A family of analytically solvable time-dependent driven conformal field theories

Xueda Wen
Physics Department, Harvard University

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Refs: arXiv: 2006.10072; 2008.01123; 2011.09491; & ongoing works

Some simple motivations:

- We know some analytically solvable examples on time-dependent driven two-level systems in quantum mechanics:
  - Landau-Zener transition (1932)
  - Rabi oscillation (1937)
  - Some descents of the two examples above

  These examples are important in, e.g., qubit control.
- We hope to find some analytically solvable examples in quantum many-body systems.

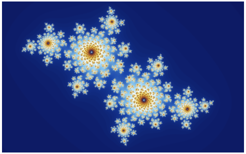
Some recent progress on quantum quench in CFTs:

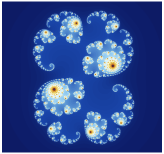
(Change the system suddenly, and then let it evolve.)

- Global and local quantum quench by changing the Hamiltonian suddenly. Cardy, Calabrese, ... 2000s
- Local quantum quench by acting local operators suddenly.
  Nozaki, Numasawa, He, Takayanagi, ... 2010s

In this talk, we are interested in the *driven* systems.

Basic idea in a picture:



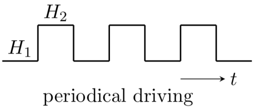


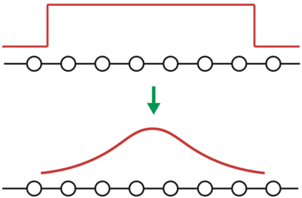
$$z_n = \underbrace{f \circ f \circ \cdots f}_{n \text{ times}}(z).$$

Iterating conformal mapping
$\Rightarrow$ periodically driven non-equilibrium dynamics

Let's start from an "experiment" on computer:

- A free-fermion lattice model at the half filling, with Hamiltonian $H_1 = \sum_{j=1}^{L-1} c_j^\dagger c_{j+1} + h.c.$
- We consider a deformed Hamiltonian $H_2 = \sum_{j=1}^{L-1} f(j)\, c_j^\dagger c_{j+1} + h.c.$, where $f(j) = 2\sin^2 \frac{\pi j}{L}$.
- We drive the system periodically with H_1 and H_2.





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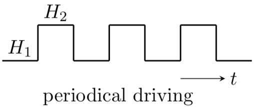
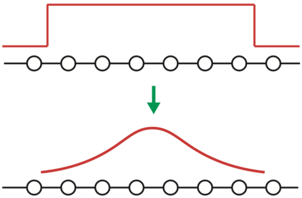
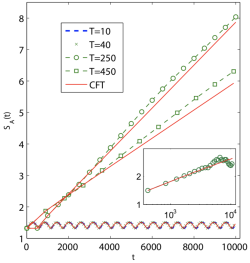
- We drive the system periodically with H_1 and H_2.
- Interestingly, there are two "different phases". [XW, Jie-Qiang Wu, arXiv:1802.07765 &1805.00031]





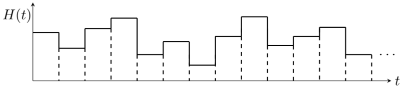


This 'experiment' stimulates our interest in exploring driven CFTs.

Remark on related histories:

- This "experiment" is inspired by sine-square deformation (SSD) that was studied in equilibrium physics.
- SSD was first proposed in numerical calculations.
  [Gendiar, Krcmar, and Nishino, 2009, 2010; Hikihara, Nishino, 2011; Shibata, Hotta, 2011, 2012; ...]
- Later, SSD of (1+1)d conformal field theories was studied.
  [Katsura, 2011,2012; Tada, Ishibashi, 2015, 2016; Okunishi, 2016; Wen, Ryu, Ludwig, 2016; Tada, 2019; Liu, Tada, 2020; ...]
- In this talk, we are interested in a generally deformed CFT, and use the deformed Hamiltonians to drive the system.

Setup for time-dependent driven CFTs



- We consider a $(1+1)$d CFT.
- Discrete time-dependent driving:

$$|\Psi_n\rangle = U_n \cdots U_2 \cdot U_1 |\Psi_0\rangle, \quad \text{with} \quad U_n = e^{-iH_n T_n}.$$

- $[H_i, H_j] \neq 0$.
- The Hamiltonian H_i in the i-th step is a CFT Hamiltonian with spatial deformation:

$$H_i = \frac{1}{2\pi} \int_0^L [f_i(x)\, T_{00}(x) + g_i(x)\, T_{01}(x)] dx.$$

$f_i(x)$ and $g_i(x)$ are a real functions. $T_{00} = T + \overline{T}$ is the energy density, and $T_{01} = T - \overline{T}$ is the momentum density.

Algebraic view of this problem:

- The Fourier components of H_i form the so-called Virasoro algebra,

$$[L_m,\, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad n, m \in \mathbb{Z},$$

  which are infinite dimensional.
- The unique finite dimensional subalgebra is

$$\mathfrak{sl}_2 \subset \text{Virasoro algebra}$$

  that are generated by $\{L_0, L_{\pm q}\}$, $q \in \mathbb{Z}$.
- For $\mathfrak{sl}_2$ deformation, $f_i(x)$ and $g_i(x)$ in the deformed Hamiltonian H_i are of the form "$a + b\sin\left(\frac{2\pi q(x+c)}{L}\right)$", where $a, b, c \in \mathbb{R}$.
- We will study $\mathfrak{sl}_2$ deformation first, and then study the general deformations.

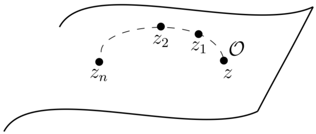
Operator evolution

- Time-dependent multi-point correlation function:

$$\langle \Psi_0 | \mathcal{O}_1(x_1, t_1) \cdots \mathcal{O}_n(x_n, t_n) | \Psi_0 \rangle$$

  where $\mathcal{O}(x, t) = e^{iHt} \mathcal{O}(x) e^{-iHt}$.

- Path integral representation (2d spacetime):



Operator evolution

- For a $\mathfrak{sl}_2$ deformed Hamiltonian H, we have

$$e^{iHt}\,\mathcal{O}(z,\bar{z})\,e^{-iHt} = \left(\frac{\partial z_{\text{new}}}{\partial z}\right)^{h}\left(\frac{\partial \bar{z}_{\text{new}}}{\partial \bar{z}}\right)^{\bar{h}}\mathcal{O}\big(z_{\text{new}},\bar{z}_{\text{new}}\big).$$

  Here z_{new} is related to z through a Möbius transformation:

$$z_{\text{new}} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}\cdot z = \frac{\alpha z + \beta}{\beta^* z + \alpha^*}, \quad \text{with } \alpha,\beta\in\mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1.$$

  The associated matrix is $\mathrm{SU}(1,1)$, which is isomorphic to $\mathrm{SL}(2,\mathbb{R})$.

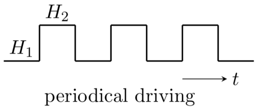
Multiple driving

- Now if we drive the system with N steps, we have

$$z_N = \underbrace{\begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_1^* & \alpha_1^* \end{pmatrix} \cdots \begin{pmatrix} \alpha_{N-1}^* & \beta_{N-1}^* \\ \beta_{N-1}^* & \alpha_{N-1}^* \end{pmatrix} \begin{pmatrix} \alpha_N^* & \beta_N^* \\ \beta_N^* & \alpha_N^* \end{pmatrix}}_{=:M_1 \cdots M_{N-1} M_N} \cdot z$$

  where α_i and β_i depend on both the hamiltonian H_i and the time interval T_i.
- We are actually performing conformal maps in time.
- Once the operator evolution is known, one can further calculate the entanglement/energy-momentum evolution, etc.

How can there be different phases?

- Now we consider the two-step driving: 

- Operator evolution after N driving periods:

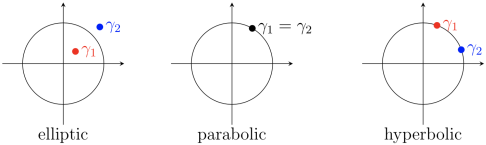
$$z_N = (M_1 \cdot M_2)^N \cdot z =: M^N \cdot z$$

where $M_1, M_2, M \in \mathrm{SU}(1,1)$.

- The phase diagram is determined as follows:

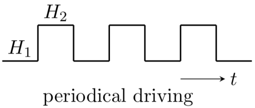
$$\begin{cases} |\mathrm{Tr}(M)| > 2, & \text{hyperbolic} \rightarrow \text{Heating phase} \\ |\mathrm{Tr}(M)| = 2, & \text{parabolic} \rightarrow \text{Phase transition} \\ |\mathrm{Tr}(M)| < 2, & \text{elliptic} \rightarrow \text{Non-heating phase} \end{cases} \tag{1}$$

They have different distributions of fixed points on the unit circle:

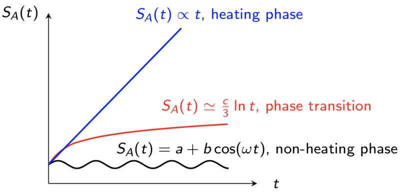


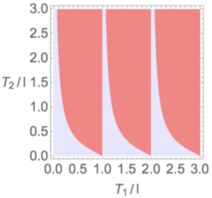
Minimal example and typical features

- Now we consider the two-step driving: (with $f_1(x) = 1$ and $f_2(x) = \sin^2 \frac{q\pi x}{L}$)



- Entanglement evolution in different phases, and phase diagram ($l = L/q$):
  [XW, Jie-Qiang Wu, arXiv: 1805.00031]

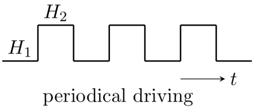




- The total energy grows exponentially fast in the heating phase, and simply oscillates in the non-heating phase.
  Fan, Gu, Vishwanath, XW, arXiv:1908.05289; Lapierre-Choo-Tauber-Tiwari-Neupert-Chitra, arXiv:1909.08618

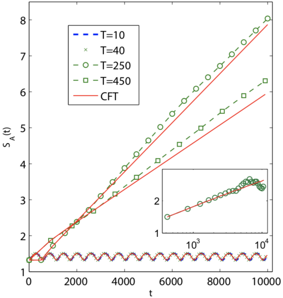
- Features of Floquet CFT at non-stroboscopic time can also be analyzed.
  Andersen, Norfjand, and Zinner, arXiv:2011.08494

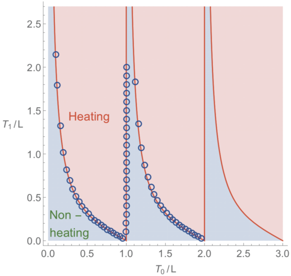
This explains our "experiments":



- Now we consider the two-step driving: (with $f_1(x) = 1$ and $f_2(x) = \sin^2 \frac{q\pi x}{L}$)

- Free fermion lattice simulation: [XW, Wu, arXiv: 1805.00031; Fan, Gu, Vishwanath, XW, arXiv:1908.05289]

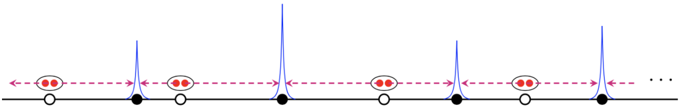




- See also [Lapierre-Choo-Tauber-Tiwari-Neupert-Chitra, arXiv:1909.08618] for comparison of XXZ spin chain and CFT calculation.

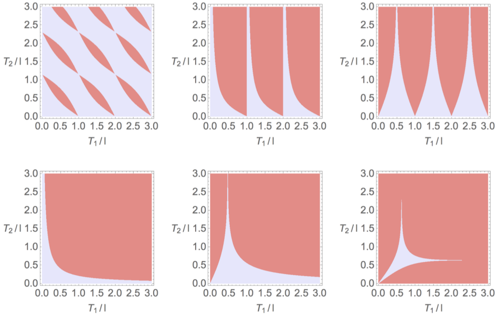
Underlying physical picture in the heating phase:

- Emergent stable (•)/unstable (∘) fixed points
  ⇒ Entanglement patterns & Energy-momentum density distribution
  Fan, Gu, Vishwanath, XW, arXiv:1908.05289
  Lapierre-Choo-Tauber-Tiwari-Neupert-Chitra, arXiv:1909.08618
  Fan, Gu, Vishwanath, XW, arXiv: 2011.09491
- Each unstable fixed point (∘) emits EPR pairs, which generates quantum entanglement between two neighboring stable fixed points (•).
- The locations of these fixed points are determined by the fixed points of hyperbolic $SU(1,1)$ matrix.



Classify the $\mathfrak{sl}_2$ deformed Floquet CFT

- Heating phase is generic, but the non-heating phase may not necessarily exist.
- We give sufficient and necessary conditions for the existence of non-heating phases when there are $N = 2$ driving Hamiltonians. For $N > 2$, we give sufficient conditions. [Han, XW, arXiv:2008.01123]



Generalizations:

- **Generalization 1:**

  Periodical driving $\longrightarrow$ Quasi-periodic/ Random driving.
  [XW, Fan, Vishwanath, Gu, arXiv: 2006.10072; Lapierre, Choo, Tiwari, Tauber, Neupert, Chitra, arXiv: 2006.100054]

Driven CFTs	Heating phase	Non-heating phase	Critical
Periodic	$\surd$	$\surd$	$\surd$
Random	$\surd^{\bullet}$	$\times$	$\times$
Fibonacci	$\surd$	$\times$	$\surd^{*}$

  $^{\bullet}$: With exceptional points *: Cantor set of measure zero

- **Generalization 2:**

$$\mathfrak{sl}_2 \text{ algebra} \rightarrow \text{Virasoro algebra}$$

  That is, we will consider an arbitrary smooth function $f(x)$ in the deformed Hamiltonian. [Lapierre, Moosavi, arXiv:2010.11268; Fan, Gu, Vishwanath, XW, arXiv:2011.09491]

On generalization 1:

- The structure of matrix product in operator evolution is similar to the transfer matrix in a tight-binding model on a lattice:

$$[H\psi]_n = \psi_{n+1} + \psi_{n-1} + V_n\psi_n, \quad n \in \mathbb{Z},$$

where V_n is on-site potential. Eigenvalue function $H\psi = E\psi$, for $E \in \mathbb{R}$. By denoting $\Psi_n = (\psi_n, \psi_{n-1})^T$, one has

$$\Psi_n = \underbrace{(T_n \cdots T_2 \cdot T_1)}_{\text{matrix product}} \Psi_1, \quad \text{where } T_n = \begin{pmatrix} E - V_n & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Lyapunov exponents & Phase diagram, etc

- Lyapunov exponent in matrix products:

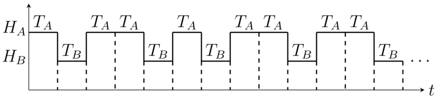
$$\lambda_L := \lim_{n\to\infty} \frac{1}{n} \log ||M_n \cdots M_2 \cdot M_1||.$$

  The sequence of $\{M_i\}$ can be periodic, quasi-periodic and random, etc.
- We have the following analogy:

	Wavefunction in lattice	Driven CFTs
$\lambda_L > 0$	localized	heating
$\lambda_L = 0$	critical	phase transition
$\lambda_L = 0$	extended	non-heating

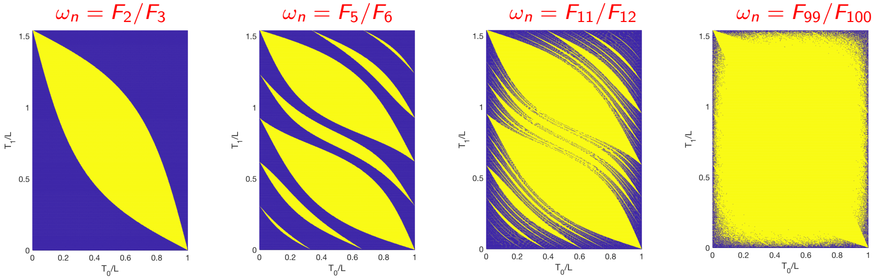
Fibonacci quasi-periodically driven CFTs

- Fibonacci driving protocol:



Fibonacci quasi-periodical driving

- One can prove that the non-heating phases form a Cantor set of measure zero as we approach the limit of Fibonacci sequence. [XW, Fan, Vishwanath, Gu, arXiv: 2006.10072; Lapierre, Choo, Tiwari, Tauber, Neupert, Chitra, arXiv: 2006.100054]



Randomly driven CFT, Anderson localization & Furstenberg theorem

- Anderson localization [P. W. Anderson, 1958]:

$$[H\psi]_n = \psi_{n+1} + \psi_{n-1} + V_n \psi_n, \quad n \in \mathbb{Z},$$
$$H\psi = E\psi.$$

Given random V_n, for arbitrary energy E, the eigenstate is localized.

- Furstenberg theorem [H. Furstenberg, 1963]:

Let $X_1, X_2, \cdots, X_n, \cdots$ denote the sequence of i.i.d. random matrices in $SL(n, \mathbb{R})$ with a distribution μ. Let G_μ denote the smallest subgroup of $SL(n, \mathbb{R})$ containing the support of μ. If G_μ is noncompact, and no subgroup of G_μ of finite index is reducible (strongly irreducible), then $\lambda_L := \lim_{n\to\infty} \frac{1}{n} \ln ||X_n \cdots X_2 X_1|| > 0$ with probability 1.

– In Andeson localization, Furstenberg theorem always holds.

– In randomly driven CFTs, Furstenberg theorem does not always hold.

Randomly driven CFT:

- We can exhaust all possible types of random driving with $\mathfrak{sl}_2$ deformation in CFT.
- If Furstenberg's theorem holds, the randomly driven CFT is in a heating phase, with $\lambda_L > 0$, and

$$\boxed{S_A(t) \sim \frac{c}{3} \cdot \lambda_L \cdot t, \qquad E(t) \sim c \cdot e^{2\lambda_L \cdot t}}$$

  The energy density peaks are randomly distributed.
- If Furstenberg's theorem does not hold, the Lyapunov exponent is $\lambda_L = 0$, with

$$\boxed{S_A(t) \sim c \cdot \sqrt{t}, \qquad E(t) \sim c \cdot e^{2\lambda_E \cdot t}}$$

  The energy density peaks are regularly (not randomly) distributed.
  [XW, Fan, Gu, Vishwanath, To appear.]

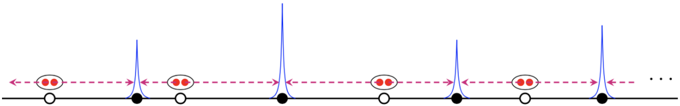
On generalization 2:

- $\mathfrak{sl}_2$ algebra $\to$ Virasoro algebra
- An arbitrary smooth deformation $f(x)$ in the driving Hamiltonian:

$$H_1 = \int_0^L dx\, f(x) T(x) + \text{anti-chiral part.}$$

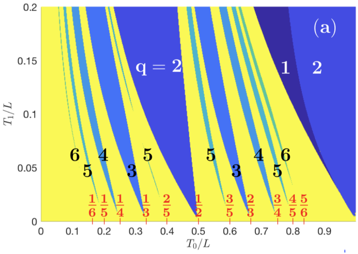
One can still obtain the operator evolution. [Fan, Gu, Vishwanath, XW, arXiv: 2011.09491]

- Heating/non-heating phases are determined by the presence/absence of fixed points in the operator evolution.



Phase diagram:

- For periodic driving, most features are similar to those in $\mathfrak{sl}_2$ deformed Flouqet CFTs.
- New features:
  - There can be distinct heating phases with different number of fixed points in the operator evolution. [Lapierre, Moosavi, arXiv:2010.11268; Fan, Gu, Vishwanath, XW, arXiv: 2011.09491]
  - One typical feature is the appearance of "Arnold tongues" in the phase diagram. [Fan, Gu, Vishwanath, XW, arXiv: 2011.09491]



Summary and Discussion:

- Time-dependent driven CFT with $\mathfrak{sl}_2$ deformation:
  - Periodic driving $\surd$
  - Quasi-periodic driving $\surd$
  - Random driving $\surd$
- Time-dependent driven CFT with general deformation:
  - Periodic driving $\surd$
  - Quasi-periodic driving ?
  - Random driving ?
- The feature of entanglement Hamiltonian in each phase ?
- Discrete driving $\rightarrow$ Continuous driving
  Das, Ghosh, Sengupta, arXiv:2101.04140
- Non-unitary/non-hermitian driving ?

  For some initial efforts along this direction, see: [Ageev, Bagrov, Iliasov, arXiv:2006.11198]
- Generalization to higher dimensional CFTs ?
- Holographic dual ?
  [MacCormack, Liu, Nozaki, Ryu, arXiv:1812.10023]
- Drivings that break conformal symmetry ?