Efficiently generating ground states is hard for postselected quantum computation

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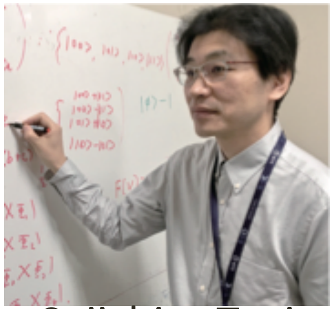
Joint work with

Yasuhiro Takahashi
(NTT)



Seiichiro Tani
(NTT)

Recent progress in theoretical physics based on quantum information theory

2nd March, 2021

Ground states of Hamiltonians

Generating ground states of (local) Hamiltonians and measuring their energy are important to explore several physical properties such as

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- Magnetic properties
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- (Classical) variational methods
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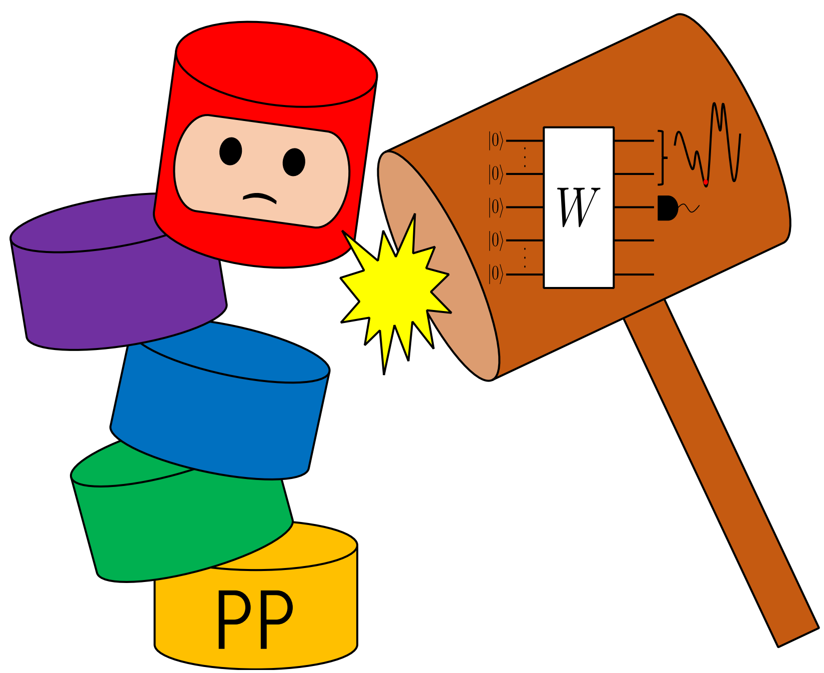
Our question: *In general*, how hard to efficiently generate them?

Our result: We give an evidence for its hardness in terms of computer science.

Our result

If there exists (a family of) postselected quantum circuits efficiently generating ground states of any 3-local Hamiltonians, then the counting hierarchy collapses to its 1st level.

Counting Hierarchy



A cartoon of our result

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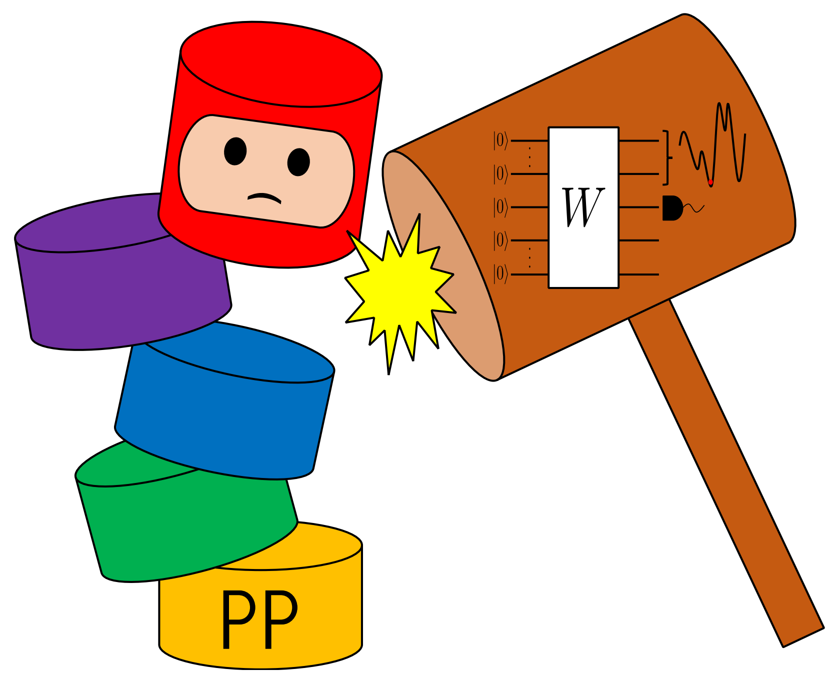

Classical complexity classes

Complexity class = A set of decision problems (i.e., YES-NO questions)

ex.) ⭕ Is 1+1equal to 2? → YES

✖ What is 1+1? → 2

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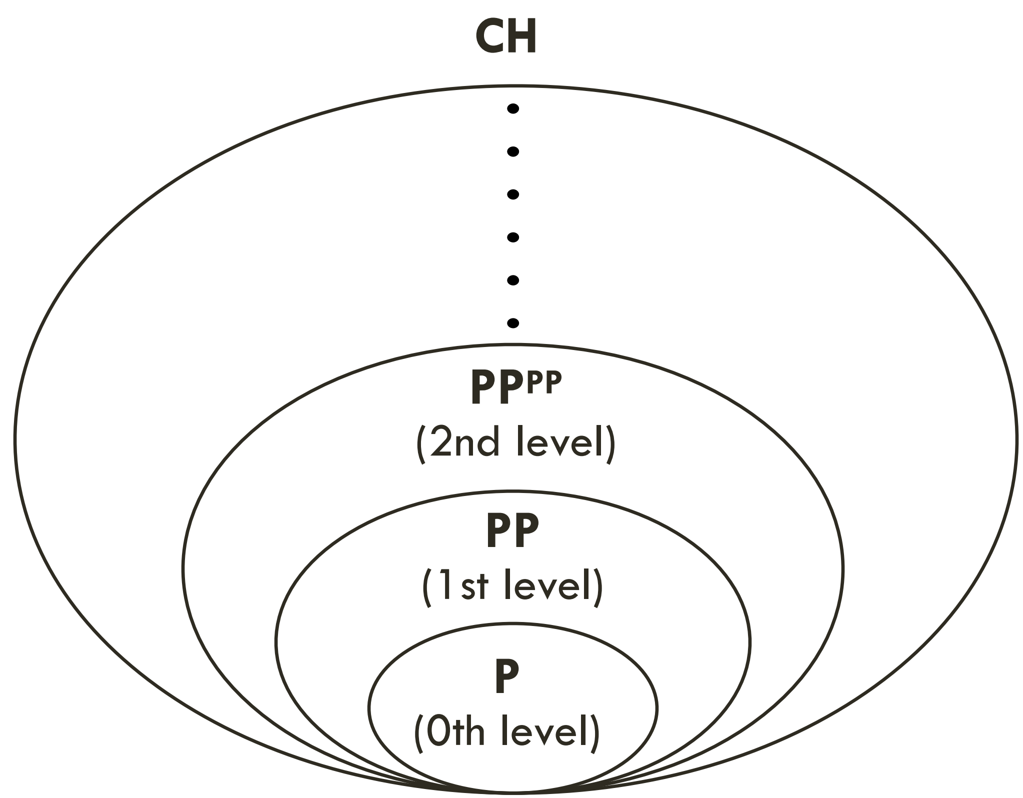
<u>Counting hierarchy (CH)</u>

An infinite tower of complexity classes that generalize **PP**



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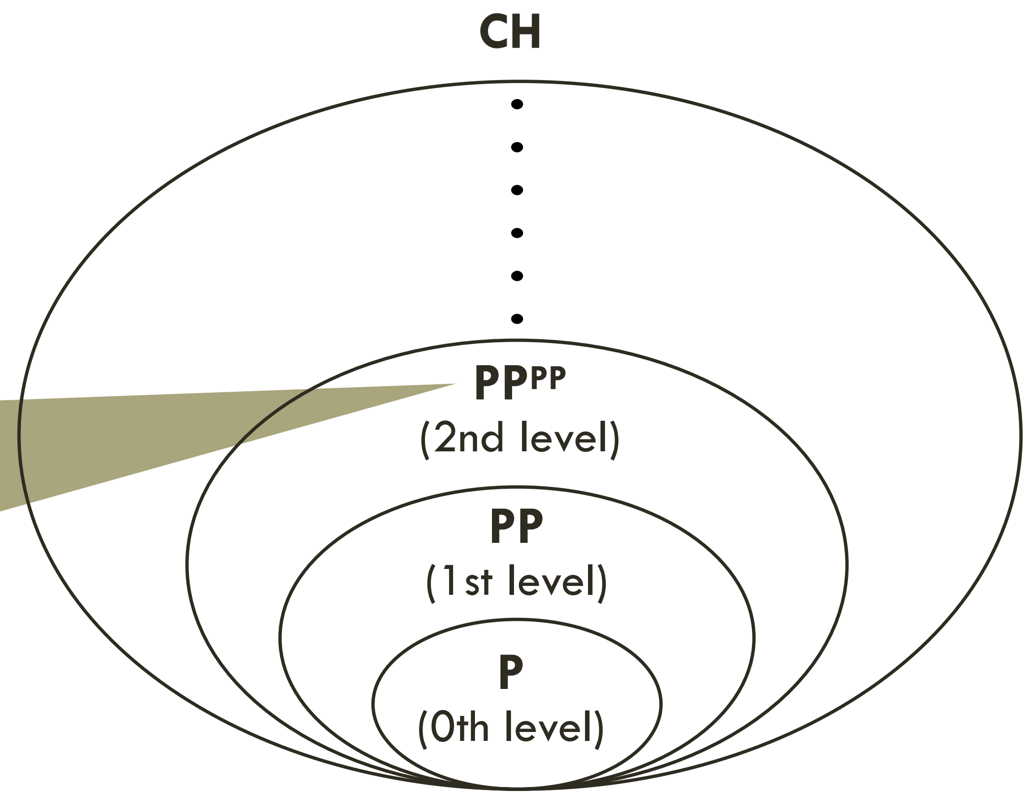

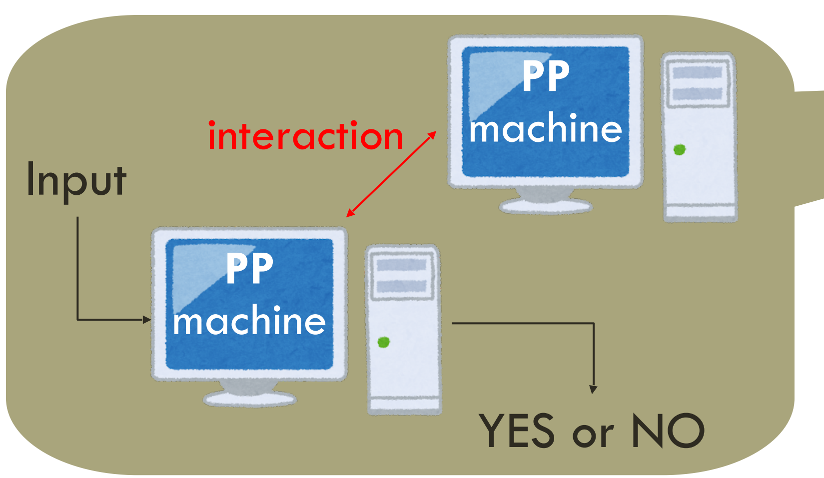

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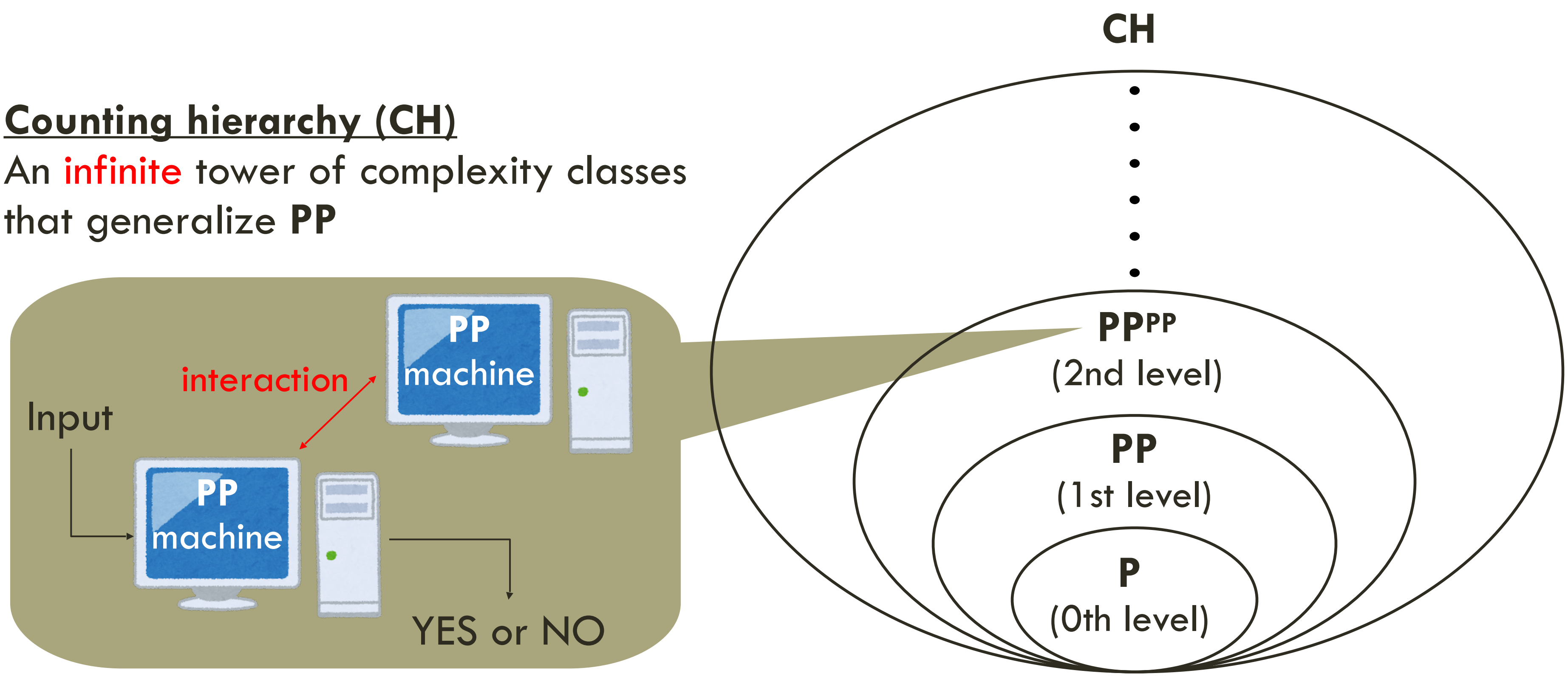

The 0th-level collapse of **CH** → **P** = **NP**: unlikely relation (strongly believed in C.S.)

Classical complexity classes

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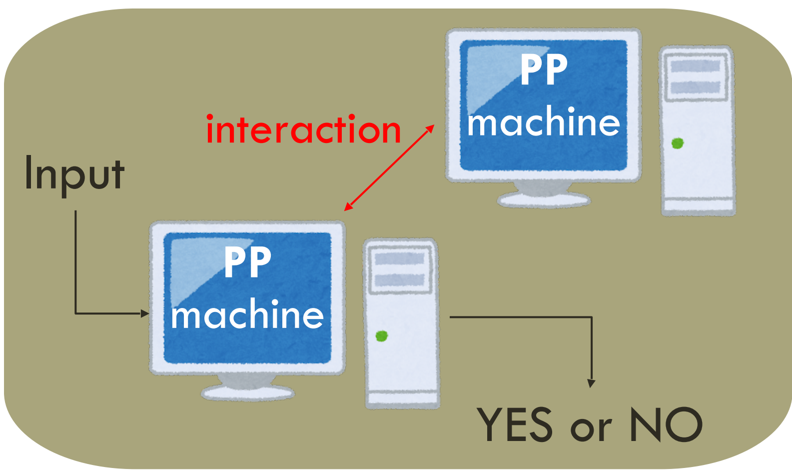

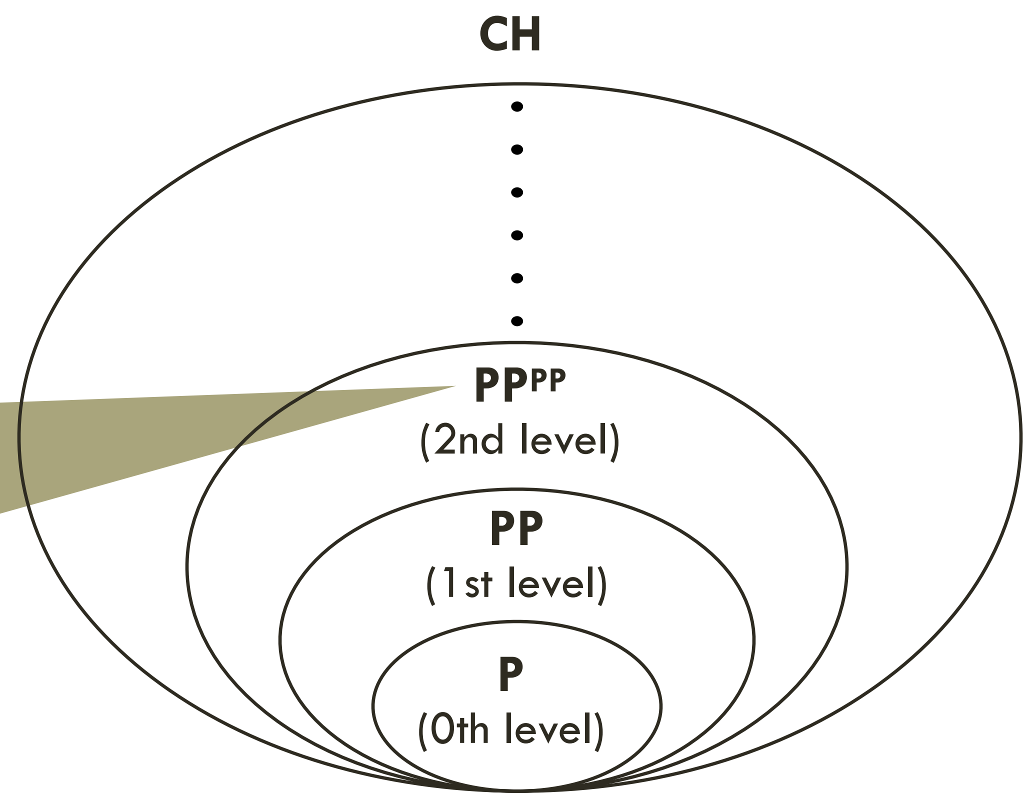

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The 1st-level collapse of **CH** → **PH** ⊆ **PP**: relation used as unlikely one in some papers (e.g., [M. Vyalyi '03])

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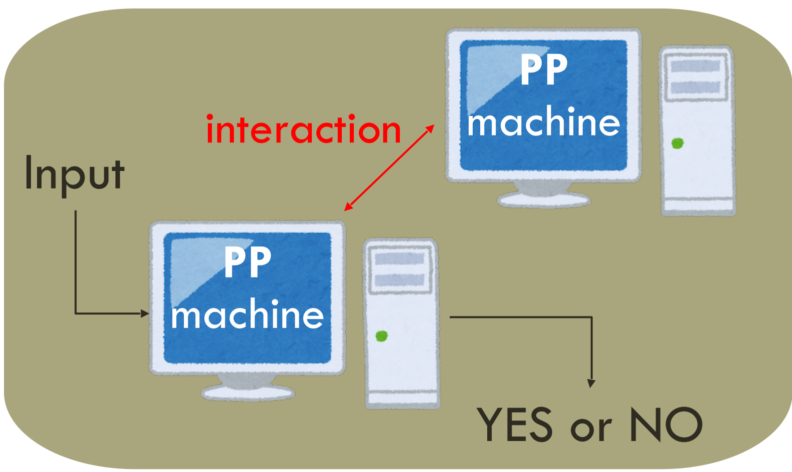

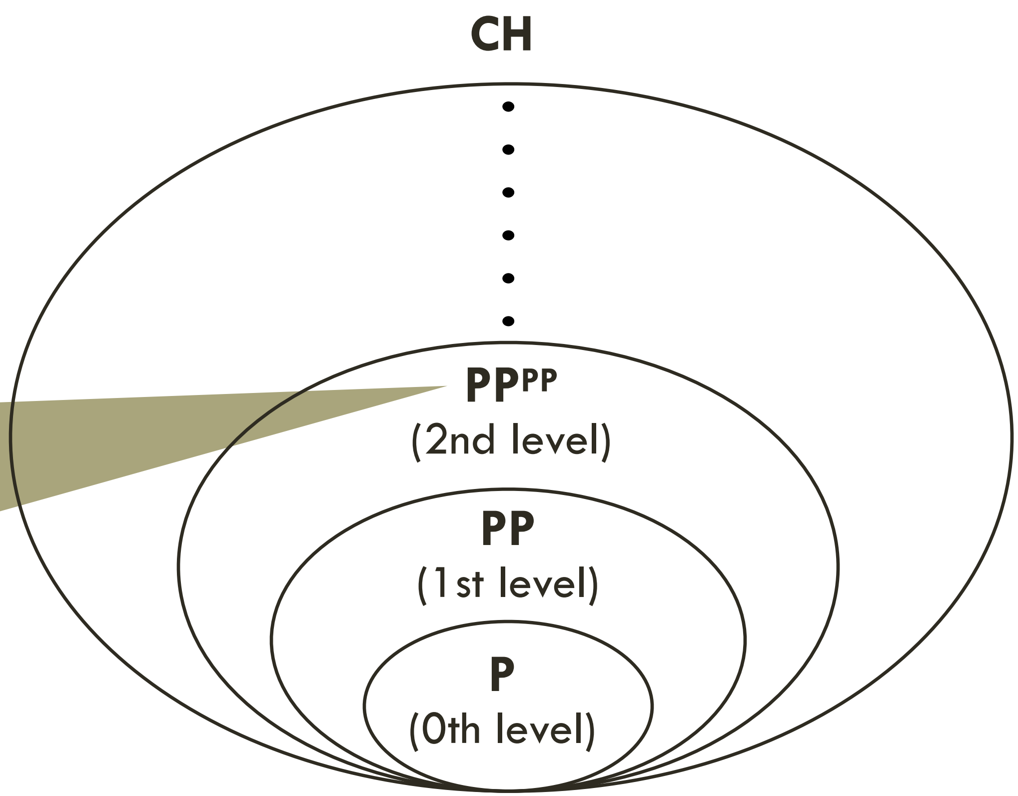

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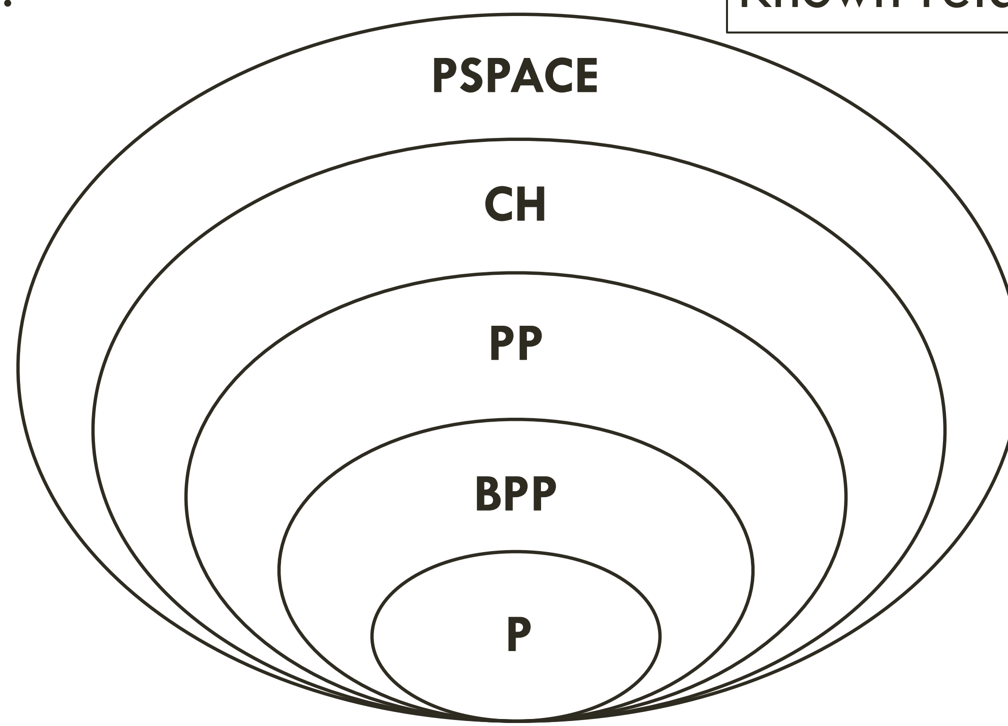
Classical complexity classes

PSPACE

Decision problems solvable by a classical computer with no error probability by using only a polynomial amount of space

The required time does not matter.

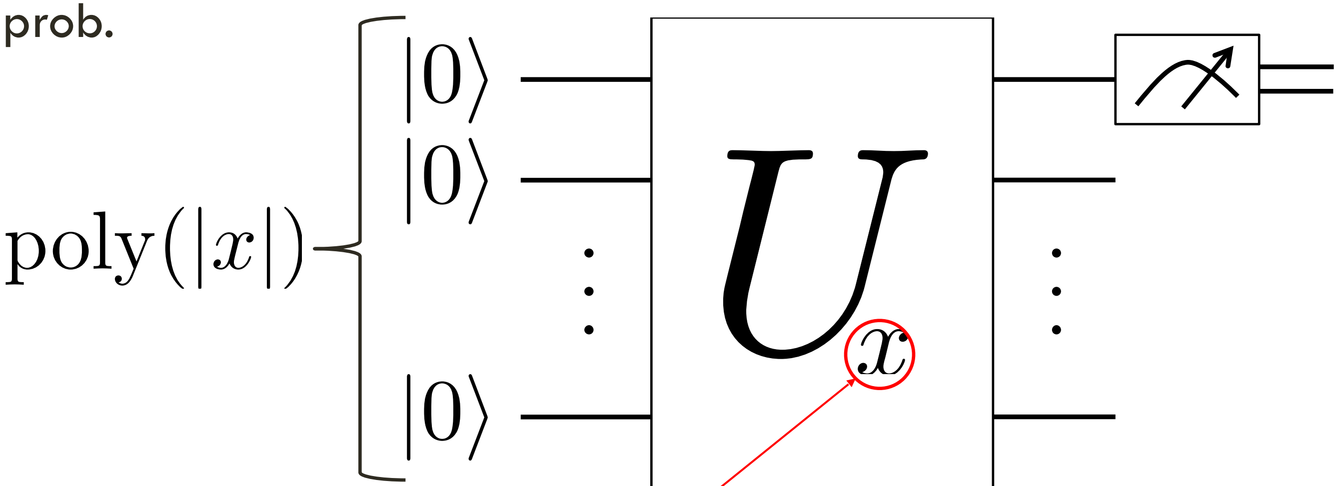
Known relations



Quantum complexity classes

BQP

decision problems efficiently solvable by a quantum computer with a bounded error prob.



$o = 1$ with prob.

$\geq 2/3$ (YES case)

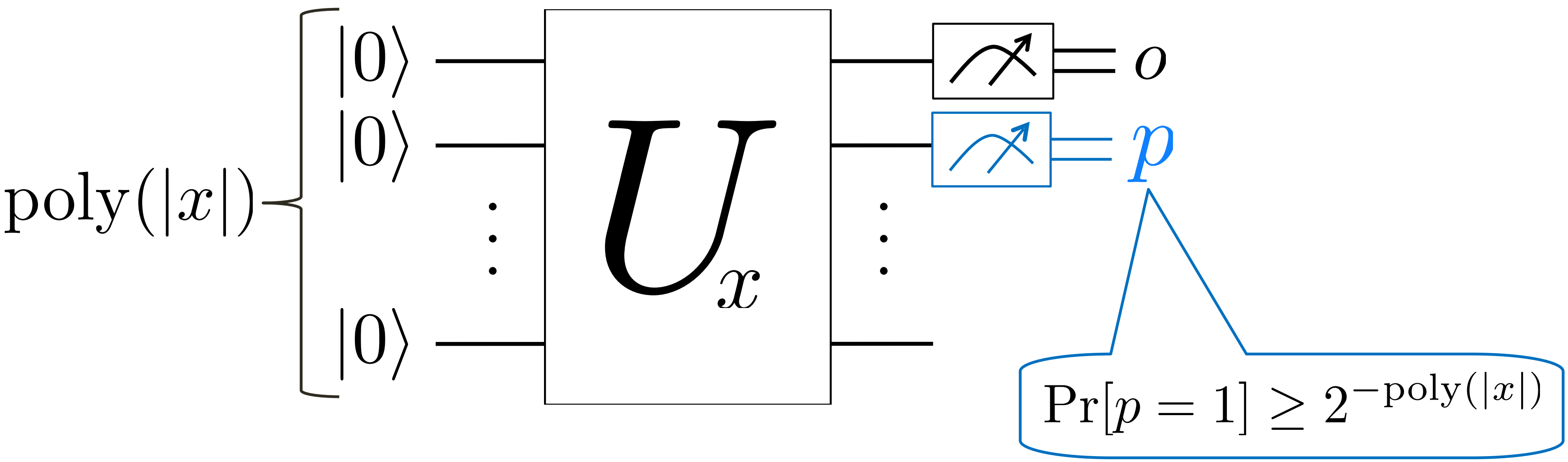
$\leq 1/3$ (NO case)

An instance of the decision problem

Quantum complexity classes

postBQP

the postselected version of **BQP**

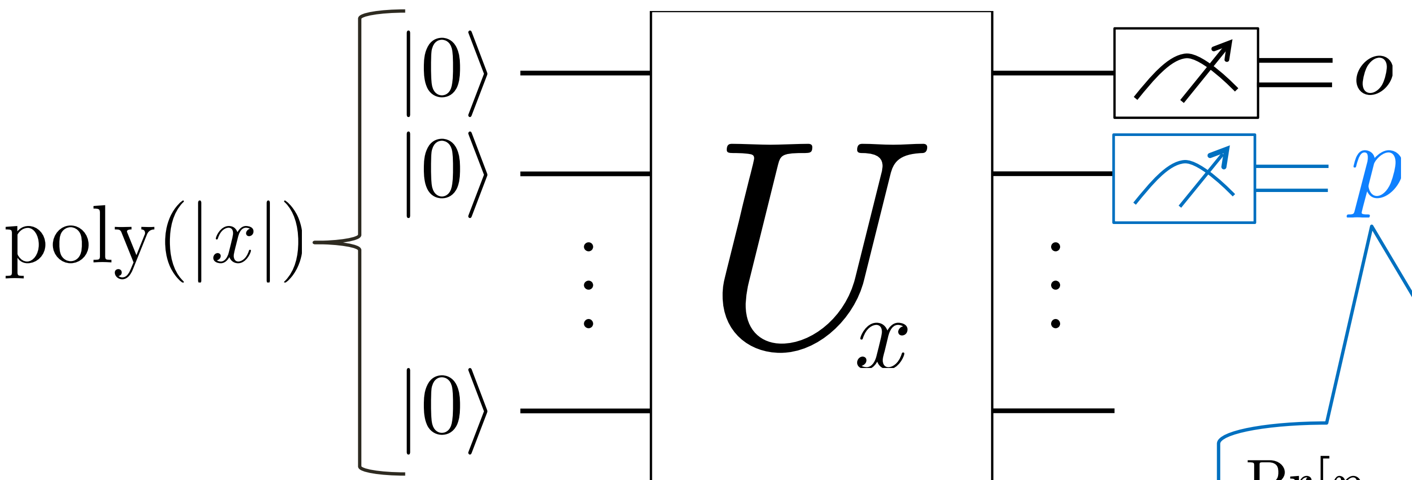


(YES case) $\Pr[o=1 \mid p=1] \geq 2/3$

(NO case) $\Pr[o=1 \mid p=1] \leq 1/3$

Quantum complexity classes

postBQP = **PP** [S. Aaronson '05]
the postselected version of **BQP**



$$\Pr[p = 1] \geq 2^{-\mathrm{poly}(|x|)}$$

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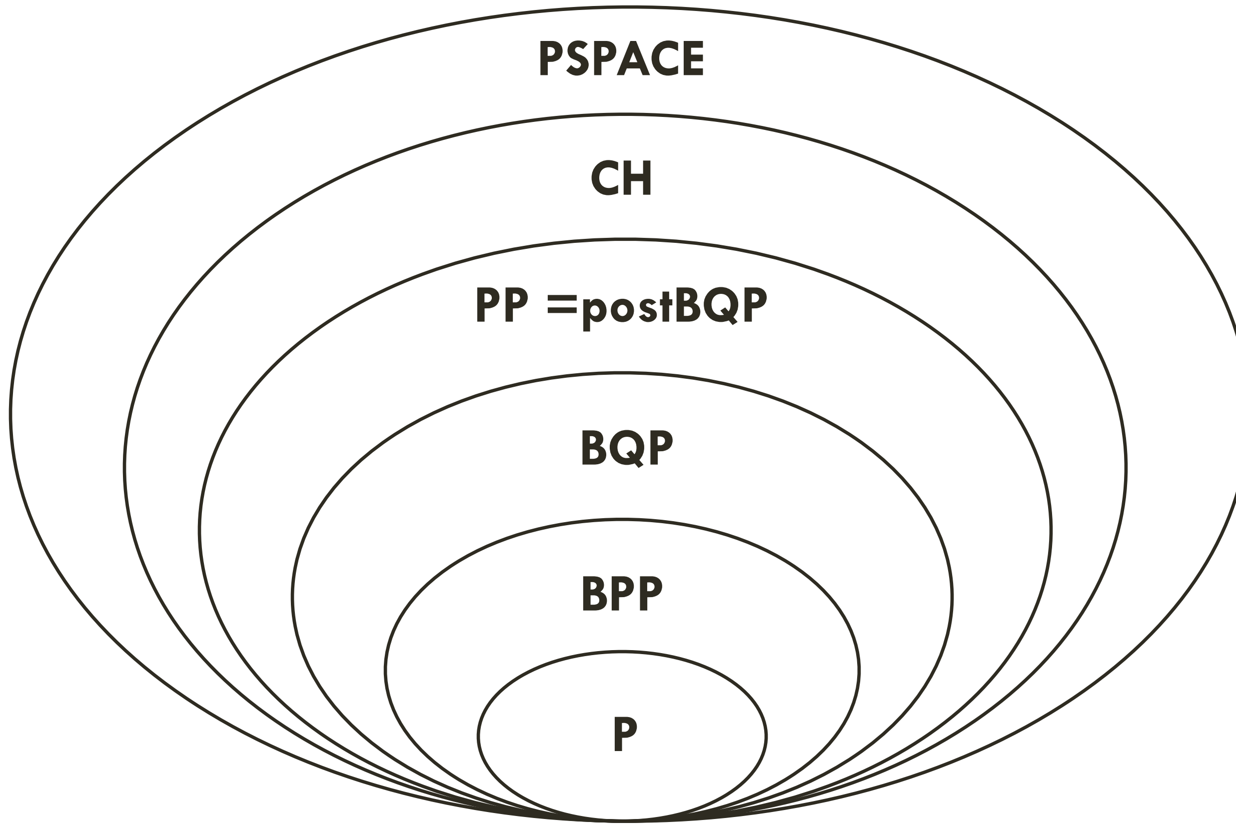
Relations between complexity classes



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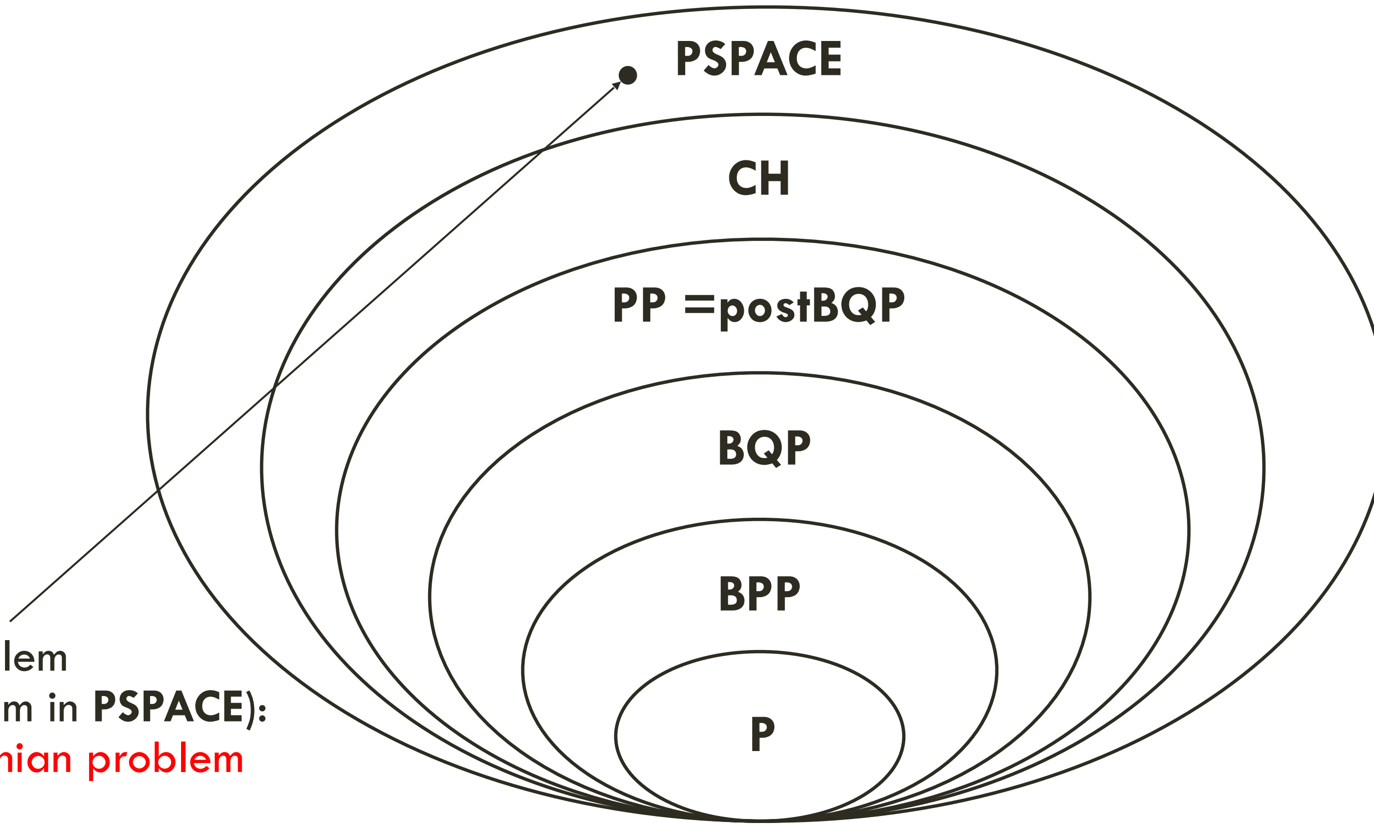

PSPACE-complete problem
(i.e., the hardest problem in **PSPACE**):
Precise 3-local Hamiltonian problem

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3-local Hamiltonian

An n-qubit Hamiltonian H represented as a sum of a polynomial number of 3-qubit Hermitian operators $\{H^{(i)}\}_i$:

$$H = \sum_{i=1}^{t} H^{(i)} \quad (||H^{(i)}|| \leq 1, t = \mathrm{poly}(n))$$

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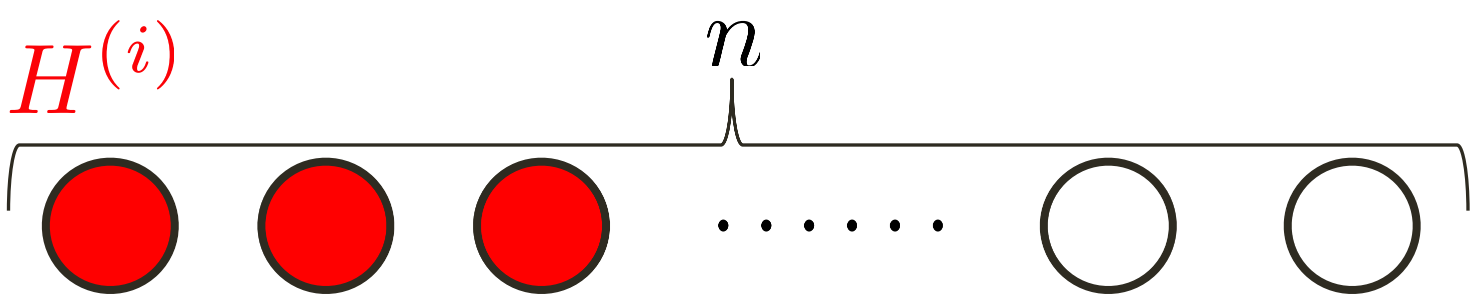

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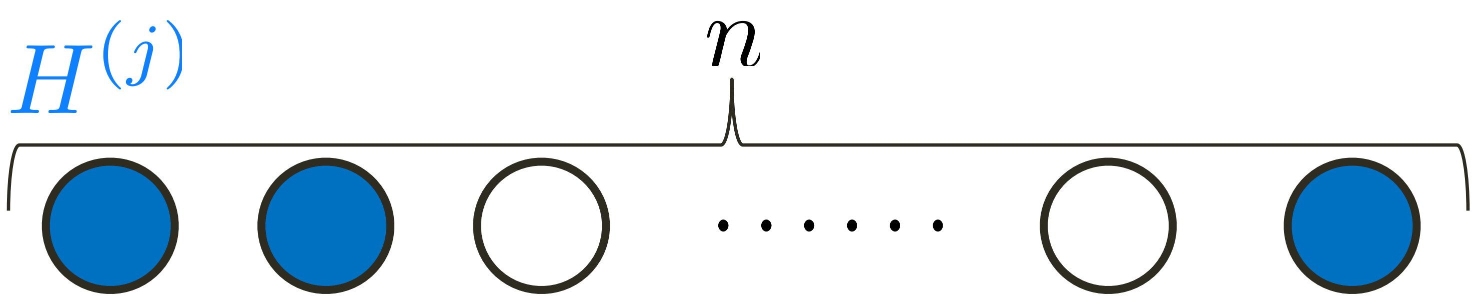


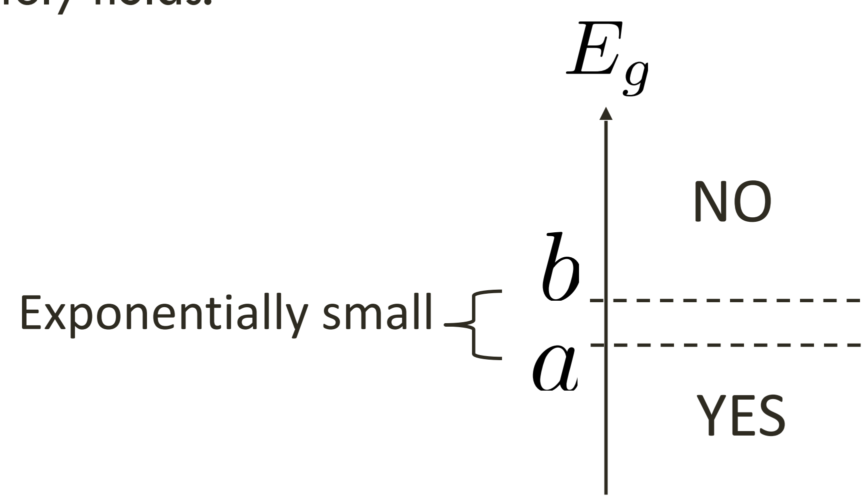
Precise 3-local Hamiltonian problem

Precise 3-local Hamiltonian problem [B. Fefferman and C. Y.-Y. Lin '18]

Given a 3-local Hamiltonian H and two real numbers a and b satisfying b-$a > 2^{-\mathrm{poly}(n)}$, decide either holds.

- (YES case) The ground-state energy E_g of H is $\leq a$.
- (NO case) $E_g \geq b$.

It is promised that one of them definitely holds.



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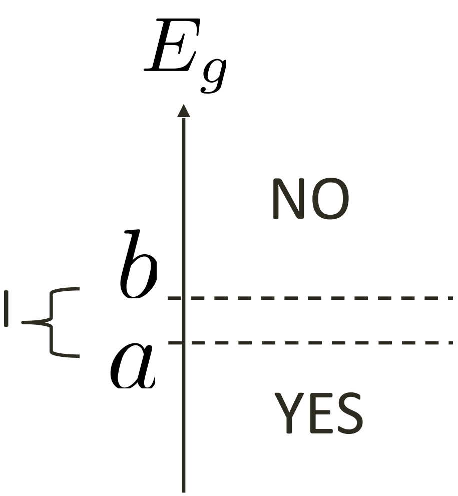
Exponentially small
= precise



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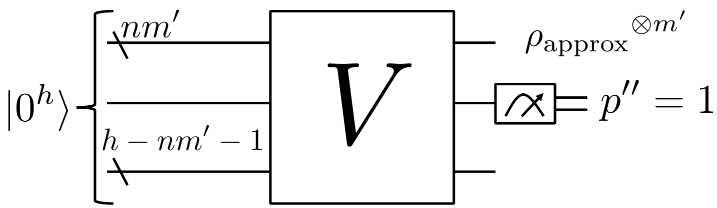
Our result improves the previous one (in a sense).

Previous hardness results are shown for universal quantum computation without the postselection. [J. Kempe, A. Kitaev, and O. Regev '06]

Sketch of the proof



The precise 3-local Hamiltonian problem is in **postBQP.**

Exponential accuracy:

$$\langle g|\rho_{\mathrm{approx}}|g\rangle = 1 - 2^{-\mathrm{poly}(n)}$$

Condition from **postBQP**:

$$\Pr[p''=1] \geq 2^{-\mathrm{poly}(n)}$$

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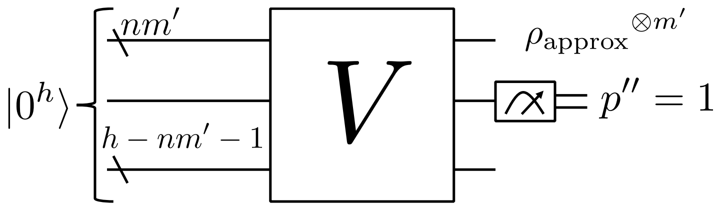

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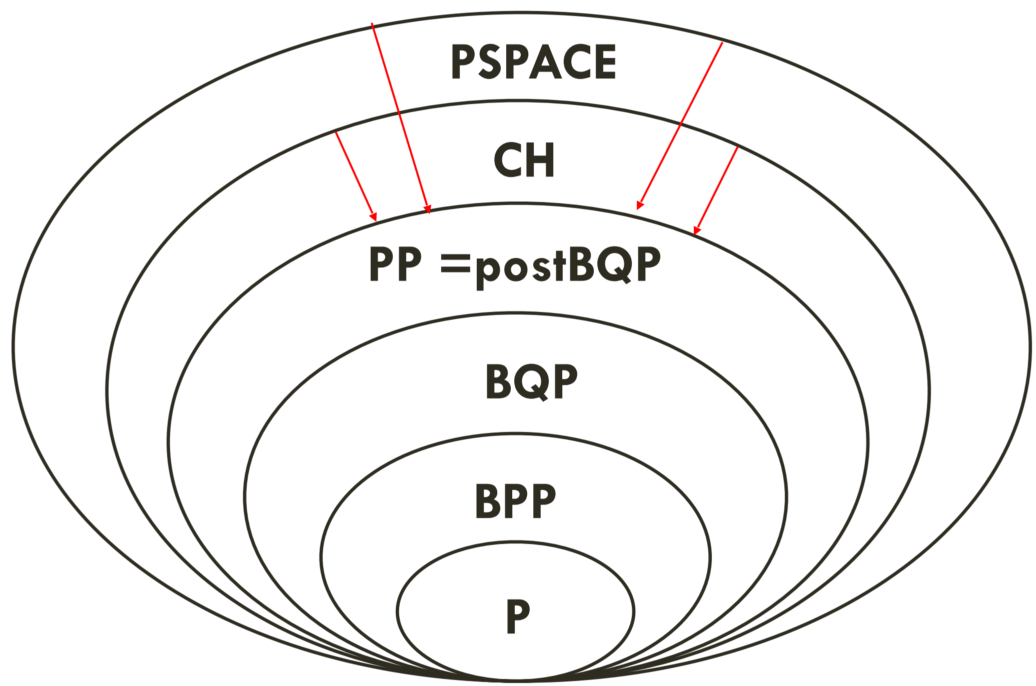


Outlook

- Can we show the hardness for a constant (or the inverse of a polynomial) precision of the approximation?

- Can we strengthen the unlikliness?

  ex.) One direction is to improve the 1st-level collapse to the 0th-level one.

- Can our result be generalized to other Hamiltonians such as

✓ 2-local Hamiltonians

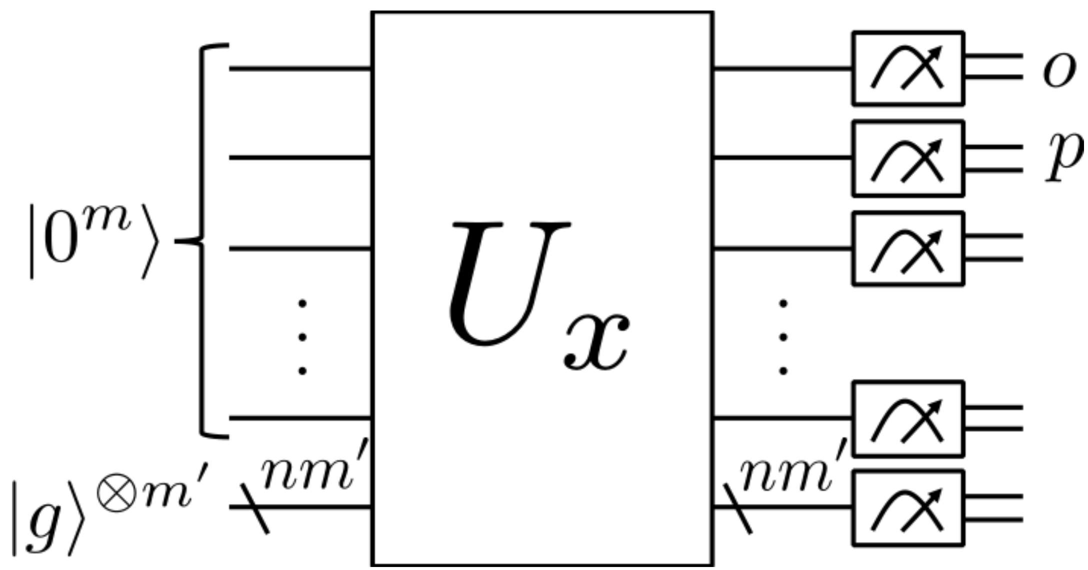
✓ Translation-invariant Hamiltonians

✓ Geometrically-local Hamiltonians?

Proof of our result

- The 1st step (Our lemma)

The following polynomial-size quantum circuit can be constructed by a classical computer in polynomial time.



For the precise 3-local Hamiltonian problem, $\Pr[o = 1 \mid p = 1] \geq 2/3$ (YES case)

$\leq 1/3$ (NO case).

Proof of our result

It is based on some previous results:

- ❖A. Y. Kitaev, A. H. Shen, and M. N. Vyalyi, *Classical and Quantum Computation* (2002).
- ❖D. Aharonov and T. Naveh, arXiv:quant-ph/0210077 (2002).
- ❖T. Morimae and H. Nishimura, QIC **17**, 959 (2017).
- ❖B. Fefferman and C. Y.-Y. Lin, in *proc. of ITCS*, p. 4:1 (2018).

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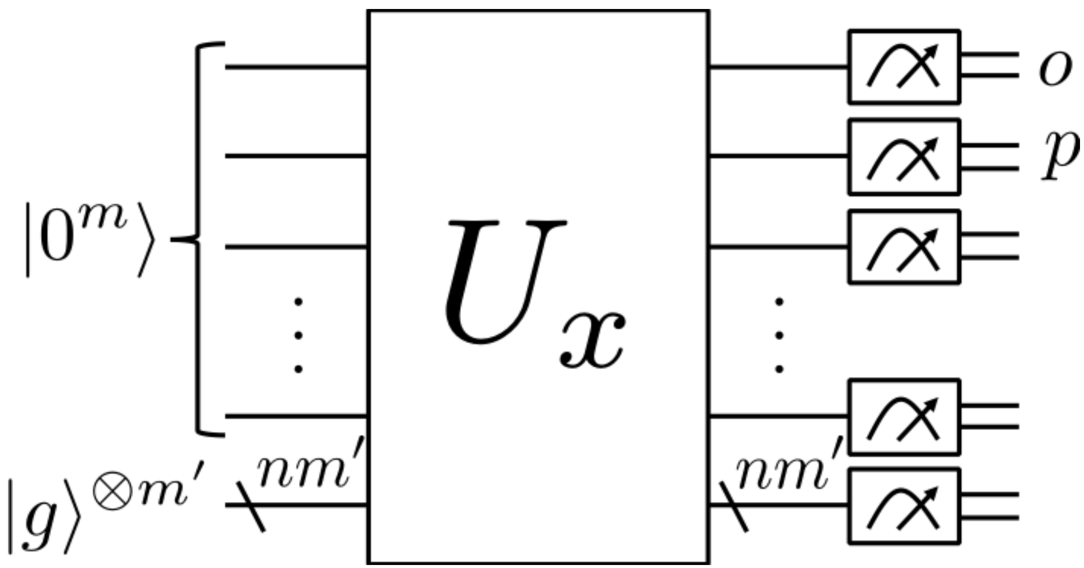

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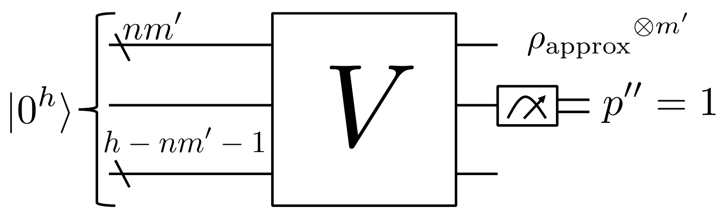
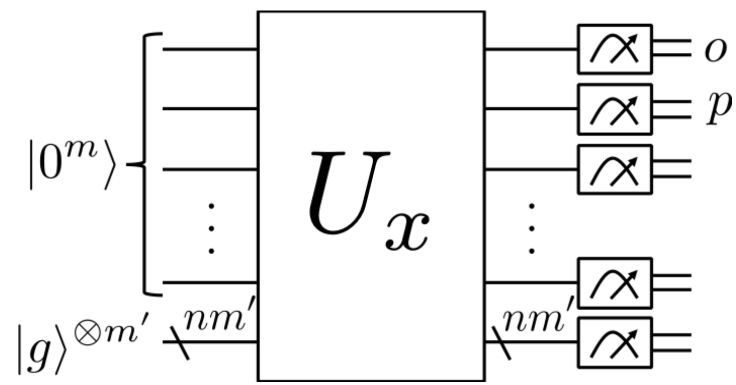


(Assumption)



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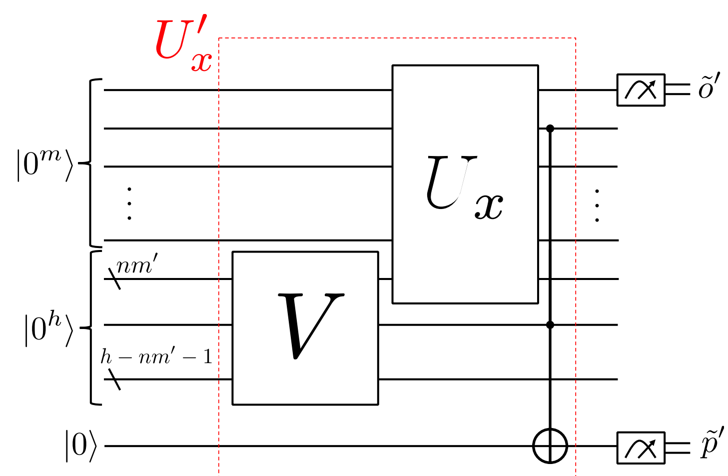
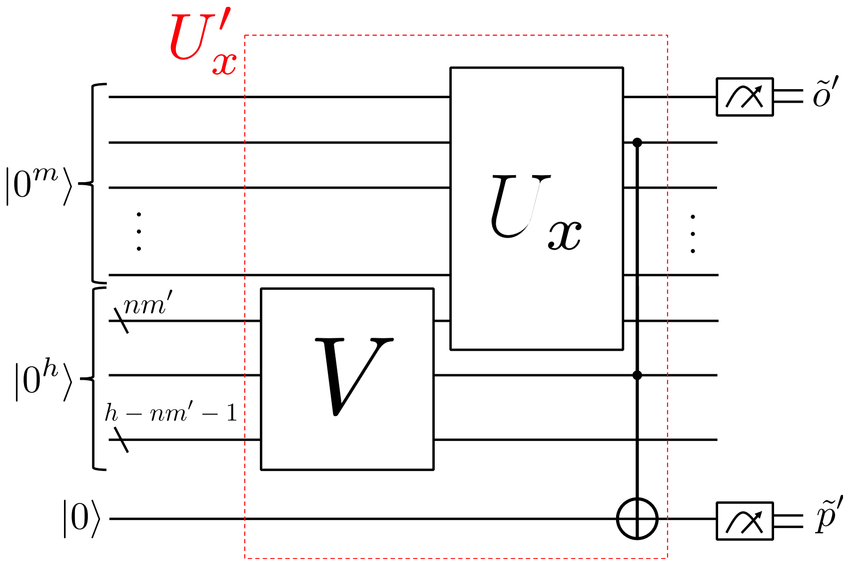


Proof of our result

- The 3rd step



Acutually-constructed
postselected quantum circuit

$$\Pr[\tilde{o}' = 1 \mid \tilde{p}' = 1]$$

Proof of our result

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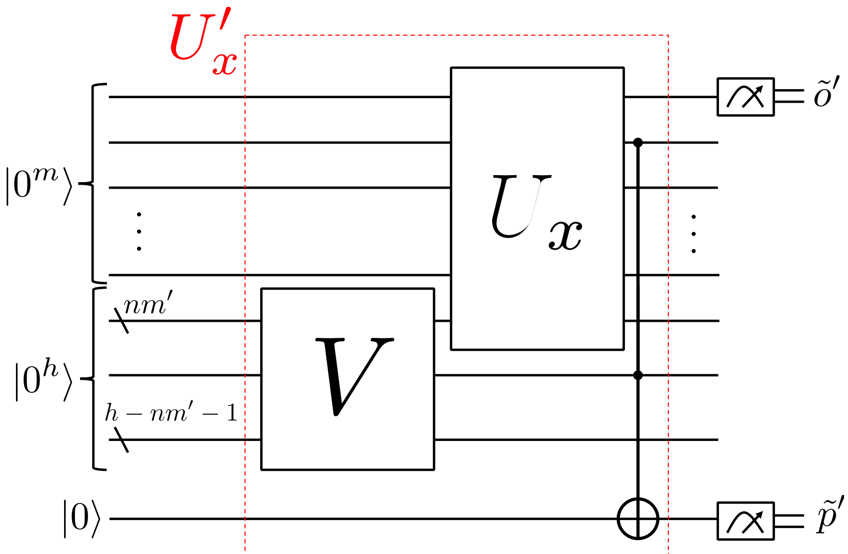

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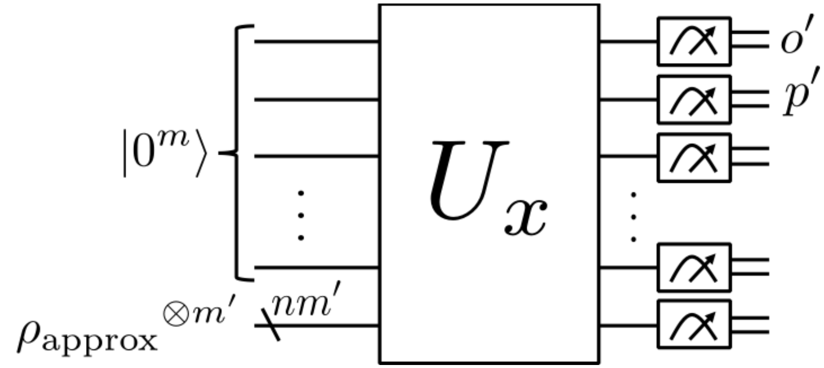

Virtual
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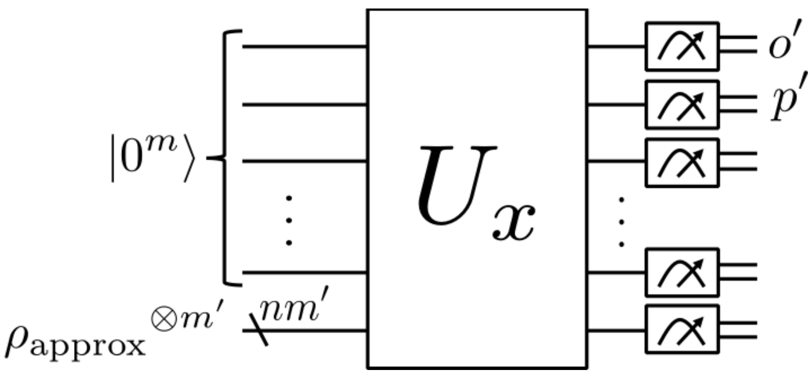
$$\Pr[\tilde{o}' = 1 \mid \tilde{p}' = 1] = \Pr[o' = 1 \mid p' = 1]$$

Proof of our result

- The final step



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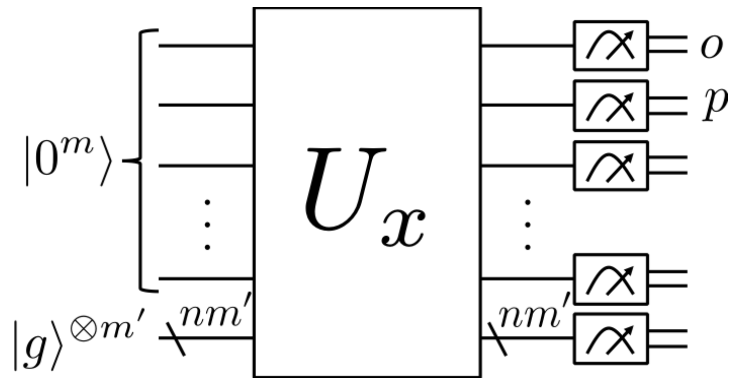

Postselected quantum circuit
solving the P3LH problem
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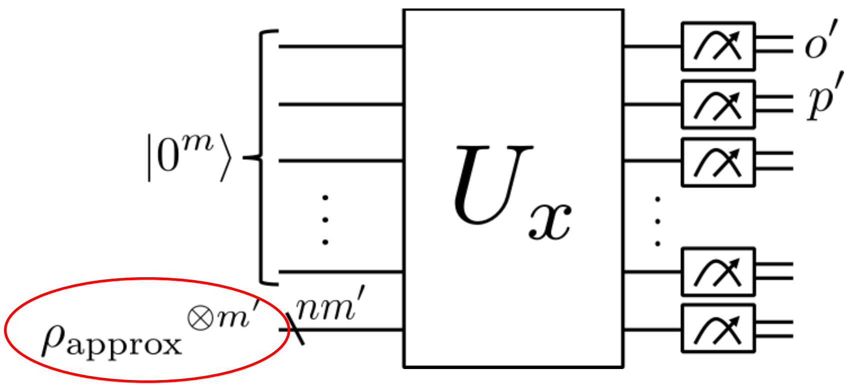

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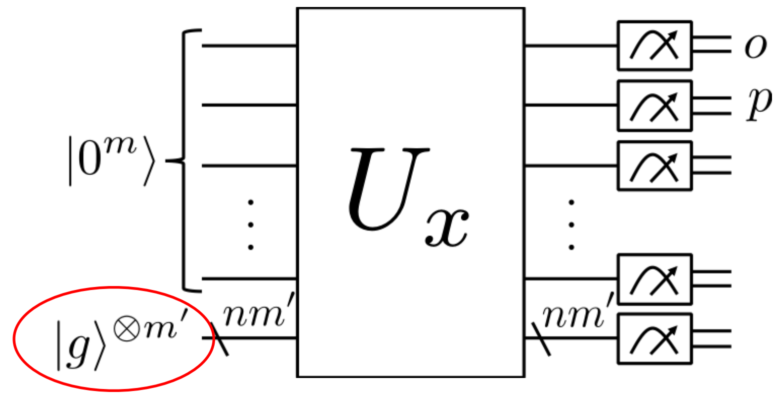

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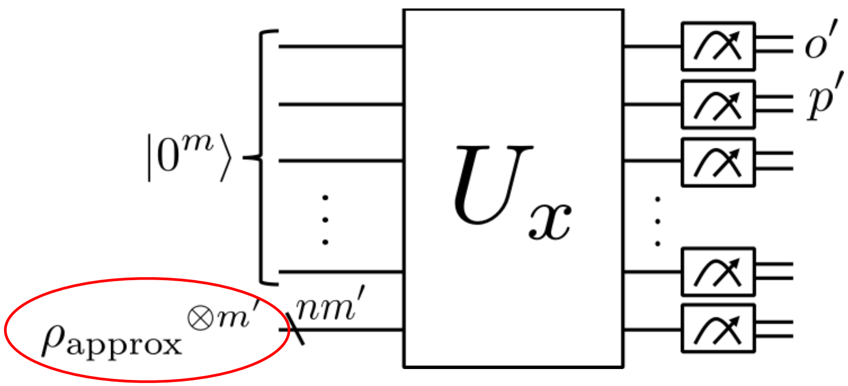

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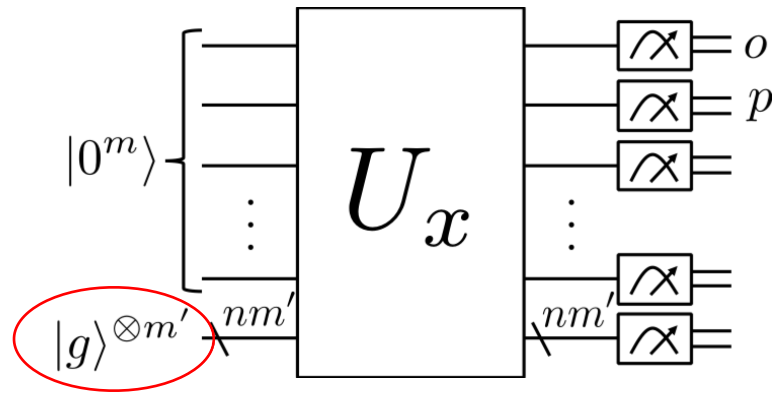

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$$\Rightarrow \quad |\Pr[o' = 1 \mid p' = 1] - \Pr[o = 1 \mid p = 1]| \leq 2^{-\text{poly}(n)}$$