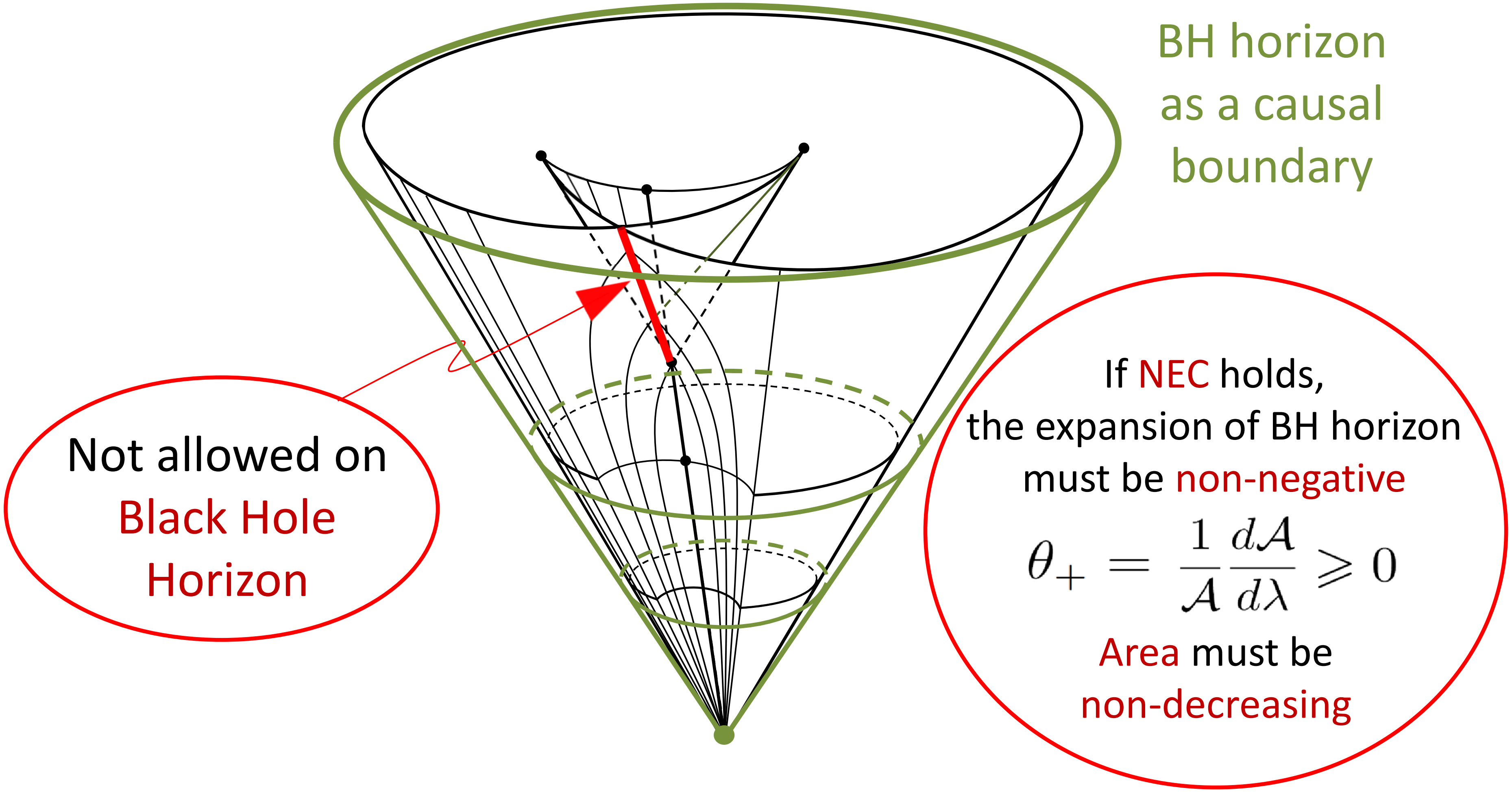
Focusing theorem

Focusing $\theta \to -\infty$ of null geodesic congruence generically occurs under NEC and Null generic condition/initial convergence $\theta_0 < 0$



Area theorem and BH 2nd Law

Causal nature of the BH event horizon does not allow focusing $\theta \to -\infty$ of the null generators toward future.



$$\theta_+ = \frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\lambda} \geqslant 0$$

Area theorem and BH 2nd Law

Causal nature of the BH event horizon does not allow focusing $\theta \to -\infty$ of the null generators toward future.

Classical Focusing w. NEC $\Rightarrow$ $\boxed{\delta \mathcal{A} \geqslant 0}$ Hawking 71

BH must have its own Entropy $S_{BH} = \dfrac{\mathcal{A}}{4G\hbar}$ $\delta S_{BH} \geqslant 0$ Bekenstein 72

Generalized Entropy $S_{gen} = S_{BH} + S_{out}$ Bekenstein 73

$$S_{out} = -\mathrm{Tr}\rho_{out} \log \rho_{out}$$

von Neumann entropy

Generalized 2nd Law $\boxed{\delta S_{gen} \geqslant 0}$ e.g. Page 76, Unruh-Wald 82, Zurek-Thorne 85, Frolov-Page 93

Null Energy Condition

Null Energy Condition (NEC) plays an important role in GR

$$T_{kk} := T_{ab}k^a k^b \geqslant 0$$

- holds for classical matter systems

It governs the focusing of null geodesic congruence, from which one can learn a lot about the spacetime geometry under consideration.

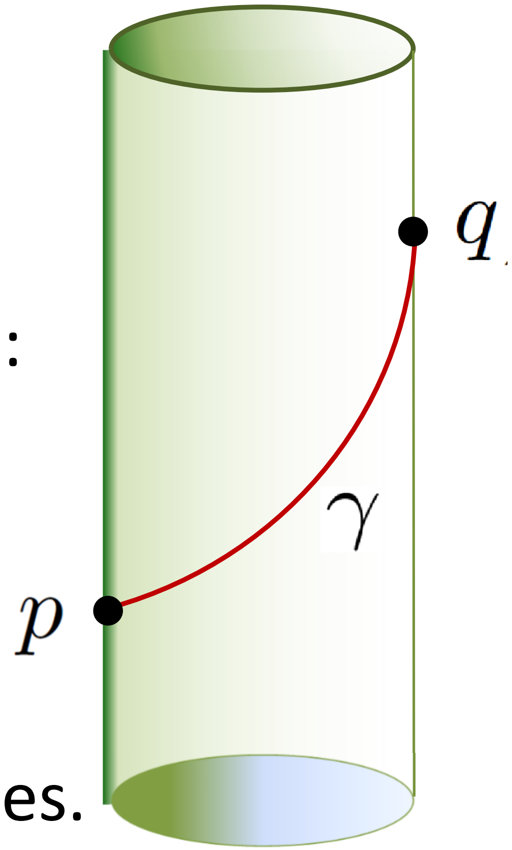
It is used in an essential way in the proof of

Singularity theorems
Area theorem
Topology censorship
No-Bulk-shortcut theorem

No-bulk-shortcut and AdS/CFT

- We are concerned with the compatibility of bulk-boundary causal structures and AdS/CFT duality
- Consider two boundary points p, q on a boundary *achronal* null geodesic γ.

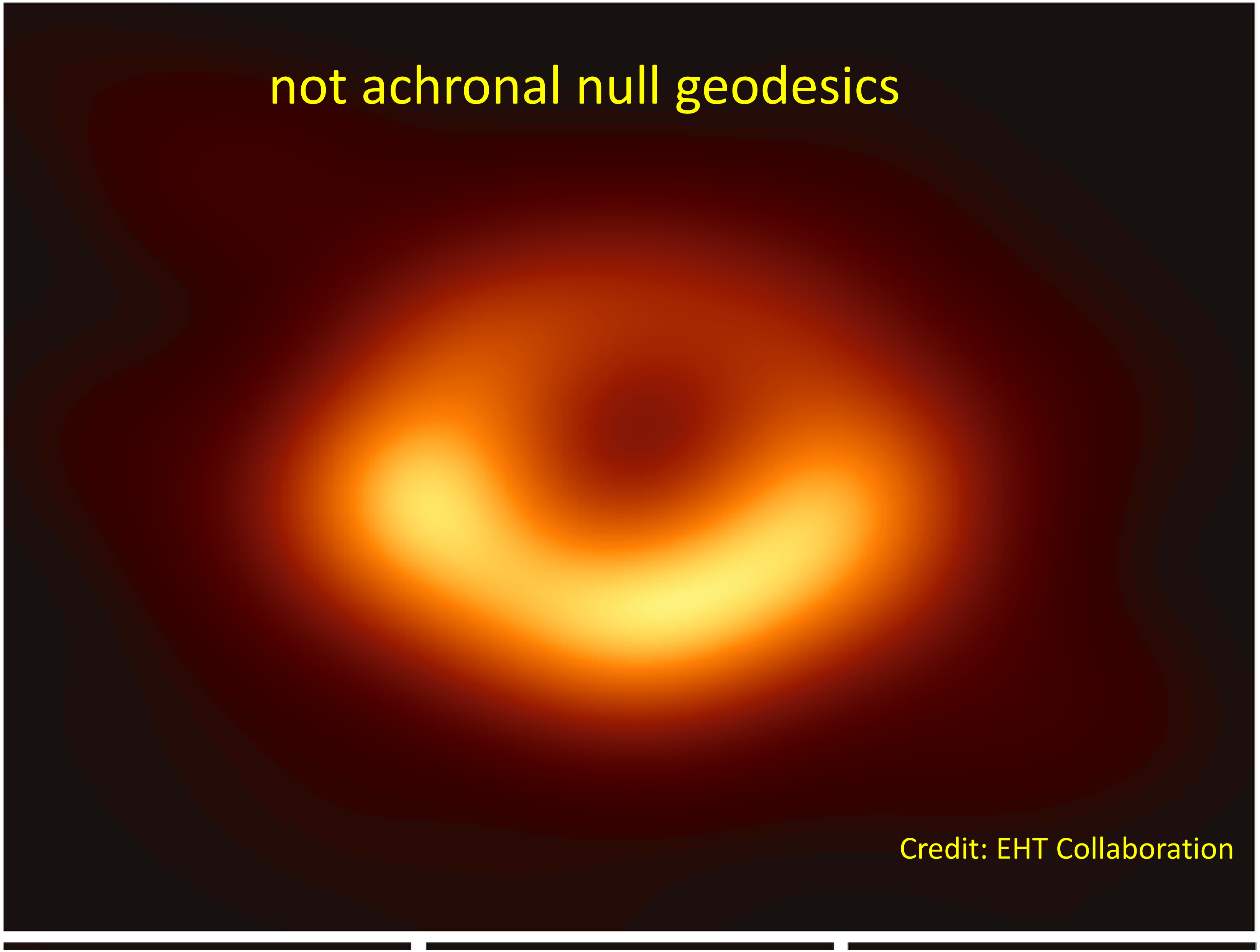
Achronal null γ is the fastest null geodesic:
No two points on γ can be connected by a boundary timelike curve .



Any null geodesic is achronal in Minkowski space but *not* always so in *curved* spacetimes.

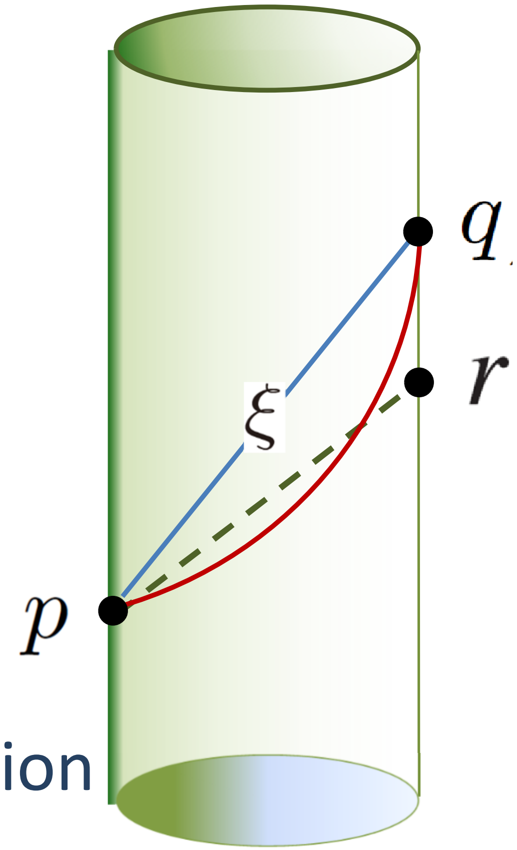
2019年4月10日

EHT Black Hole Shadow



No-bulk-shortcut and AdS/CFT

- We are concerned with the compatibility of bulk-boundary causal structures and AdS/CFT duality
- Consider two boundary points p, q on a boundary achronal null geodesic.
- Suppose there is a bulk-shortcut (timelike) curve ξ. Then p can be connected w/ a boundary point r strictly past to q by a bulk causal curve (dashed-line).



- Then bulk propagator could carry information faster than boundary propagator. This is *not* a sensible causal property for bulk-boundary duality

Basic questions

(i) Given a boundary metric, what kind of effects on the boundary theory does a sensible bulk causal property (i.e., no bulk-shortcut) give rise to?

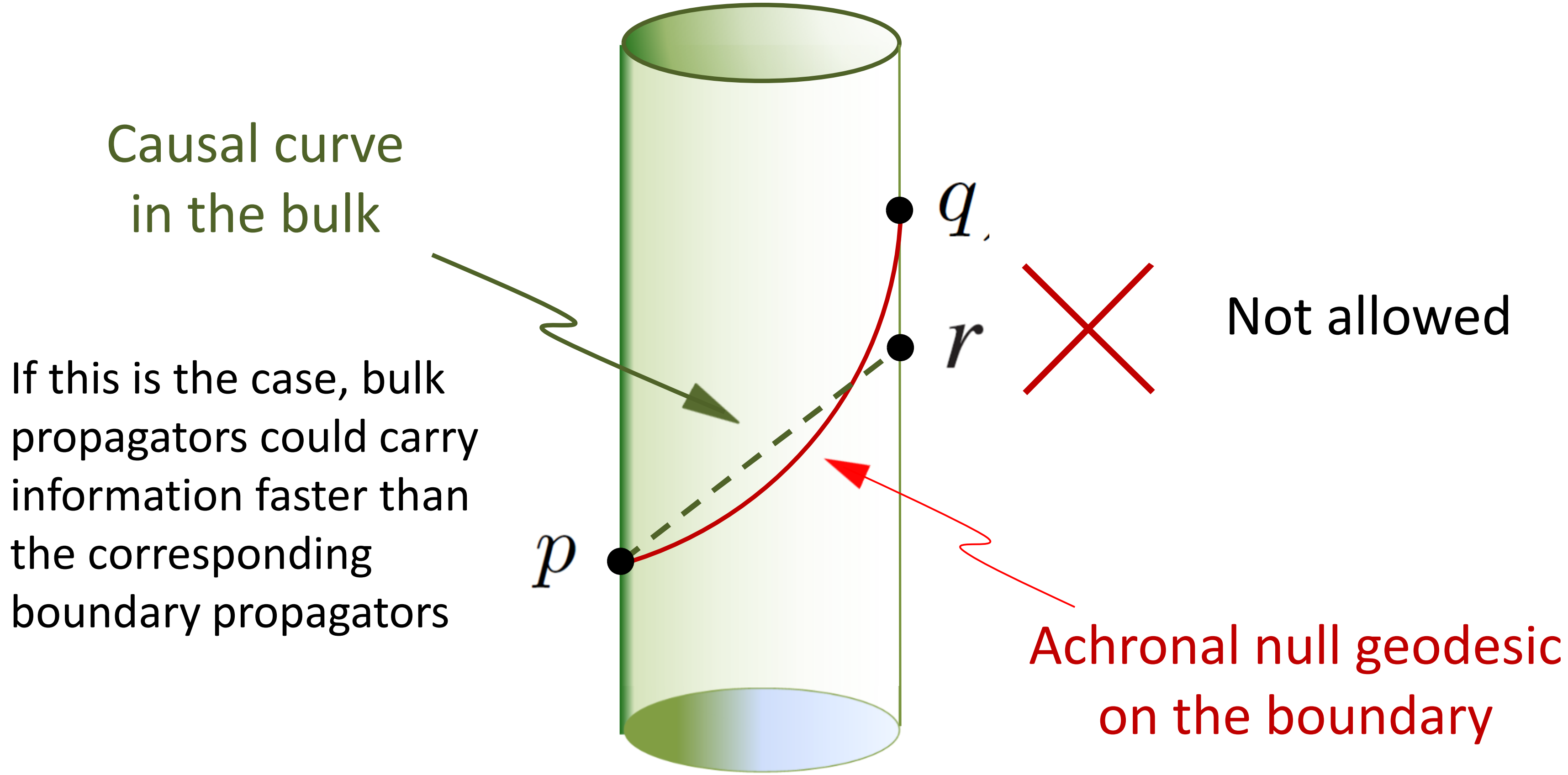
(ii) What kind of boundary metric admits bulk-shortcut?

- Key to answering these questions is the Averaged Null Energy Condition (ANEC)

No-bulk-shortcut Theorem

No acausal propagation (Gao-Wald 00)

Bulk NEC ⇒ No shortcut through the bulk



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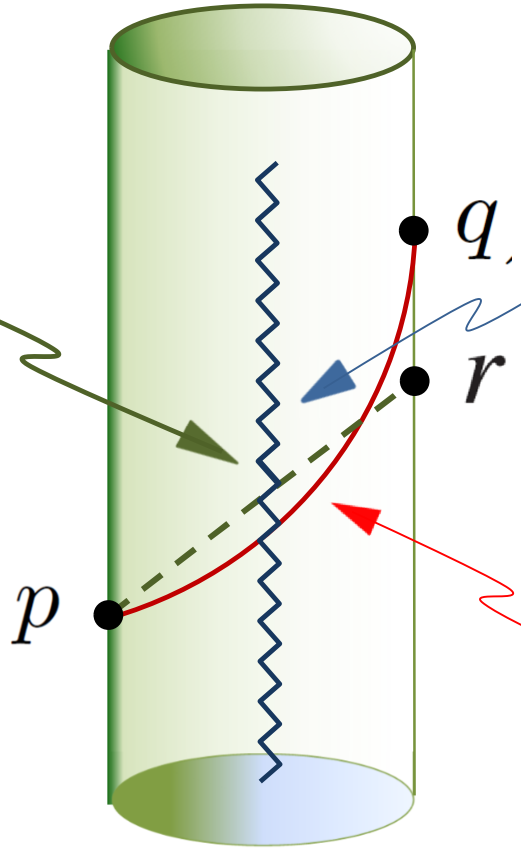


This can be the case when there exists a naked singularity in the bulk

AI-Maeda-Mefford 19

Violation of NEC and Non-local conditions

NEC (any local energy conditions) can be violated by quantum fields.

$$T_{kk} \cancel{\geqslant} 0 \qquad \langle T_{ab} \rangle k^a k^b \cancel{\geqslant} 0$$

- by e.g. Hawking-radiation

Averaged Null Energy Conditions (ANEC)

$$\int \langle T_{ab} \rangle k^a k^b d\lambda \geqslant 0$$

Non-local: defined along any complete null geodesic

e.g. Wald-Yurtsever 91

ANEC can be violated by Quantum Fields.

$$\int \langle T_{ab} \rangle k^a k^b d\lambda \cancel{\geqslant} 0$$

e.g. Visser 96

Achronal averaged Null Energy Conditions (AANEC)

e.g. Graham and Olum 07

- However, quantum field effects violate NEC and ANEC.

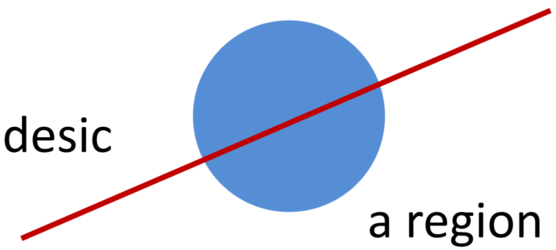
- Counter-examples of ANEC:

A local violation of NEC can be enhanced by conformal transformations. This can occur since ANEC itself is not conformally invariant.

Urban-Olum 10

AI-Maeda-Mefford 19

a null geodesic



a region where NEC is locally violated

Basic questions

(i) Given a curved boundary metric, what kind of effects on the boundary theory does No-bulk-shortcut property give rise to?

(ii) What kind of boundary metric admits bulk-shortcut?

- Key to answering these questions is the Averaged Null Energy Condition (ANEC)

Purpose

- Taking the assertion of No-bulk-shortcut theorem as our guiding principle, we discuss the compatibility of bulk-boundary causality and AdS/CFT duality.
- We derive ANEC with a weight function on boundary curved spacetime.
- In odd-dimensions, it is conformally invariant (CANEC)

ANEC from AdS/CFT

ANEC (CANEC) from AdS/CFT

- (d+1)-asymptotic AdS metric and Fefferman-Graham expansion:

$$ds^2 = \frac{dz^2 + \hat{g}_{\mu\nu}(z,\, x)\, dx^\mu dx^\nu}{z^2}$$

$$\hat{g}_{\mu\nu}(z,\, x) = \sum_{n=0}^{\infty} g_{(n)\mu\nu}(x)\, z^n + \underline{z^d \ln z^2\, h_{\mu\nu}(x)}$$

Only when d is even

- Boundary stress-energy tensor:

de Haro-Solodukhin-Skenderis 01

$$\langle T_{\mu\nu} \rangle = \frac{d l^{d-1}}{16\pi\, G_{d+1}} g_{(d)\mu\nu} + X_{\mu\nu}$$

$X_{\mu\nu}$: gravitational anomaly

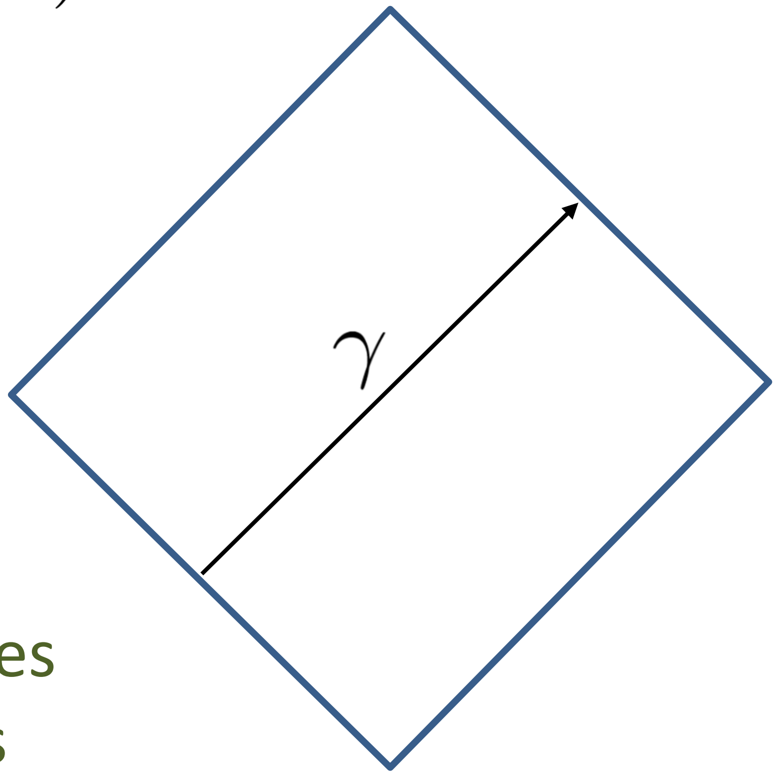
(⇨ vanishes in odd-dimensions)

c.f. ANEC in Minkowski space from AdS/CFT

Kelly-Wall 14

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + z^d t_{\mu\nu} + O(z^{d+1})$$

$$\langle T_{\mu\nu} \rangle = \frac{d l^{d-1}}{16\pi G} t_{\mu\nu}$$



In Minkowski space, all null curves are complete and achronal (fastest null curve).

In curved space, achronal null curves are limited: e.g. horizon generators

The above formulas involve the curved background metric, curvature, source terms and become more complicated in higher dimensions.

ANEC in curved (closed) universe from AdS/CFT

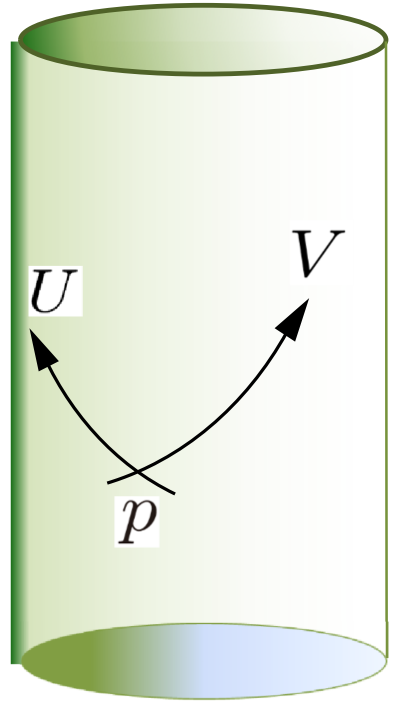
Iizuka-AI-Maeda 19, 20

4-dim vacuum AdS-bulk $R_{ab} - \frac{1}{2} R g_{ab} = -\Lambda g_{ab}$

(in the bulk, NEC is satisfied ➡ No-bulk-shortcut)

3-dim. boundary metric of static compact universe $g_{(0)\mu\nu}$

$$ds_3^2 = -2\, dU dV + \sin^2\left(\frac{V-U}{\sqrt{2}}\right) d\varphi^2$$



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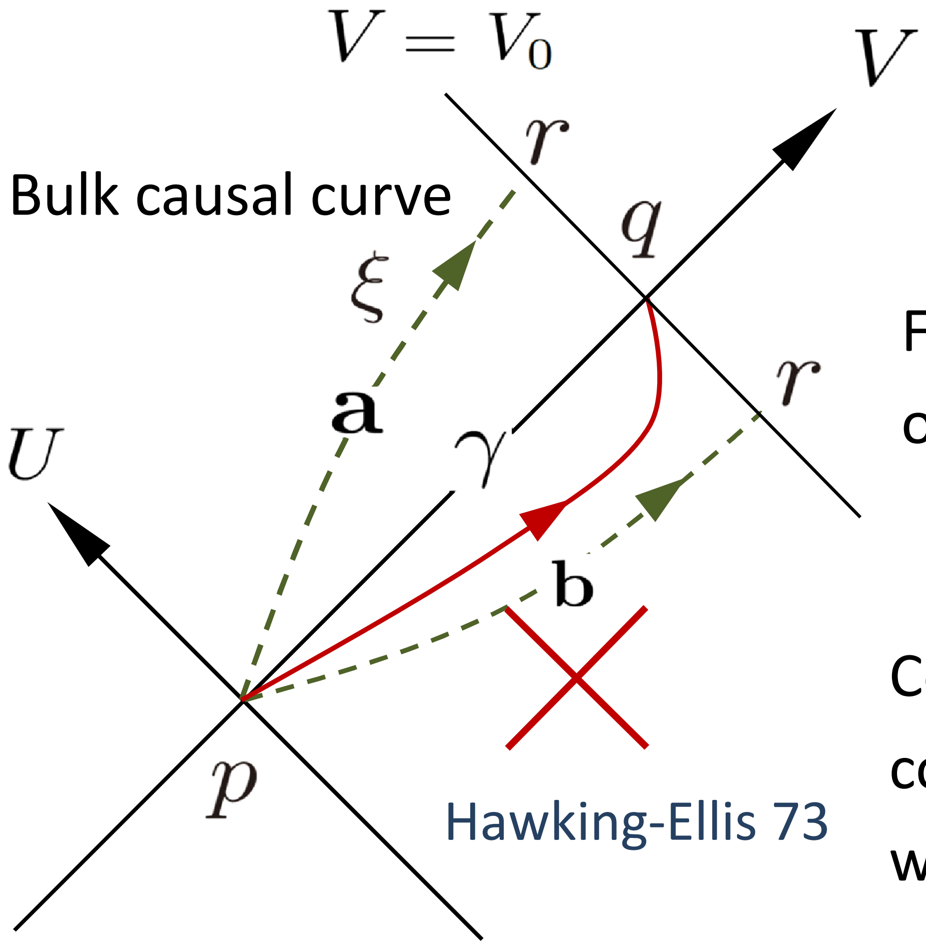
Consider a boundary achronal null geodesic w/ tangent

$$l^\mu = (\partial_V)^\mu \qquad (0 \le \lambda \le \sqrt{2}\pi)$$

Null-null component of boundary Ricci tensor is

$$\hat{R}_{\mu\nu} l^\mu l^\nu = \frac{1}{2}$$



Boundary achronal null geodesic

$$\gamma : p \longrightarrow q$$

From No-shortcut theorem, the value of U at r must be that

$$U(\sqrt{2}\pi) \geq 0$$

Consider a bulk causal curve ξ that connects two boundary points p , r w/ tangent vector:

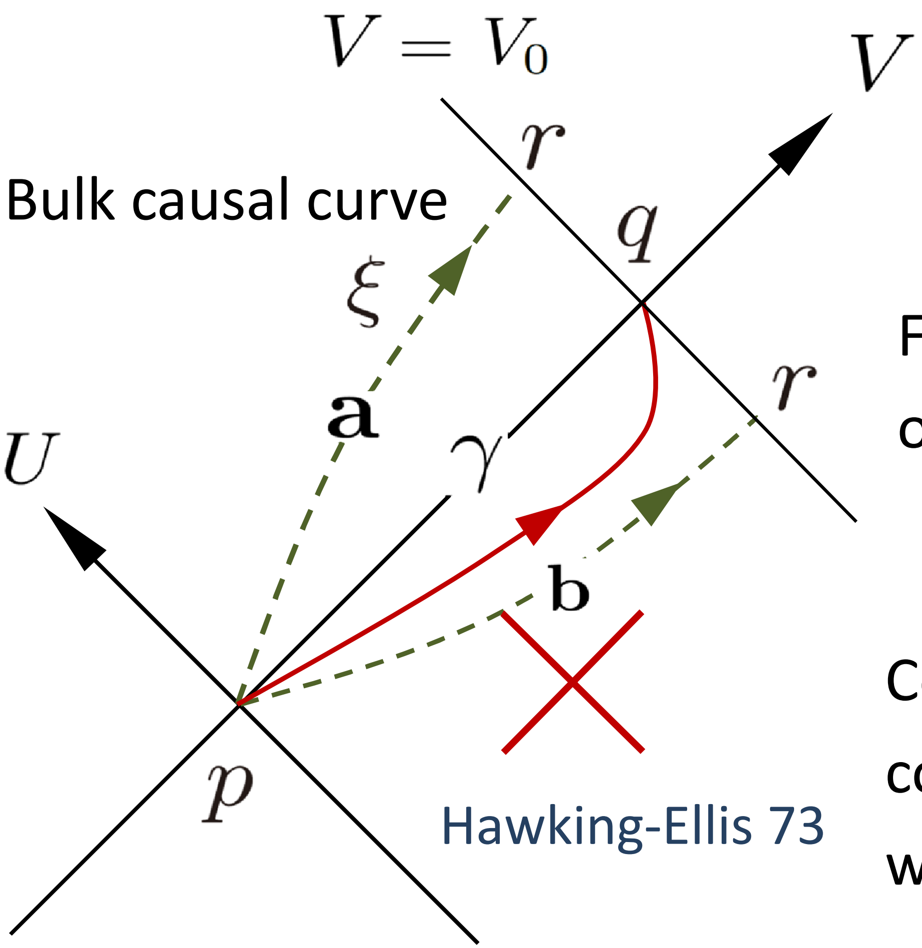
$$K^a = \left(\frac{dz}{d\lambda}, \frac{dU}{d\lambda}, \frac{dV}{d\lambda}, \frac{d\varphi}{d\lambda} \right)$$

$$z = \epsilon z_1 + \epsilon^2 z_2 + \cdots$$

$$K^U = \epsilon^2 \frac{du_2}{d\lambda} + \epsilon^3 \frac{du_3}{d\lambda} + \cdots$$

Since ξ is a causal curve

$$\hat{g}_{ab} K^a K^b \leq 0$$



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$$K^U = \epsilon^2 \frac{du_2}{d\lambda} + \epsilon^3 \frac{du_3}{d\lambda} + \cdots$$

2nd-order of ϵ yields

$$\ddot{z}_1 = -\frac{1}{2} z_1 \;\Rightarrow\; z_1 = \sin \frac{\lambda}{\sqrt{2}}$$

thus z_1 is just like a boundary Jacobi field η

- From $\hat{g}_{ab} K^a K^b \leq 0$ and 3rd-order expansion, it follows:

$$U(\sqrt{2}\pi) \geq \frac{\epsilon^3}{2}\int_0^{\sqrt{2}\pi} z_1^3 \, \underline{g_{(3)\mu\nu}} l^\mu l^\nu d\lambda \geq 0$$

- Holographic renormalized stress-energy formula yields $g_{(3)\mu\nu}$ corresponds to $\langle T_{\mu\nu} \rangle$

- Boundary Raychaudhuri equation implies that z_1 can be viewed as the boundary Jacobi field η

$$\ddot{z}_1 = -\hat{R}_{\mu\nu} l^\mu l^\nu z_1 \quad \longleftrightarrow \quad \ddot{\eta} = -\hat{R}_{\mu\nu} l^\mu l^\nu \eta$$

Bulk Boundary

Thus we obtain: $$\boxed{\int_{\lambda_-}^{\lambda_+} \eta^3 \langle T_{\mu\nu} \rangle l^\mu l^\nu d\lambda \geq 0}$$

- This is invariant under conformal transformation:

$$\tilde{g}_{(0)\mu\nu} = \Omega^2 g_{(0)\mu\nu} \qquad \frac{d\tilde{\lambda}}{d\lambda} = \Omega^2 \qquad \tilde{\eta} = \Omega\,\eta \qquad \langle \tilde{T}_{\mu\nu} \rangle = \Omega^{-1} \langle T_{\mu\nu} \rangle$$

- One can derive this conformally invariant ANEC (CANEC) in the case of more generic 3-dim. or 5-dim Boundary spacetime w/ positive spatial curvature

$$\boxed{\int_{\lambda_-}^{\lambda_+} \eta^3 \langle T_{\mu\nu} \rangle l^\mu l^\nu d\lambda \geq 0 \qquad \int_{\lambda_-}^{\lambda_+} \eta^5 \langle T_{\mu\nu} \rangle l^\mu l^\nu d\lambda \geq 0}$$

3-dim. bndry case　　　　5-dim. bndry case

- These inequalities are given in terms only of the boundary quantities .
- These inequalities can be used to restrict the extent of the negative value of the null energy $\langle T_{\mu\nu} \rangle l^\mu l^\nu$
- The conf. transf. trick used for ANEC cannot invalidate CANEC .

- So far, we have focused on the odd-dimensional boundary case.

- In even-dimensions, one has to deal with conformal anomalies, which make relevant formulas more complicated.

- Boundary stress-energy tensor: de Haro-Solodukhin-Skenderis 01

$$\langle T_{\mu\nu} \rangle = \frac{d l^{d-1}}{16\pi\, G_{d+1}} g_{(d)\mu\nu} + X_{\mu\nu}$$

$X_{\mu\nu}$: gravitational anomaly

- Using a similar holographic method, one can derive a lower bound for weighted ANEC in even-dimensions.
- d=2 case: We can obtain the ANEC.
- d=4 case:

$$4\pi G\int_{v_-}^{v_+}\eta^4\langle T_{\mu\nu}\rangle\, l^\mu l^\nu \geq \frac{1}{12}\int_{v_-}^{v_+}\eta^4\Big[\theta_+^2\left(R_{uv}+\frac{R_{\theta\theta}}{r^2}\right)-\frac{\theta_+^2}{2}\cdot\mu(r)$$

$$+\frac{1}{2}\theta_+\theta_- R_{vv}+\left(\frac{3}{2r^2}-\frac{R_{\theta\theta}}{r^2}-2R_{uv}\right)R_{vv}\Big]$$

$\theta_\pm$: Expansion of boundary null geodesics

$\mu(r) := \dfrac{1}{r^2}+\theta_+\theta_-$: Quasi-local mass density

$R_{\mu\nu}$: Boundary Ricci curvature

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- d=4 case:

$$4\pi G\int_{v_-}^{v_+}\eta^4\langle T_{\mu\nu}\rangle\, l^\mu l^\nu \geq \frac{1}{12}\int_{v_-}^{v_+}\eta^4\Big[\theta_+^2\Big(R_{uv}+\frac{R_{\theta\theta}}{r^2}\Big)-\frac{\theta_+^2}{2}\cdot\mu(r)$$

$$+\frac{1}{2}\theta_+\theta_-R_{vv}+\Big(\frac{3}{2r^2}-\frac{R_{\theta\theta}}{r^2}-2R_{uv}\Big)R_{vv}\Big]$$

Einstein-Static cylinder case

$$\geq \frac{1}{8}\int_{v_-}^{v_+}\eta^4 > 0$$

- Schwarzschild-AdS bulk and boundary ANEC

$$ds^2 = -\left(r^2 + 1 - \frac{M}{r^2}\right)dt^2 + \left(r^2 + 1 - \frac{M}{r^2}\right)^{-1} dr^2 + r^2(d\rho^2 + \sin^2\rho\, d\Omega^2)$$

- The boundary weighted ANEC implies that the bulk mass parameter must be non-negative:

$$\Rightarrow \quad M \geq 0$$

- -- suggests No-bulk-shortcut condition to be connected with the positive mass theorem in asymptotically AdS spacetime.

- A higher curved boundary spacetime in which ANEC is violated.

Iizuka-AI-Maeda 19, 20

- C.f. conformal symmetry implies ANEC for CFT in dS and AdS

and a similar lower bound for ANEC in closed universe (Lorentizan cylinder) is obtained.

Rosso 20

- For some closed universe the minimum of the ANEC can be negative.

Fischetti-Hickling-Wiseman 16

Summary

- We have pursued the question of what is a sensible causal interaction between the bulk and conformal boundary in the context of AdS/CFT duality with Gao-Wald's No-bulk-shortcut property as our guiding principle for the appropriate choice of boundary geometry.

- We have derived, in the case of generic 3-dim. or 5-dim boundaries w/ compact positive spatial curvature, a conformally invariant ANEC (CANEC), which is expressed in terms only of the boundary quantities.

  No-bulk-shortcut $\Rightarrow$ CANEC

- In the even-dimensional case, one has to deal with conformal anomalies. We have derived ANEC in 2-dimensions and a lower bound for ANEC with a certain weight function in 4-dimension.