*Progress of code development*

# *2nd-order Einstein-Boltzmann solver for CMB anisotropy*

Takashi Hiramatsu

*Yukawa Institute for Theoretical Physics (YITP)*
*Kyoto University*

Collaboration with Ryo Saito (APC), Atsushi Naruko (TITech), Misao Sasaki (YITP)

# Current status

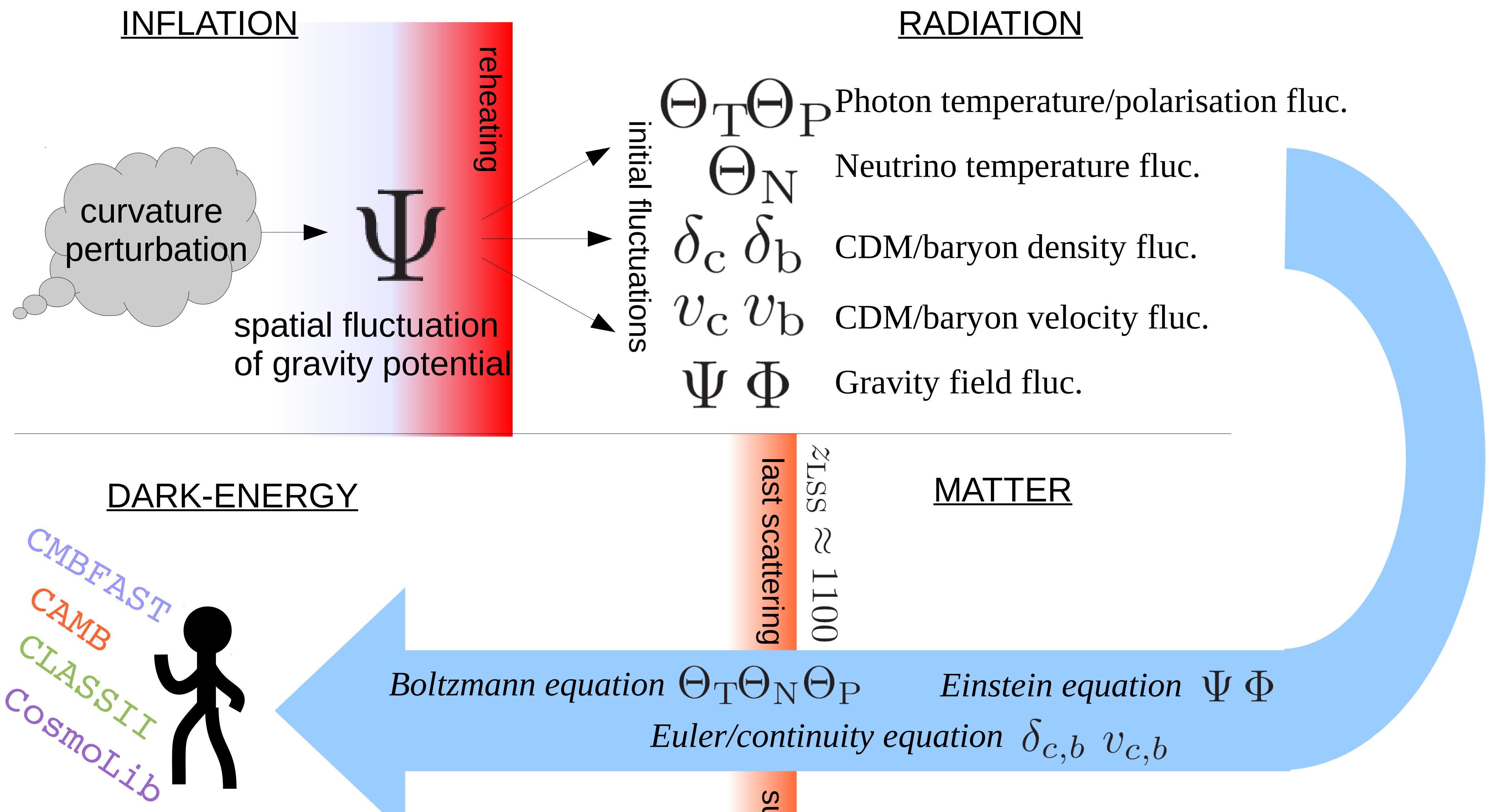


| *1st-order perturbations* | *2nd-order perturbations* |
| --- | --- |
| Implementing basic equations | |
| History of electron density | |
| line-of-sight integral<br>+angular power spectrum | Implementing basic equations 1 |
| Qualitative check | |
| Quantitative check using CAMB ← | NOW |
| NOW → | 2nd-order line-of-sight integral<br>R.Saito, Naruko, Hiramatsu, Sasaki, arXiv:1409.2464 |
| | Implementing basic equations 2 |
| | Observables |
| | Speed-up + Optimisation |

1st-order 95%

2nd-order 15%

# Evolution of fluctuations




CMBFAST : Seljak, Zaldarriaga, APJ469 (1996) 437
CAMB : Lewis, Challinor, APJ538 (2000) 473
CLASS II : Blas, Lesgourgues, Tram, JCAP 1107 (2011) 034
CosmoLib : Huang, JCAP 1206 (2012) 012

# Photons/Neutrinos

Takashi Hiramatsu

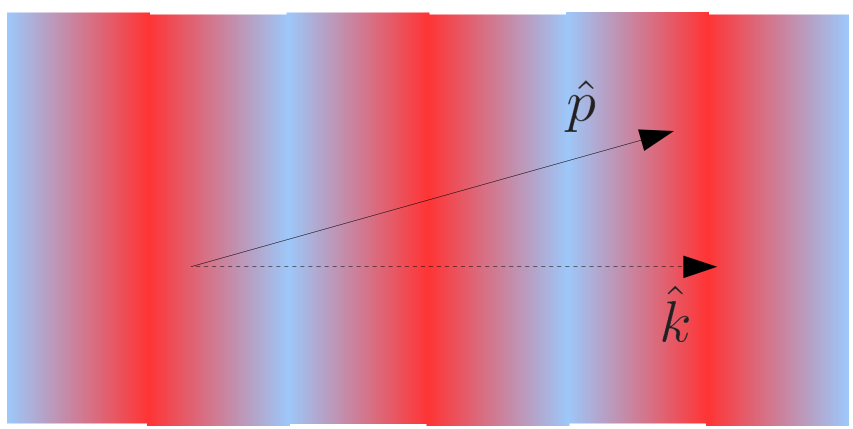
They are described by the Bose-Einstein/Fermi-Dirac distribution functions:

$$f_{\gamma,\nu}(\mathbf{x},\mathbf{p},\eta) = \left[\exp\left\{\frac{p}{T_{\gamma,\nu}(\eta)\left[1+\Theta_{\mathrm{T,N}}(\mathbf{x},\hat{p},\eta)\right]}\right\} \pm 1\right]^{-1}$$

Temperature fluctuation has three independent variables

$$\Theta(\mathbf{x},\hat{p},\eta) \longrightarrow \Theta(\mathbf{k},\hat{p},\eta) \longrightarrow \Theta(k,\mu,\eta) \longrightarrow \Theta_\ell(k,\eta)$$

Fourier trans. Directional cosine Multipole exp.




$$\mu \equiv \frac{\mathbf{k}\cdot\hat{p}}{k}$$

Multipole expansion

$$\Theta_\ell(k,\eta) = \frac{1}{2(-i)^\ell}\int_{-1}^{1} \mathcal{P}_\ell(\mu)\Theta(k,\mu,\eta)\,d\mu$$

YITP YUKAWA INSTITUTE FOR THEORETICAL PHYSICS
Takashi Hiramatsu

Distribution function satisfies the Boltzmann equation :

Liouville term — Collision term

$$\frac{df}{d\eta} = C[f] \longrightarrow \begin{cases} \dot{\Theta}_{\rm T} + ik\mu\Theta_{\rm T} + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}\mathcal{P}_2(\mu)\Pi\right] \\ \dot{\Theta}_{\rm P} + ik\mu\Theta_{\rm P} = -\dot{\tau}\left[-\Theta_{\rm P} + \frac{1}{2}(1-\mathcal{P}_2(\mu))\Pi\right] \\ \dot{\Theta}_{\rm N} + ik\mu\Theta_{\rm N} + \dot{\Phi} + ik\mu\Psi = 0 \end{cases}$$

$$\Rightarrow \Theta_\ell^{\rm T,P,N}(k,\eta)$$

$$\Pi = \Theta_{T2} + \Theta_{P0} + \Theta_{P2}$$

S. Dodelson, "*Modern Cosmology*"

The efficiency of collision term is controlled by $\dot{\tau}(\eta)$ , time-derivative of optical depth

$$\dot{\tau}(\eta) = -n_e(\eta)\sigma_T a$$

$\sigma_T$ : Thomson cross-section

| | |
|---|---|
| Deep in radiation dominant epoch | Extremely large $\dot{\tau}(\eta) \longrightarrow$ tight-coupling |
| Last-scattering surface | $\dot{\tau}(\eta)$ suddenly decays |
| Late-time Universe | No collision, free-streaming |
| Reionisation | $\dot{\tau}(\eta)$ is revived, but not so significant |

# CDM/baryons

Takashi Hiramatsu

EB equations for CDM/baryons
+ photon coupling.

$$\xrightarrow{\text{Fluid approx.}}$$

Relativistic Fluid equations
for CDM and baryons
with source terms.

$$n_{c,b}(\mathbf{x},\eta) = \int \frac{d^3p}{(2\pi)^3} f_{c,b} = n^{(0)}_{c,b}(1 + \underline{\delta_{c,b}}) \qquad n_{c,b}\underline{v^i_{c,b}}(\mathbf{x},\eta) = \int \frac{d^3p}{(2\pi)^3} \frac{p\hat{p}^i}{E} f_{c,b}$$

$$\Big\downarrow \text{Fourier transform}$$

CDM
baryon

$$\begin{cases} \dot{\delta}_{\rm c} = -ikv_{\rm c} - 3\dot{\Phi} \\ \dot{\delta}_{\rm b} = -ikv_{\rm b} - 3\dot{\Phi} \\ \dot{v}_{\rm c} = -\mathcal{H}v_{\rm c} - ik\Psi \\ \dot{v}_{\rm b} = -\mathcal{H}v_{\rm b} - ik\Psi + \dfrac{\dot{\tau}}{R}(v_{\rm b} + 3i\Theta_{\rm T1}) \end{cases} \qquad R(\eta) = \frac{3\rho_B}{4\rho_\gamma}$$

# Gravity

Takashi Hiramatsu

Conformal Newton gauge : $ds^2 = -a^2(1+2\Psi)d\eta^2 + a^2(1+2\Phi)dx^2$

$$G^{(1)}_{00} = \frac{8\pi}{M_{\rm pl}^2} T^{(1)}_{00} \longrightarrow \dot{\Phi} = -\frac{k^2}{3\mathcal{H}}\Phi + \mathcal{H}\Psi + \frac{\mathcal{H}_0^2}{2\mathcal{H}}\delta\Omega_0$$

$$\delta\Omega_0 = \Omega_{\rm c}\delta_{\rm c} + \Omega_{\rm b}\delta_{\rm b} + 4\Omega_\gamma\Theta_{\rm T0} + 4\Omega_\nu\Theta_{\rm N0}$$

$$\Omega_i \equiv \frac{\rho_i}{\rho_c}$$

$$G^{(1),ij}_{ij} = \frac{8\pi}{M_{\rm pl}^2} T^{(1),ij}_{ij} \longrightarrow \Psi = -\Phi - \frac{12\mathcal{H}_0^2}{k^2}\Omega_{\rm r}\Theta_{r,2}$$

(non-dynamical)

$$\Omega_{\rm r}\Theta_{r,2} = \Omega_\gamma\Theta_{\rm T2} + \Omega_\nu\Theta_{\rm N2}$$

NOTE : CAMB, CMBFAST use sychronous gauge : $ds^2 = -a^2 d\eta^2 + a^2(\delta_{ij} + h_{ij})dx^2$

# 1st-order perturbation equations

Takashi Hiramatsu

Photon temperature

$$\begin{cases} \dot{\Theta}_{\mathrm{T}0} = -k\Theta_{\mathrm{T}1} - \dot{\Phi} \\ \dot{\Theta}_{\mathrm{T}1} = \frac{1}{3}k(-2\Theta_{\mathrm{T}2} + \Theta_{\mathrm{T}0}) + \dot{\tau}\left(\Theta_{\mathrm{T}1} + \frac{1}{3}v_b\right) + \frac{1}{3}k\Psi \\ \dot{\Theta}_{\mathrm{T}2} = \frac{1}{5}k(-3\Theta_{\mathrm{T}3} + 2\Theta_{\mathrm{T}1}) + \dot{\tau}\left(\Theta_{\mathrm{T}2} - \frac{1}{10}\Pi\right) \\ \dot{\Theta}_{\ell} = \frac{1}{2\ell+1}k\left[-(\ell+1)\Theta_{\ell+1} + \ell\Theta_{\ell-1}\right] + \dot{\tau}\Theta_{\ell} \end{cases}$$

Photon polarisation

$$\begin{cases} \dot{\Theta}_{\mathrm{P}0} = -k\Theta_{\mathrm{P}1} + \dot{\tau}\left(\Theta_{\mathrm{P}0} - \frac{1}{2}\Pi\right) \\ \dot{\Theta}_{\mathrm{P}1} = \frac{1}{3}k(-2\Theta_{\mathrm{P}2} + \Theta_{\mathrm{P}0}) + \dot{\tau}\Theta_{\mathrm{P}1} \\ \dot{\Theta}_{\mathrm{P}2} = \frac{1}{5}k(-3\Theta_{\mathrm{P}3} + 2\Theta_{\mathrm{P}1}) + \dot{\tau}\left(\Theta_{\mathrm{P}2} - \frac{1}{10}\Pi\right) \end{cases}$$

Massless neutrino temperature

$$\begin{cases} \dot{\Theta}_{\mathrm{N}0} = -k\Theta_{\mathrm{N}1} - \dot{\Phi} \\ \dot{\Theta}_{\mathrm{N}1} = \frac{1}{3}k(-2\Theta_{\mathrm{N}2} + \Theta_{\mathrm{N}0}) + \frac{1}{3}k\Psi \\ \dot{\Theta}_{\mathrm{N}2} = \frac{1}{5}k(-3\Theta_{\mathrm{N}3} + 2\Theta_{\mathrm{N}1}) \end{cases}$$

CDM, baryon

$$\begin{cases} \dot{\delta}_{\mathrm{c}} = -ikv_{\mathrm{c}} - 3\dot{\Phi} \\ \dot{\delta}_{\mathrm{b}} = -ikv_{\mathrm{b}} - 3\dot{\Phi} \\ \dot{v}_{\mathrm{c}} = -\mathcal{H}v_{\mathrm{c}} - ik\Psi \\ \dot{v}_{\mathrm{b}} = -\mathcal{H}v_{\mathrm{b}} - ik\Psi + \frac{\dot{\tau}}{R}(v_{\mathrm{b}} + 3i\Theta_{\mathrm{T}1}) \end{cases}$$

Gravity

$$\dot{\Phi} = -\frac{k^2}{3\mathcal{H}}\Phi + \mathcal{H}\Psi + \frac{\mathcal{H}_0^2}{2\mathcal{H}}\delta\Omega_0$$

Large Boltzmann hierarchy, say $\ell \lesssim 2000$, is required, but it is too hard to calculate...

- Truncating the Boltzmann hierarchy at $\ell \sim 15$ to ensure accurate calculation of $\Theta_{\ell\leq 2}, \Phi, \delta_b, \delta_c, v_b, v_c$

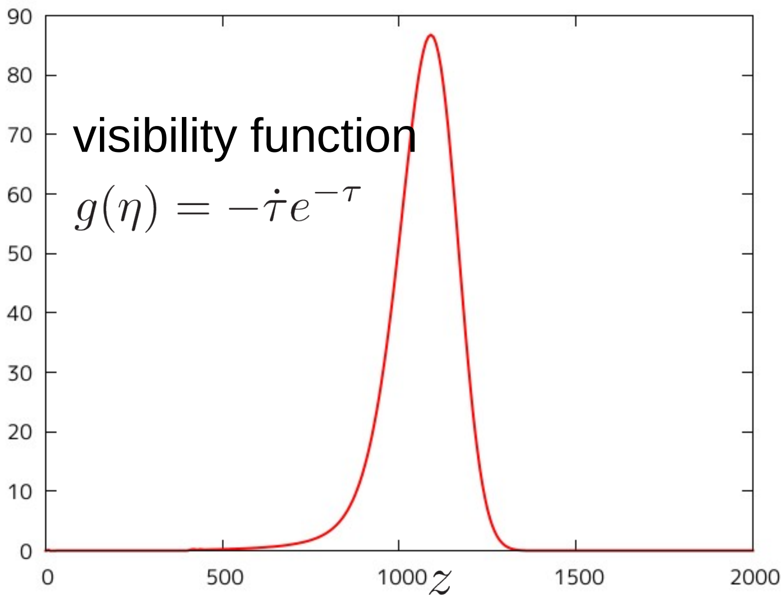
  cf. $\Theta_\ell \sim \dfrac{k\eta}{2\tau}\Theta_{\ell-1}$

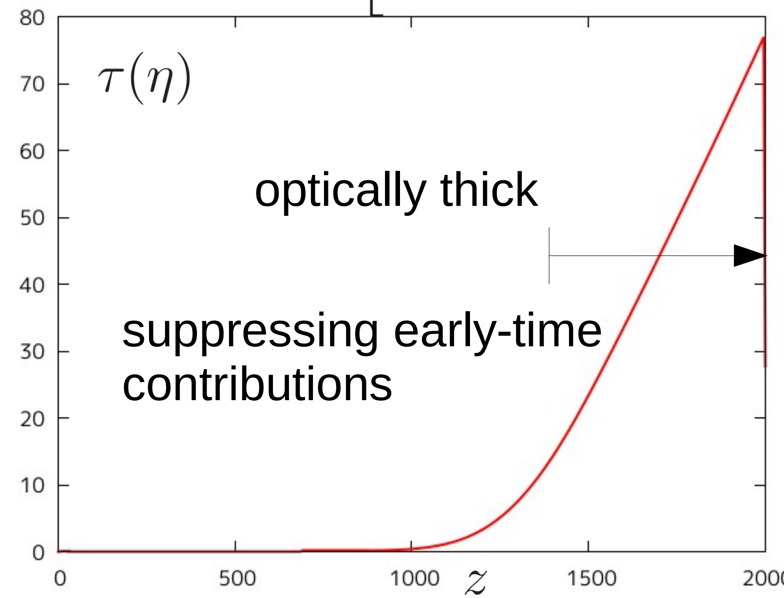
- For larger $\ell$ , it is useful to use the integral representation, *line-of-sight integral*.

**Seljak, Zaldarriaga, APJ 469 (1996) 437**

$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0} d\eta\, S(k,\eta) j_\ell[k(\eta_0-\eta)]$$

$$S(k,\eta) = g(\eta)\left[\Psi + \left(\Theta_0 + \frac{1}{4}\Pi\right)\right] + \frac{i}{k}\frac{d}{d\eta}[g(\eta)v_b(k,\eta)] + \frac{3}{4k^2}\frac{d^2}{d\eta^2}[g(\eta)\Pi] + e^{-\tau}\left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta)\right]$$



# Line-of-sight integral

Takashi Hiramatsu

Sourced by fluctuations at LSS ( including Sachs-Wolfe effect )

$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0} d\eta\, g(\eta)\left[\Theta_0(k,\eta)+\Psi(k,\eta)+\frac{1}{4}\Pi(k,\eta)\right] j_\ell[k(\eta_0-\eta)]$$

Monopole

$$+\int_0^{\eta_0} d\eta\, g(\eta) v_b(k,\eta)\frac{1}{k}\frac{d}{d\eta} j_\ell[k(\eta_0-\eta)]$$

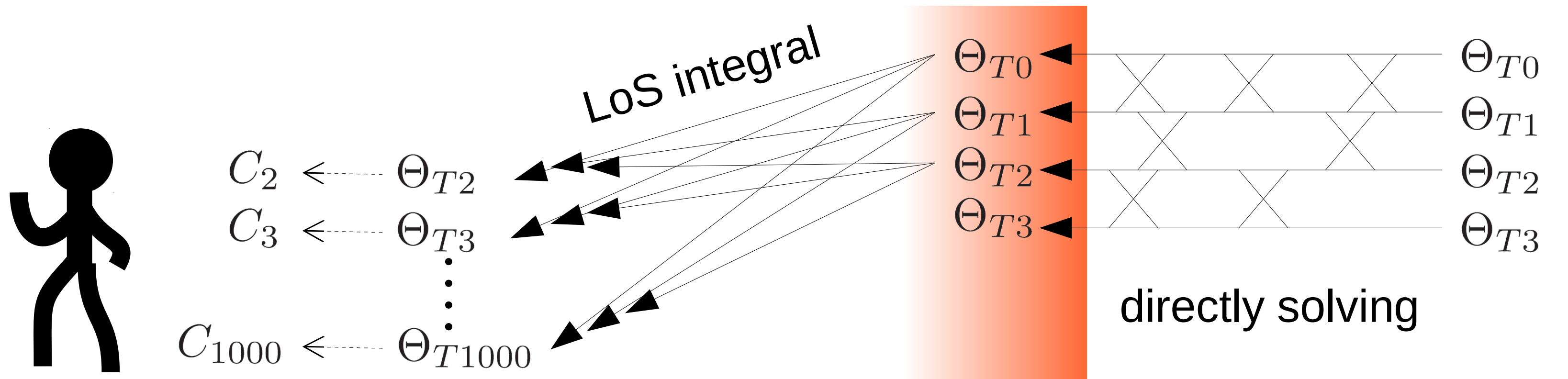
Dipole

$$+\int_0^{\eta_0} d\eta\, g(\eta)\frac{3}{4}\Pi(k,\eta)\frac{1}{k^2}\frac{d^2}{d\eta^2} j_\ell[k(\eta_0-\eta)]$$

Quadrupole

$$+\int_0^{\eta_0} d\eta\, e^{-\tau}\left[\dot\Psi(k,\eta)-\dot\Phi(k,\eta)\right] j_\ell[k(\eta_0-\eta)]$$

Integrated Sachs-Wolfe effect



Deep in radiation dominant epoch where all modes are larger than horizon scale.

$z_{\rm in} = 1.44 \times 10^6$  $k_{\max} = 1200 H_0$ which crosses the horizon at $z \approx 1.3 \times 10^5$

RD,
Unchanged potential,
Similarly fluctuated,
Superhorizon,
Tight-coupling,
Negligible photon's quadrupole,
Non-negligible neutrino's.

$$\Theta_{T1} = \Theta_{N1} = -\frac{v_b}{3} = -\frac{v_c}{3} = \frac{k}{6\mathcal{H}}\Psi$$

$$\Theta_{T0} = \Theta_{N0} = \delta_c = \delta_b = -\frac{1}{2}\Psi$$

$$\Phi = -\left(1 + \frac{2}{5} f_\nu\right)\Psi \qquad f_\nu = \rho_\nu / \rho_R$$

Promordial perturbations

$$\zeta = -\frac{i k_i \delta T^0{}_i H}{k^2(\rho + P)} - \Psi$$

during inflation $\zeta = -aH\frac{\delta\phi}{\dot{\phi}}$ = almost flat

after inflation $\zeta = -\frac{3\mathcal{H}\Theta_{T1}}{k} - \Psi$

$\longrightarrow$ $\Psi = -\frac{2}{3}\zeta$ = almost flat

# Initial conditions

$$\begin{cases} \delta_c = \delta_b = 3\Theta_{T0} = 3\Theta_{N0} = \zeta \\ v_b = v_c = -3\Theta_{T1} = -3\Theta_{N1} = \dfrac{k}{3\mathcal{H}}\zeta \\ \Phi = \dfrac{2}{3}\left(1 + \dfrac{2}{5}f_\nu\right)\zeta \\ \Psi = -\dfrac{2}{3}\zeta \end{cases}$$

@ $z = z_{\rm in} = 1.44 \times 10^6$

All quantities can be written by the primordial curvature perturbation.
Hence it is convenient to define the transfer functions such as

$$\Phi(k,\eta) = \mathcal{T}_\Phi(k,\eta)\zeta(k,\eta_{\rm in})$$

The governing equations are reinterpreted into those of the transfer functions.

$$\langle \zeta(\mathbf{k})\zeta^*(\mathbf{k})\rangle = (2\pi)^3\delta_D^3(\mathbf{k}+\mathbf{k}')P_\zeta(k)$$

$$P_\zeta(k) = \frac{2\pi^2}{k^3}\Delta^2(k_{\rm pivot})\left(\frac{k}{k_{\rm pivot}}\right)^{n_s-1}$$

→ Not used until calculating $C_\ell$

$$\Delta^2(k_{\rm pivot}) = 2.46 \times 10^{-9} \qquad k_{\rm pivot} = 0.002\ {\rm Mpc}^{-1}$$

$$n_s = 0.96$$

Flat FLRW model

$$H^2 = H_0^2 \left[\Omega_M (1+z)^3 + \Omega_R (1+z)^4 + \Omega_\Lambda\right]$$

$$\begin{cases} \Omega_\Lambda = 1 - \Omega_M - \Omega_R \\ \Omega_R = \dfrac{4\pi^3 g_* T_0^4}{45 H_0^2 M_{\rm pl}^2} \\ g_* = 2 + \dfrac{7}{4}\left(\dfrac{4}{11}\right)^{4/3} N_{\rm eff} \end{cases}$$

<u>fiducial parameters</u>

$$T_0 = 2.725 \ [\mathrm{K}]$$
$$N_{\rm eff} = 3.04$$
$$h = 0.7$$
$$h^2 \Omega_{\rm CDM} = 0.114$$
$$h^2 \Omega_{\rm B} = 0.0226$$

It can be easily extended to include non-flat case.

NOTE : we use $\sigma = \log a(\eta)$ as the time variable instead of $\eta$ in solving EB equations. Then we don't have to solve the Friedmann equation.

Number density of free electrons

$\longrightarrow \dot{\tau} = -n_{\rm e}\sigma_{\rm T}a$ controls ...

- Strength of Photon-Baryon coupling

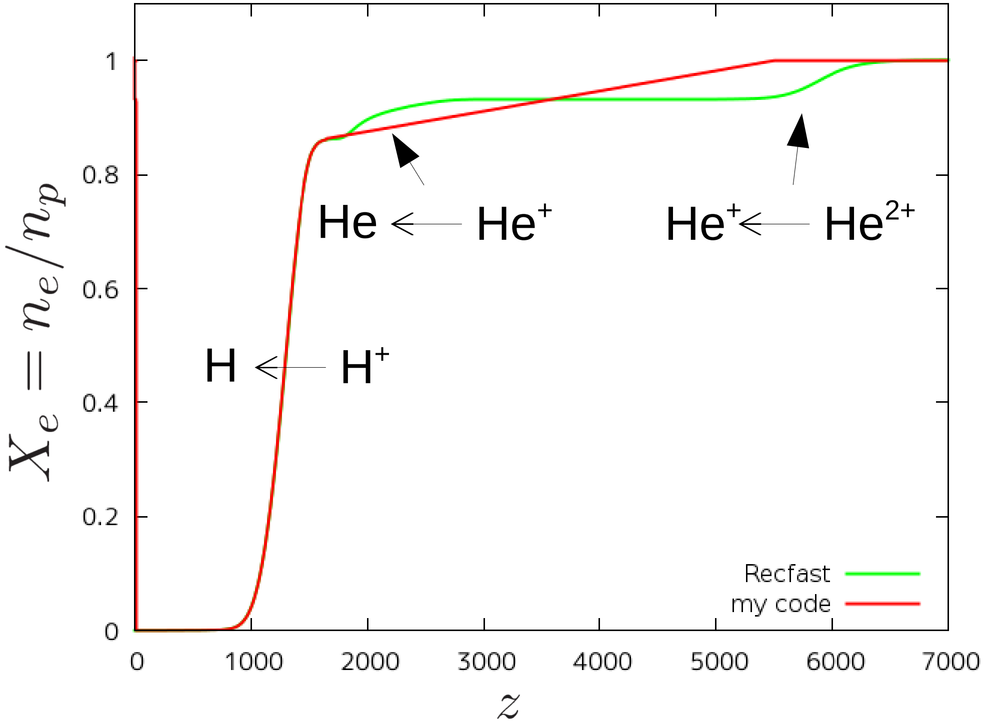
cf. $\dot{v}_{\rm b} = -\mathcal{H}v_{\rm b} - ik\Psi + \frac{\dot{\tau}}{R}(v_{\rm b} + 3i\Theta_{\rm T1})$

- Opacity of the Universe (affects ISW)

At recombination : instanteneous / polynomial fitting of $n_e$ / Weinberg's textbook / Recfast output

Peebles, APJ 153 (1968) 1

Weinberg : $$\frac{dX}{dT} = \frac{\alpha n}{HT}\left(1 + \frac{\beta}{\Gamma_{2s} + 8\pi H/\lambda_\alpha^3 n(1-X)}\right)^{-1}\left[X^2 - (1-X)/S\right]$$




CAMB uses more realistic model by Seager, but the difference is small.

Seager,Sasselov,Scott, APJS 128 (2000) 407

# Supporting actors : Recombination/Ionisation



Number density of free electrons

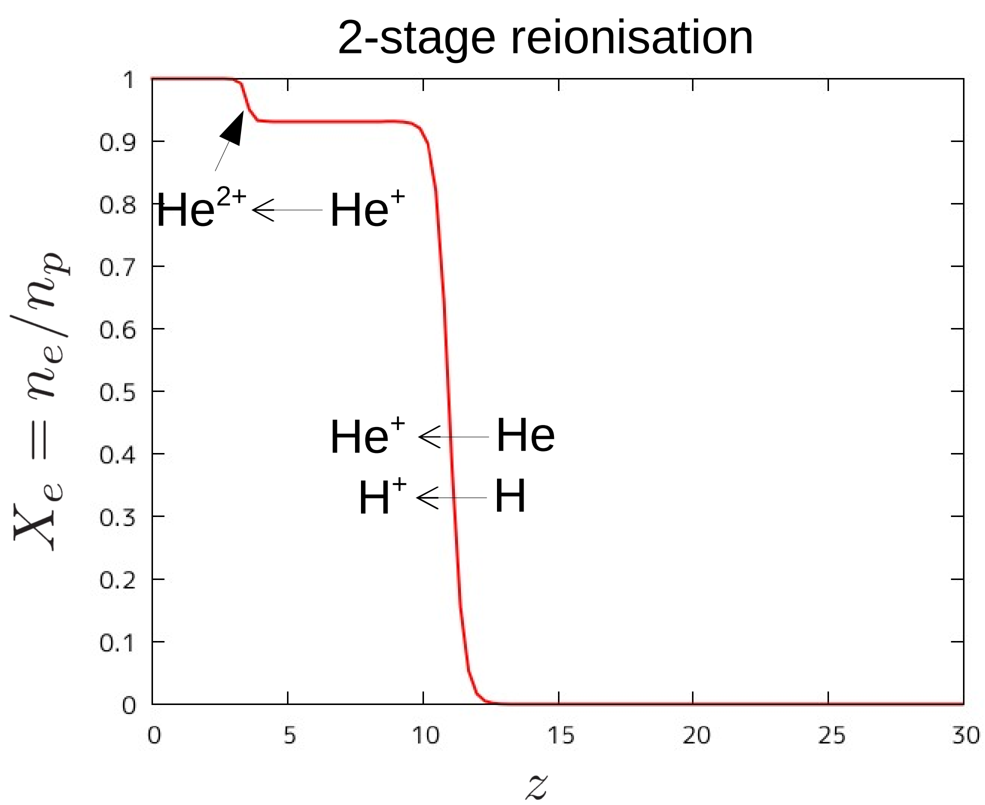
$\rightarrow \dot{\tau} = -n_{\mathrm{e}}\sigma_{\mathrm{T}}a$ controls ...

- Strength of Photon-Baryon coupling

  cf. $\dot{v}_{\mathrm{b}} = -\mathcal{H}v_{\mathrm{b}} - ik\Psi + \frac{\dot{\tau}}{R}(v_{\mathrm{b}} + 3i\Theta_{\mathrm{T1}})$

- Opacity of the Universe (affects ISW)

At reionisation : 2 stages / instanteneous / none




Same scheme as CAMB's.....(maybe)

During each stage, the fraction is interpolated by `tanh()` function.

$$z_{\mathrm{reion1}} = 11.0$$
$$\Delta z_{\mathrm{reion1}} = 0.5$$
$$z_{\mathrm{reion2}} = 3.5$$
$$\Delta z_{\mathrm{reion2}} = 0.2$$

$$\downarrow$$

$$\tau = 8.67 \times 10^{-2}$$

Use three different methods to maintain an accuracy of $\mathcal{O}(10^{-6})$ for $\ell, x < 10^5$

$$\begin{cases} \text{Descending reccurence :} & x > \ell \\ \text{Debye's expansion :} & x < \ell, \ell > 20 \\ \text{Taylor expansion :} & x < 0.5, \ell \leq 20 \end{cases}$$

[Descending]

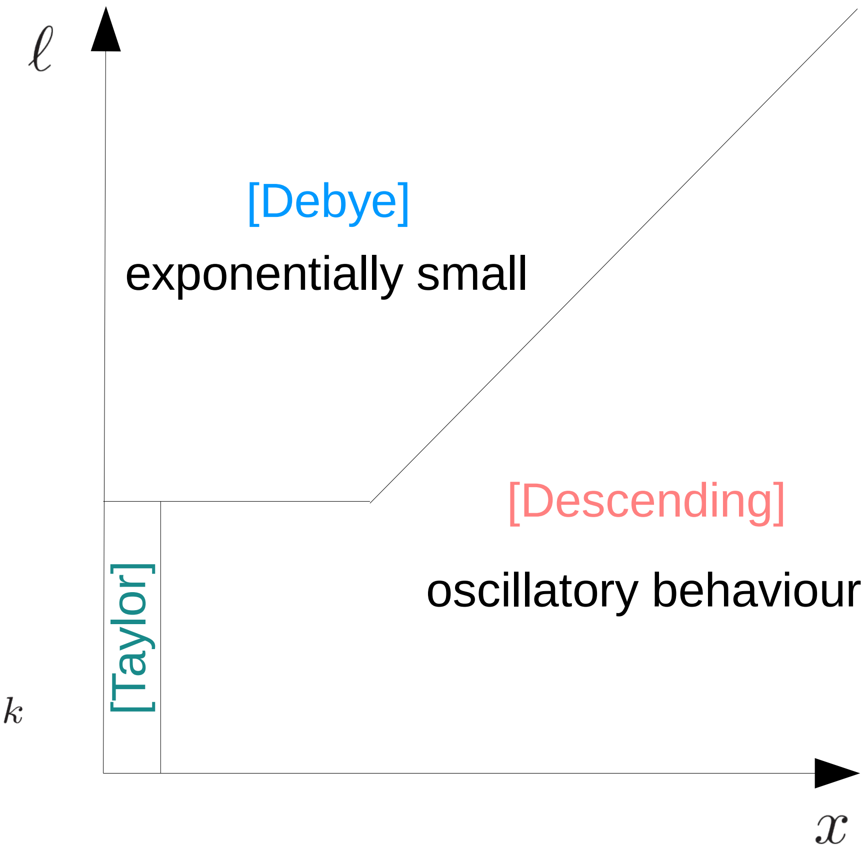
$$j_\ell(x) = \frac{2\ell+3}{x} j_{\ell+1}(x) - j_{\ell+2}(x)$$

[Debye]

$$J_\nu(x) \sim \frac{e^{\nu(\tanh\alpha - \alpha)}}{\sqrt{2\nu\pi\tanh\alpha}} \sum_{k=0}^{\infty} \frac{U_k(\coth\alpha)}{\nu^k}$$

[Taylor]

$$j_\ell(x) = \frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^\ell \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\ell+k+3/2)}\left(\frac{x}{2}\right)^{2k}$$

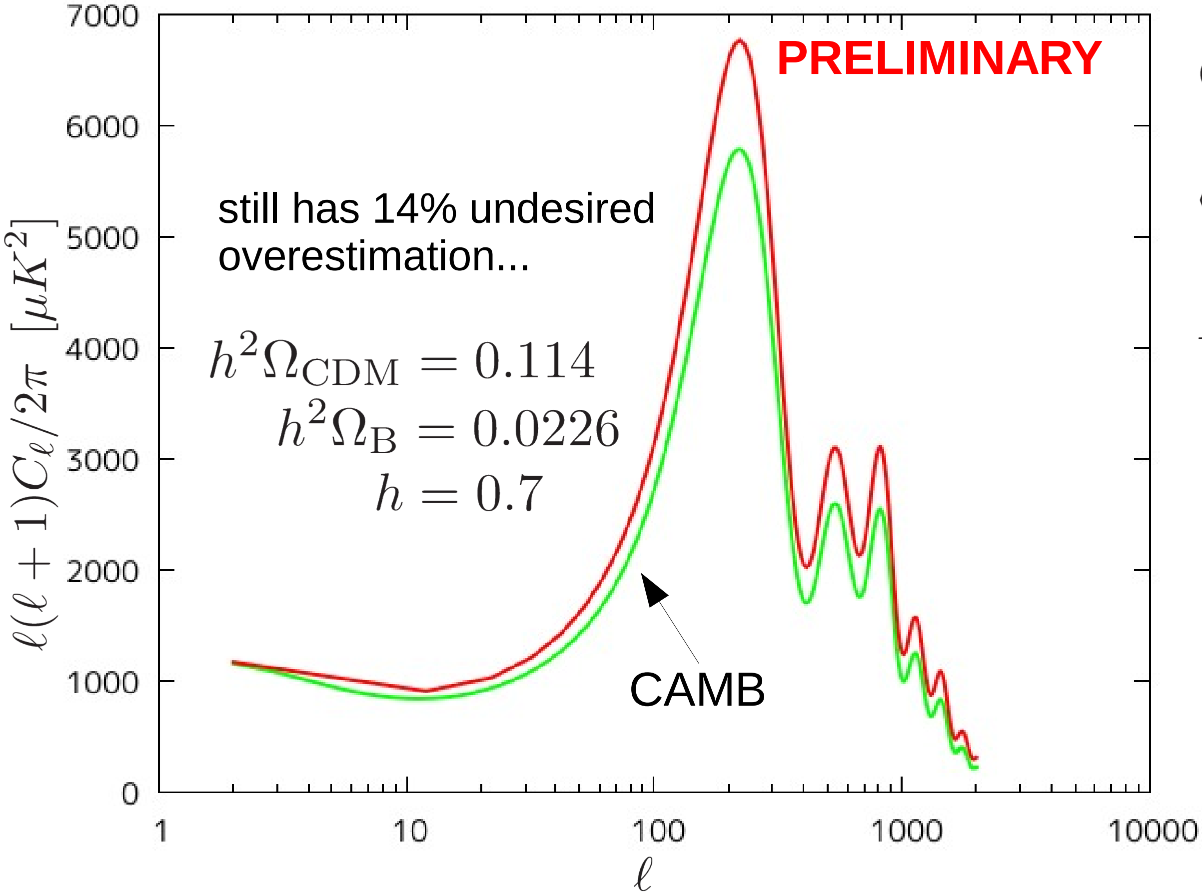



Once storing $j_\ell(x_n)$ for $0 \leq x \leq k_{\max}\eta_0$, $j_\ell\left[k(\eta_0 - \eta)\right]$ with arbitrary argument is given by the partitioned polynomial interpolation.

# Angular power spectrum (1st-order)

Angular power spectrum

$$C_\ell = \frac{2}{\pi}\int_0^\infty dk\, k^2 \mathcal{T}_{\Theta_\ell}(k,\eta)^2 P_\zeta(k,\eta_{\rm in})$$




$$\Theta_\ell(k,\eta) = \mathcal{T}_{\Theta_\ell}(k,\eta)\zeta(k,\eta_{\rm in})$$

$$\langle\zeta(\mathbf{k})\zeta^*(\mathbf{k})\rangle = (2\pi)^3\delta_D^3(\mathbf{k}+\mathbf{k}')P_\zeta(k)$$

$$P_\zeta(k) = \frac{2\pi^2}{k^3}\Delta^2(k_{\rm pivot})\left(\frac{k}{k_{\rm pivot}}\right)^{n_s-1}$$

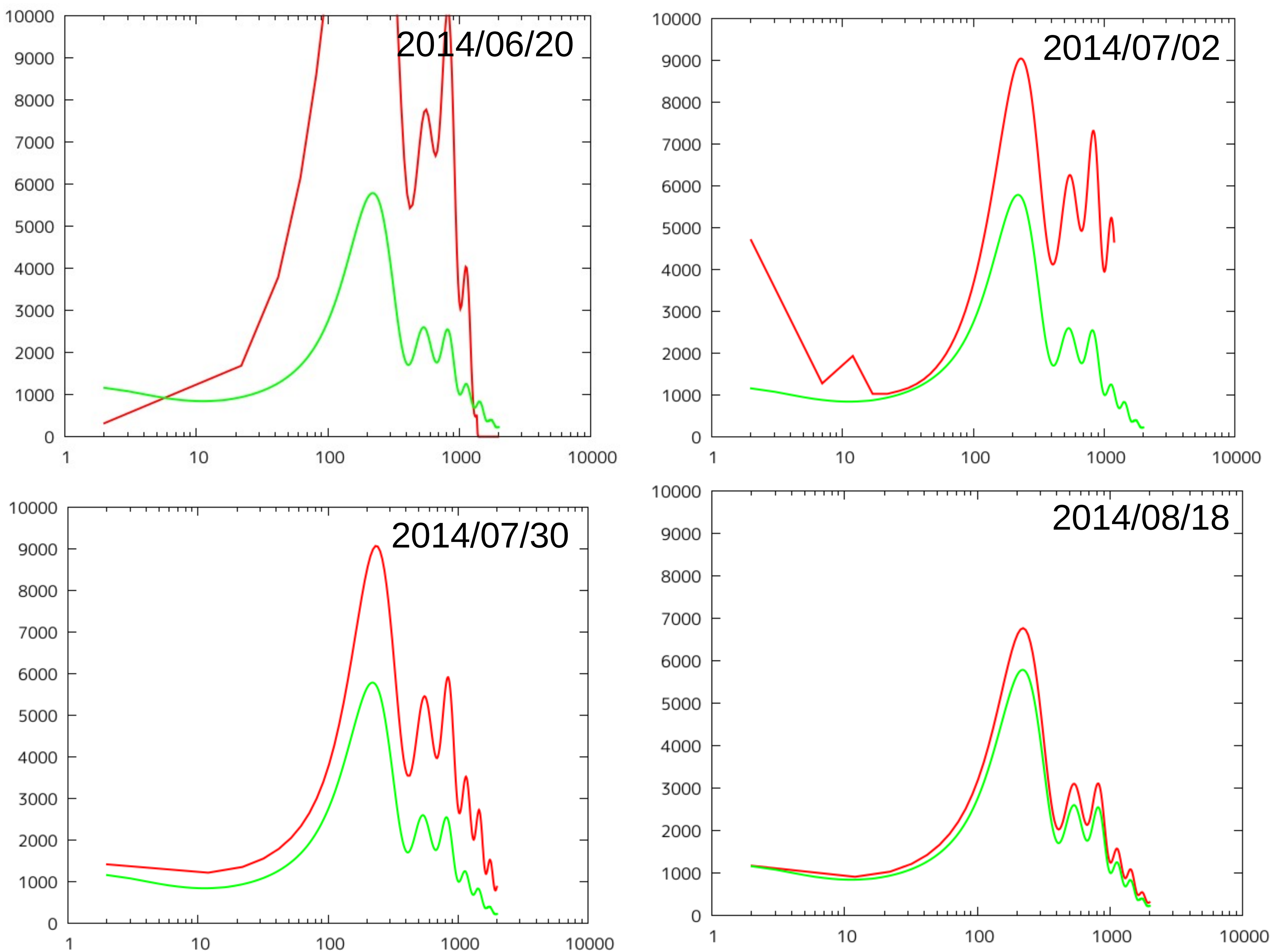
$$\Delta^2(k_{\rm pivot}) = 2.46\times10^{-9}$$
$$k_{\rm pivot} = 0.002\ {\rm Mpc}^{-1}$$
$$n_s = 0.96$$

We have some existing codes named CMBFAST, CAMB, CosmoLib, CLASS, etc..


CMBFAST : Seljak, Zaldarriaga, APJ469 (1996) 437
CAMB : Lewis, Challinor, APJ538 (2000) 473
CosmoLib : Huang, JCAP 1206 (2012) 012
CLASS II : Blas, Lesgourgues, Tram, JCAP 1107 (2011) 034

# Angular power spectrum (1st-order)

Takashi Hiramatsu



# Go to 2nd-order



2nd-order contributions appear in...

- Einstein-Boltzmann equations for 2nd-order quantities sourced by $[\text{1st-order}]^2$

  CDM+Baryon+Gravity have been implemented,
  but Baryon-Photon/Gravity-Photon couplings are not considered yet.

- Line-of-sight integral sourced by $[\text{1st-order}]^2$

  Formulations have been completed by R.Saito.

  **R.Saito, Naruko, Hiramatsu, Sasaki, arXiv:1409.2464**

Existing 2nd-order Boltzmann solver

`CMBquick` — **CMBquick : Creminelli, Pitrou, Vernizzi, arXiv:1109.1822**

`SONG` — **SONG : Pettinari, Filder, Crittenden, Koyama, Wands; Pettinari, arXiv:1405.2280 (thesis)**

`CosmoLib2nd` — **CosmoLib2nd : Huang, Vernizzi, arXiv:1212.3573**

# Expansion up to 2nd-order

Takashi Hiramatsu

Poisson gauge + neglecting 1st-order $B_i$ and $h_{ij}$

$$ds^2 = -a^2 e^{2\Psi} d\eta^2 - 2a^2 B_i dx^i d\eta + a^2 \left( e^{2\Phi}\delta_{ij} + h_{ij} \right)$$

$$\partial_i B^i = 0 \qquad B_i = B^{(x)} e_i^{(x)} + B^{(y)} e_i^{(y)}$$

$$\partial_i h^{ij} = 0 \qquad h_{ij} = h^{(+)} e_{ij}^{(+)} + h^{(\times)} e_{ij}^{(\times)}$$

$$\delta_{ij} h^{ij} = 0$$

Expanding up to 2and-order

$$\Psi = \Psi^{(1)} + \Psi^{(2)} + \cdots \qquad \delta_i = \delta_i^{(1)} + \delta_i^{(2)} + \cdots$$

$$\Phi = \Phi^{(1)} + \Phi^{(2)} + \cdots \qquad v_i = v_i^{(1)} + v_i^{(2)} + \cdots$$

$$B^{(p)} = 0 + B^{(p)(2)} + \cdots \qquad (i = b, c)$$

$$h^{(p)} = 0 + h^{(p)(2)} + \cdots$$

# 2nd-order perturbation equtaions

Takashi Hiramatsu

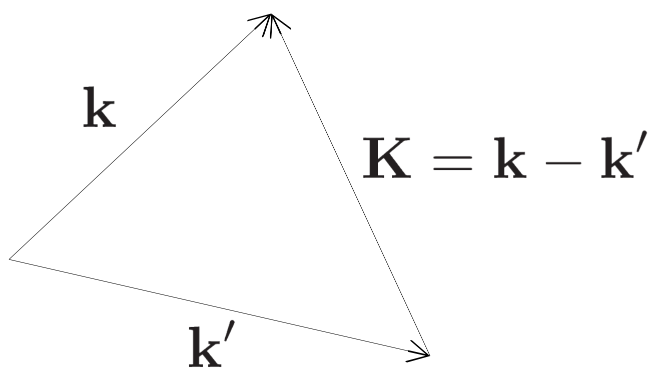
CDM + gravity (contribution from radiation is work in progress....)

$$\Psi^{(2)} + \Phi^{(2)} = \mathcal{Q}_\Psi\left[\Phi^{(1)}\Phi^{(1)}, \Psi^{(1)}\Psi^{(1)}, \Phi^{(1)}\Psi^{(1)}, V^{(1)}V^{(1)}\right]$$

'Q'uadratic terms of 1st-order quantities

$$\begin{aligned} &= c_{1,1}(k,k',K)\Phi^{(1)}(k')\Phi^{(1)}(K) \\ &\quad + c_{1,2}(k,k',K)\Psi^{(1)}(k')\Psi^{(1)}(K) \\ &\quad + c_{1,3}(k,k',K)\Phi^{(1)}(k')\Psi^{(1)}(K) \\ &\quad + \kappa^2 a^2 \rho^{(0)}(1+w)c_{1,4}(k,k',K)V^{(1)}(k')V^{(1)}(K) \\ &\quad - \frac{3}{2k^4}\kappa^2 a^2 \mathcal{F}\left[\widehat{T}^{R(2)i}{}_j{}^{,j}{}_{,i}\right] \end{aligned}$$

Pending until deciding how to treat 2nd-order radiation (temperature ? brightness ?)




$c_{i.j}(k,k',K)$ is a fractional expression like $c_{1,1} = \dfrac{3(K^2-k'^2)^2 - k^2(3K^2-k'^2) + 2k^4}{4k^4}$

and $\rho^{(0)}V^{(1)}(k')V^{(1)}(K) = \rho_b^{(0)}v_b^{(1)}(k')v_b^{(1)}(K) + \rho_c^{(0)}v_c^{(1)}(k')v_c^{(1)}(K)$

# 2nd-order perturbation equtaions

Takashi Hiramatsu

CDM + gravity (contribution from radiation is work in progress....)

$$\Phi^{(2)\prime} - \mathcal{H}\Psi^{(2)} + \frac{k^2}{3\mathcal{H}}\Phi^{(2)} - \frac{\kappa^2 a^2}{6\mathcal{H}}\rho^{(0)}\delta^{(2)} = \mathcal{Q}_\Phi\left[\Psi^{(1)}\Psi^{(1)}, \Phi^{(1)\prime}\Psi^{(1)}, \Phi^{(1)\prime}\Phi^{(1)\prime}, \Phi^{(1)\prime}\Phi^{(1)}, V^{(1)}V^{(1)}\right] - \frac{\kappa^2 a^2}{6\mathcal{H}}T^{R(2)0}{}_0$$

$$B^{(A)\prime} + 2\mathcal{H}B^{(A)} = \mathcal{Q}_B\left[\Phi^{(1)}\Phi^{(1)}, \Psi^{(1)}\Psi^{(1)}, \Phi^{(1)}\Psi^{(1)}, \Psi^{(1)}\Phi^{(1)}, V^{(1)}V^{(1)}\right] + \frac{2i\kappa^2 a^2}{k^2}e^i(\hat{k})\mathcal{F}\left[\widehat{T}^{R(2)}{}_{ij}{}^{,j}\right]$$

$$h^{(A)\prime} + 2\mathcal{H}h^{(A)} + k^2 h^{(A)} = \mathcal{Q}_h\left[\Phi^{(1)}\Phi^{(1)}, \Psi^{(1)}\Psi^{(1)}, \Phi^{(1)}\Psi^{(1)}, \Psi^{(1)}\Phi^{(1)}, V^{(1)}V^{(1)}\right] + 2\kappa^2 a^2 e^{ij}(\hat{k})\mathcal{F}\left[\widehat{T}^{R(2)}{}_{ij}\right]$$

$$V_\mathrm{c}^{(2)\prime} + \mathcal{H}V_\mathrm{c}^{(2)} + k\Psi^{(2)} = \mathcal{Q}_{V_\mathrm{c}}\left[V_\mathrm{c}^{(1)}\delta^{(1)\prime}, V_\mathrm{c}^{(1)}\Phi^{(1)\prime}, V_\mathrm{c}^{(1)}\Phi^{(1)}, V_\mathrm{c}^{(1)}\Psi^{(1)}, V_\mathrm{c}^{(1)}\delta^{(1)}, \Psi^{(1)}\delta^{(1)}, V_\mathrm{c}^{(1)\prime}\delta^{(1)}, V_\mathrm{c}^{(1)\prime}\Phi^{(1)}, V_\mathrm{c}^{(1)\prime}\Psi^{(1)}, V_\mathrm{c}^{(1)\prime}V_\mathrm{c}^{(1)}\right]$$

$$\delta_\mathrm{c}^{(2)\prime} + 3\Phi^{(2)\prime} - kV_\mathrm{c}^{(2)} = \mathcal{Q}_{\delta_\mathrm{c}}\left[\delta_\mathrm{c}^{(1)}\Phi^{(1)\prime}, \delta_\mathrm{c}^{(1)}V_\mathrm{c}^{(1)}, V_\mathrm{c}^{(1)\prime}V_\mathrm{c}^{(1)}, V_\mathrm{c}^{(1)}V_\mathrm{c}^{(1)}, \Psi^{(1)}V_\mathrm{c}^{(1)}, \Phi^{(1)}V_\mathrm{c}^{(1)}\right]$$

# 2nd-order line(curve)-of-sight integral

R.Saito, Naruko, Hiramatsu, Sasaki, arXiv:1409.2464

$$\delta I^{(\mathrm{II})} = \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \mathcal{T}^{(\mathrm{II})}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{n}_{\mathrm{obs}})\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)$$

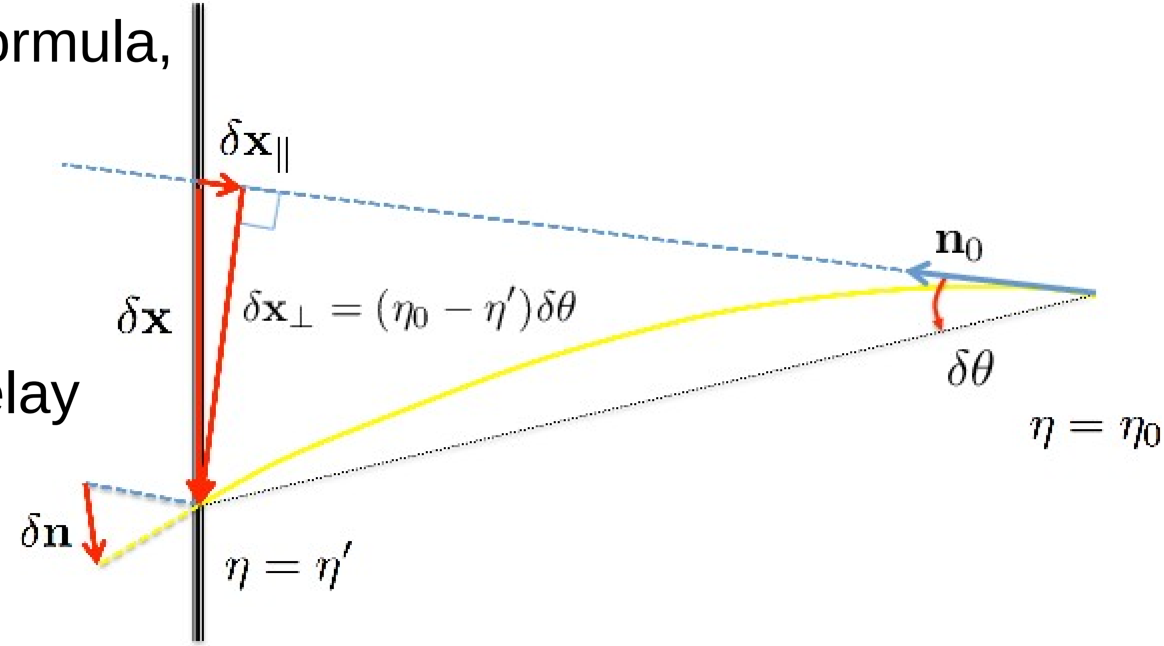
$$\mathcal{T}^{(\mathrm{II})}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{n}_{\mathrm{obs}}) = F(\mathbf{n}_{\mathrm{obs}}) \int_0^{\eta_0} d\eta' \, F_S(\hat{k}_1) S(k_1, \eta') e^{i\mathbf{k}_1 \cdot \mathbf{n}_{\mathrm{obs}}(\eta_0 - \eta')}$$

$$\times \int d\eta_1 \, F_T(\hat{k}_2) T(k_2, \eta_1, \eta') e^{i\mathbf{k}_2 \cdot \mathbf{n}_{\mathrm{obs}}(\eta_0 - \eta_1)}$$

We found 7 combinations in this formula,

SW x ISW
SW x Lensing
SW x Time-delay
SW x Deflection

ISW x ISW
ISW x Lensing
ISW x Time-delay



R.Saito, Naruko, Hiramatsu, Sasaki, arXiv:1409.0000

## SW x Lensing

$$S(k_1,\eta') = k_1 g(\eta')\left[\Theta_{T0}+4\Psi\right] + \frac{d}{d\eta'}\left(\frac{4g(\eta')v_{\rm b}}{k_1}\right) + \mathcal{P}_2\left(\frac{1}{ik_1}\frac{d}{d\eta'}\right)\left[g(\eta')\Pi\right]$$

$$T(k_2,\eta_1,\eta') = k_2(\eta_1-\eta')\left[\Psi(k_2,\eta_1)-\Phi(k_2,\eta_1)\right]$$

$$F(\mathbf{n}_{\rm obs})F_S(\hat{k}_1)F_T(\hat{k}_2) = -\sum_{\lambda=\pm}(i\epsilon^\lambda\cdot\hat{k}_1)(i\epsilon^\lambda\cdot\hat{k}_2)$$

## Bispectrum

$$B^{m_1m_2m_3}_{\ell_1\ell_2\ell_3} = 2[1+(-1)^{\ell_1+\ell_2+\ell_3}]\mathcal{G}^{m_1m_2m_3;(+1)(-1)0}_{\ell_1\ell_2\ell_3}\int_0^{\eta_0} d\eta'\, b^S_{\ell_1}(\eta')b^T_{\ell_2}(\eta') + 2\ \mathrm{sym.}$$

$$b^S_{\ell_1}(\eta') = \frac{2}{\pi}\sqrt{\frac{\ell_1(\ell_1+1)}{2}}\int dk_1\, k_1^2 P_\zeta(k_1)\mathcal{T}_{\Theta_{\ell_1}}(k_1)\frac{S(k_1,\eta')}{k_1(\eta_0-\eta')}j_{\ell_1}[k_1(\eta_0-\eta')]$$

$$b^T_{\ell_2}(\eta') = \frac{2}{\pi}\sqrt{\frac{\ell_2(\ell_2+1)}{2}}\int_{\eta'}^{\eta_0} d\eta_1\int dk_2\, k_2^2 P_\zeta(k_2)\mathcal{T}_{\Theta_{\ell_2}}(k_1)\frac{T(k_2,\eta_1,\eta')}{k_2(\eta_0-\eta_1)}j_{\ell_2}[k_2(\eta_0-\eta_1)]$$

# Overview of my code

Takashi Hiramatsu

- Full scratch development, completely independent of existing codes

- C++

- Parallelised by OpenMP

- Time evolution : 1-stage 2nd-order implicit Runge-Kutta (Gauss-Legendre) method (implementing up to 4th-order schemes)

- Line-of-sight Integration : Trapezoidal rule

- Interpolation scheme : Polynomial approximation (up to $\mathcal{O}(h^5)$)

- Ready for implementing a variety of recombination/reionisation simulators

- We are now suffered from a small mismatch between results of our code and CAMB at the 1st-order.
- Implemented 2nd-order perturbations only for gravity and matter.
- Moved to evaluate the bi-spectrum using the 2nd-order LOS integral.