Second order CMB anisotropies

Kazuya Koyama

Institute of Cosmology and Gravitation,
University of Portsmouth

Motivation

- The Universe is non-Gaussian

  CMB anisotropies are inevitably non-Gaussian

  - Non-linear physics at recombination
  - Non-linearity of Einstein's gravity "Einstein's signature in the CMB"

  The intrinsic non-Gaussianity is expected to be $f_{NL} \sim O(1)$

- Primordial non-Guassianity

  Planck constraints

$$-9.8 < f_{NL}^{local} < 14.3 \quad (95\%\mathrm{CL})$$

This constraint is still much weaker than theoretical expectations

$f_{NL} < O(1)$

History

- Analytic approximations

  "Consistency relation" in the squeezed limit

  Bartolo et.al. 0407505; 1109.2043; Boubekeur et.al. 0905.0980;
  Creminelli & Zaldariaga 0405428; Creminelli et.al. 1109.1822
  Lewis 1204.5018

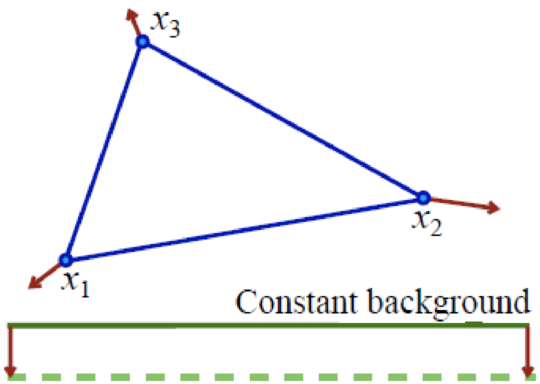
- 2^{nd} order Boltzmann equation

  Bartolo et.al. 0604416; 0610110; Pitrou 0706.4383; 0809.3245;
  Beneke & Fidler 1003.1834; Naruko et.al. 1304.6929

- Numerical codes

  Pitrou et.al. 1003.0481; Huang & Vernizzi 1212.3573; 1311.6105;
  Pettinari et.al. 1302.0832; 1406.2981; Fidler et.al. 1401.3296
  Su et.al. 1212.6968

Squeezed limit



- Consistency relation

$$ds^2 = a^2(\eta)\left[-d\eta^2 + e^{2\zeta(x^i)}dx_i dx_i\right]$$

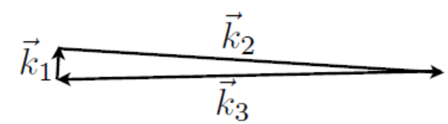
The effect of the long mode ζ_L is to rescale the coordinates

$$\langle \zeta(\vec{x}_2)\zeta(\vec{x}_3)\rangle_{\zeta_L} = \xi(|\vec{x}_3-\vec{x}_2|) + \zeta_L\left[\left(\vec{x}_3-\vec{x}_2\right)\cdot\nabla\xi(|\vec{x}_3-\vec{x}_2|)\right]$$

$$\vec{x}_\pm = \vec{x}_3 \pm \vec{x}_2$$

$$\langle \zeta(\vec{x}_1)\zeta(\vec{x}_2)\zeta(\vec{x}_3)\rangle = \langle \zeta(\vec{x}_1)\zeta(\vec{x}_+)\rangle\left[\vec{x}_-\nabla\xi(|\vec{x}_-|)\right]$$

$$= \int\frac{d^3k_L}{(2\pi)^3}\int\frac{d^3k_S}{(2\pi)^3}e^{ik_L(x_1-x_+)}P(k_L)P(k_S)\left[k_s\cdot\nabla_{k_s}\right]e^{ik_s x_-}$$



$$\Rightarrow \left\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\right\rangle = -(2\pi)^3\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)P(k_1)\frac{d\left(k_s^3P(k_s)\right)}{d\ln k_s}$$

Analytic approximation

- Ricci focusing

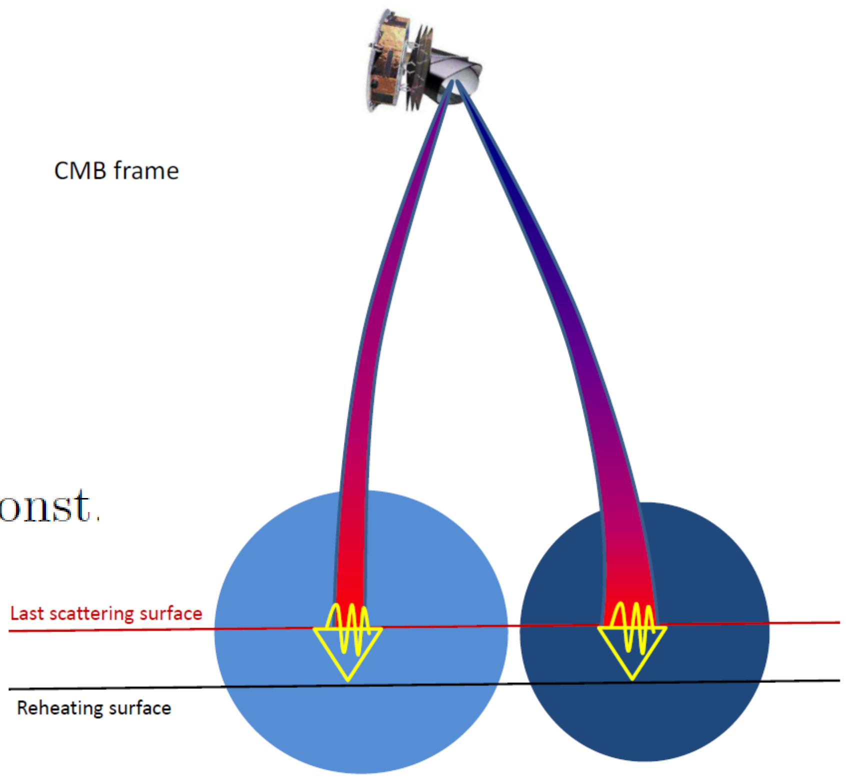
uniform radiation density gauge

Lewis 1204.5018

$$ds^2 = a^2(\eta)\left[-d\eta^2 + e^{2\zeta(x^i)}dx_i dx_i\right]$$

$$\zeta(x) \;=\; \zeta_L(x) + \zeta_S(x)$$

the observed angular size of perturbation becomes changes due to long-wavelength perturbations



$$a(\eta_r)(1+\zeta(x^i))\Delta x = \text{physical scale} = \text{const.}$$

in this gauge recombination happens at

$\eta = \eta_*$ everywhere

Analytic approximation

- Redshift modulation

$$ds^2 = a^2(\eta)[(1+2\Psi)d\eta^2 - (1-2\Phi)\delta_{ij}dx^i dx^j]$$

Lewis 1204.5018

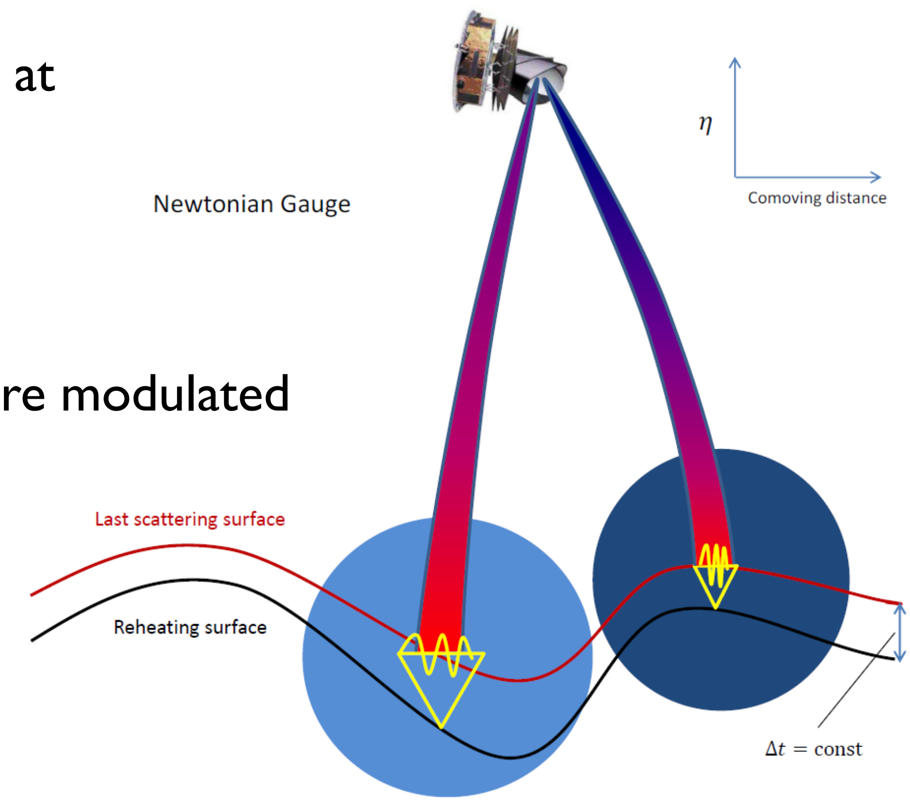
In this gauge, recombination happens at

$$\eta = \eta_* + \delta\eta \qquad \mathcal{H}\delta\eta = \frac{\Delta_\gamma}{4}$$

Newtonian Gauge

Observed small scales anisotropies are modulated by this variation in redshiftining

This is also the origin of linear large scale anisotropies



Analytic approximations (squeezed limit)

- Ricci focusing

  rescaling of the spatial coordinate (in 2D)

$$\left\langle \Theta(\vec{n}_1)\Theta(\vec{n}_2)\Theta(\vec{n}_3)\right\rangle = \left\langle \Theta(\vec{n}_1)\zeta_L \right\rangle \left[\vec{n}_- \cdot \nabla_{n_-} \left\langle \Theta(\vec{n}_2)\Theta(\vec{n}_3)\right\rangle \right]$$

$$b_{l_1 l_2 l_3} \approx -C_{l_1}^{T\zeta_0^*} \frac{1}{l^2} \frac{\mathrm{d}}{\mathrm{d}\ln l} (l^2 \tilde{C}_l) \qquad \ell_1 << \ell_2, \ell_3$$

- Redshift modulation $\quad T(\eta, x^i) = \bar{T}(\eta)(1 + \Theta(\eta, x^i))$

$$\Theta = \Theta_s + (\mathcal{H}\delta\eta + \Psi)(1 + \Theta_s)$$
$$= \Theta_L + \Theta_s + \Theta_L \Theta_s \qquad \Theta_L = \mathcal{H}\delta\eta + \Psi = \frac{\Delta_\gamma}{4} + \Psi = \frac{\Psi}{3}$$

$$b_{l_1 l_2 l_3} \approx C_{l_1} \left(\tilde{C}_{l_2} + \tilde{C}_{l_3} \right) \qquad \ell_1 << \ell_2, \ell_3$$

2^{nd} order Boltzmann equation

- Boltzmann equation for distribution function $f(x,p)$

$$\frac{\partial f}{\partial \eta} + \frac{dx^i}{d\eta}\frac{\partial f}{\partial x_i} + \frac{dq}{d\eta}\frac{\partial f}{\partial q} + \frac{dn^i}{d\eta}\frac{\partial f}{\partial n_i} = \mathfrak{C}[f]$$

$$\frac{\partial}{\partial \eta} f + n^i \frac{\partial f}{\partial x^i} + \frac{dp}{d\eta}\frac{\partial f}{\partial p} + \frac{dn^i}{d\eta}\frac{\partial f}{\partial n^i} + \left(\frac{dx^i}{d\eta} - n^i\right)\frac{\partial f}{\partial x^i}$$

redshifts lensing time-delay

- Brightness temperature

$$1 + \Delta(\eta, \mathbf{x}, \mathbf{n}) = \frac{\int dq\, q^3\, f(\eta, \mathbf{x}, q\mathbf{n})}{\int dq\, q^3\, f^{(0)}(q)}$$

Fourie transformation and spherical harmonic expansion

2^{nd} order Boltzmann equation

- Multipole decomposition $n = (\ell, m)$

$$\dot{\Delta}_n + k\,\Sigma_{nm}\Delta_m + \mathcal{M}_n + \mathcal{Q}^L_n \;=\; \mathfrak{C}_n$$

  - Free streaming term $\dot{\Delta}_n + k\,\Sigma_{nm}\Delta_m$

    couples neighbouring moments, generating higher moments over time

  - Second order metric sources (2^{nd} order ISW, SW)

$$\mathcal{M} \;=\; 4\,(n^i\partial_i\Psi - \dot{\Phi} - n^i\dot{\omega}_i + n^i n^j\dot{\gamma}_{ij}) \;-\; 4(\Psi - \Phi)\,n^i\partial_i\Psi \;-\; 8\Phi\dot{\Phi}$$

  - Propagation effects (lensing, redshift and time delay)

$$\mathcal{Q}^L \;=\; (\Psi + \Phi)\,n^i\partial_i\Delta \;+\; 4\,(n^i\partial_i\Psi - \dot{\Phi})\,\Delta \;-\; (\delta^{ij} - n^i n^j)\,\partial_j(\Psi + \Phi)\,\frac{\partial\Delta}{\partial n^i}$$

Collision terms

- Collision term $\mathfrak{C}_n = -|\dot{\kappa}| \left(\Delta_n - \Gamma_{nm}\Delta_m - \mathcal{Q}_n^C \right)$ $|\dot{\kappa}|$: Compton scattering rate

  the first two terms are the same as the first order

  $$\Gamma_{nn'} \Delta_{n'} \xrightarrow{\mathcal{I}} \delta_{\ell 0} \mathcal{I}_0^0 + \delta_{\ell 1} 4 u_{[m]} + \delta_{\ell 2} \left(\mathcal{I}_m^2 - \sqrt{6} \mathcal{E}_m^2 \right) / 10$$

  the quadratic source is made of convolutions over photon density and electron velocity

  sources exist at any multipole moments but high multipoles are suppressed

- Tight coupling

  tight coupling between electron and photon suppresses the free streaming term thus higher order moments $L > 2$

  after recombination, higher order multipoles are generated

  $$\ell \sim k \left(\tau - \tau_{\rm rec} \right)$$

Line of sight integration

- Line of sight integration

$$\dot{\Delta} + (i\,\boldsymbol{k}\cdot\boldsymbol{n} + \dot{\kappa})\Delta = S$$

$$\Rightarrow \quad \Delta(\tau_0, \boldsymbol{k}, \boldsymbol{n}) = \int_{\tau_{\rm in}}^{\tau_0} \mathrm{d}\tau\, e^{\,i\,\boldsymbol{k}\cdot\boldsymbol{n}\,(\tau-\tau_0)}\; e^{-\kappa(\tau,\tau_0)}\; \mathcal{S}(\tau, \boldsymbol{k}, \boldsymbol{n})$$

- Multi-pole decomposition

$$\mathcal{I}^{\ell}_{m}(\tau_0, \boldsymbol{k}) = \int_{\tau_{\rm in}}^{\tau_0} \mathrm{d}\tau\, e^{-\kappa} \sum_{L=0}^{L_{\rm max}} J_{L\ell m}(k\,r)\;\; S^{\mathcal{I}}_{Lm}(\tau, \boldsymbol{k})$$

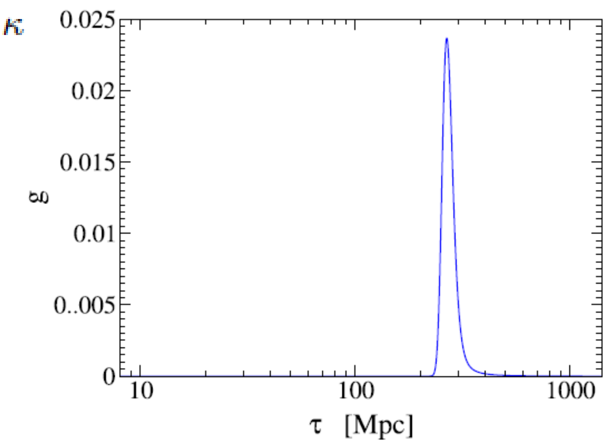
$$J_{L\ell m}(x) \equiv (-1)^m\,(2\,\ell+1) \sum_{\ell_1=|\ell-L|}^{\ell+L} i^{\,\ell-\ell_1-L}\,(2\,\ell_1+1) \begin{pmatrix} \ell & \ell_1 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell_1 & L \\ -m & 0 & m \end{pmatrix} j_{\ell_1}(x)$$

the second sources $\quad S_n = -\mathcal{M}^{(2)}_n - \mathcal{Q}^L_n + |\dot{\kappa}|(\Gamma_{nm}\Delta^{(2)}_m + \mathcal{Q}^C_n)$

Sources

- Scattering contributions

$$g(\tau) \equiv \dot{\kappa}\, e^{-\kappa}$$



$$\Delta_n(\tau_0) \supset \int_{\tau_{\rm in}}^{\tau_0} {\rm d}\tau \;\; J_{nn'}(kr) \;\; g(\tau) \left(\Gamma_{n'n''} \Delta_{n''} \; + \; \mathcal{Q}^{\mathfrak{C}}_{n'} \right)$$

$\Gamma_{nn'} \Delta_{n'}$ due to geometry of Thomson scattering
this contains only up to quadrupole $L_{\rm max} = 2$

$\mathcal{Q}^{\mathfrak{C}}_{n}$ Quadratic source include the photon density thus there are contributions from $L > 2$ but they are tight coupling suppressed
(in numerical codes, $L_{\rm max}$ needs to be treated as a parameter)

thanks to the visibility function, the evolution of photon hierarchy can be stopped right after recommbination

Sources

- Second order metric

$$\Delta_n(\tau_0) \supset \int_{\tau_{\rm in}}^{\tau_0} {\rm d}\tau \;\; J_{nn'}(kr)\; \mathcal{M}_n$$

$$\begin{aligned}\mathcal{M}_n \;=\; & -\,\delta_{L0}\,4\left[\,\dot\Phi\,+\,2\,\dot\Phi\,\Phi\,\right] \\ & -\,\delta_{L1}\,4\left[\,\delta_{m0}\,k\,\Psi\,+\,k_1^{[m]}\,\Psi\,(\Phi\,-\,\Psi)\,-\,i\,\dot\omega_{[m]}\,\right]\,-\,\delta_{L2}\,4\,\dot\gamma_{[m]}\end{aligned}$$

No visibility function thus the source needs to be evolved until today
However radiation contributions can be ignored

- Propagation sources

$$\Delta_n(\tau_0) \supset \int_{\tau_{\rm in}}^{\tau_0} {\rm d}\tau \;\; J_{nn'}(kr)\; \mathcal{Q}_n^L$$

$$\sum_{\pm} \pm\, k_1^{[m_2]}\,(\Psi\,+\,\Phi)\,\mathcal{I}_{m_1}^{\ell\pm1}\,R^{\pm,l}_{m_1\,m}$$

No visibility function thus the source needs to be evolved until today
Sources include the photon density thus $L_{\rm max}$ needs to be infinity in principle

There is no real advantage in using the line of sight integration!

Transformation of variable

- Redshift terms

Huang & Vernizzi 1212.3573; 1311.6105;
Pettinari et.al. 1302.0832; 1406.2981;
Fidler et.al. 1401.3296

transformation of variable

$$\tilde{\Delta} = \Delta - \Delta\Delta/2 \qquad \dot{\Delta} = -n_i \partial^i \Delta - 4\left(n^i \partial_i \Psi - \dot{\Phi}\right) + \mathfrak{C}$$

$$\dot{\tilde{\Delta}} = \dot{\Delta} - \Delta\dot{\Delta} = -n_i \partial^i \tilde{\Delta} - \mathcal{M} + \mathfrak{C}(1-\Delta)$$
$$-\mathcal{Q}^L + \cancel{4\left(n^i \partial_i \Psi - \dot{\Phi}\right)\Delta} \, ,$$

$$\mathcal{Q}^L = (\Psi + \Phi)\, n^i \partial_i \Delta + \cancel{4\left(n^i \partial_i \Psi - \dot{\Phi}\right)\Delta} - \left(\delta^{ij} - n^i n^j\right) \partial_j (\Psi + \Phi) \frac{\partial \Delta}{\partial n^i}$$

The redshift term is cancelled while the collision term is modified (but still suppressed by the visibility function)

Lensing and time-delay cannot be removed in this way. We do not include them in the source

Bispectrum

- Transfer function

$$\Delta_{\ell m}(\tau, \boldsymbol{k}) = \mathcal{T}^{(1)}_{\ell m}(\tau, \boldsymbol{k})\, \Phi(\tau_{\rm in}, \boldsymbol{k})$$
$$+ \int \frac{\mathrm{d}\boldsymbol{k_1}'\, \mathrm{d}\boldsymbol{k_2}'}{(2\pi)^3}\, \delta(\boldsymbol{k_1}' + \boldsymbol{k_2}' - \boldsymbol{k})\ \mathcal{T}^{(2)}_{\ell m}(\tau, \boldsymbol{k_1}', \boldsymbol{k_2}', \boldsymbol{k})\ \Phi(\tau_{\rm in}, \boldsymbol{k_1}')\, \Phi(\tau_{\rm in}, \boldsymbol{k_2}')\,.$$

- Bispectrum

$$\langle\, \Delta^3 \,\rangle = \int \frac{\mathrm{d}\boldsymbol{k_1}\, \mathrm{d}\boldsymbol{k_2}\, \mathrm{d}\boldsymbol{k_3}}{(2\,\pi)^9}\ \langle\, \Delta_{\ell_1 m_1}(\tau_0, \boldsymbol{k_1})\, \Delta_{\ell_2 m_2}(\tau_0, \boldsymbol{k_2})\, \Delta_{\ell_3 m_3}(\tau_0, \boldsymbol{k_3}) \,\rangle$$

$$\langle\, \Delta^3 \,\rangle_{\rm intr} = \int \frac{\mathrm{d}\boldsymbol{k_1}\, \mathrm{d}\boldsymbol{k_2}\, \mathrm{d}\boldsymbol{k_3}}{(2\,\pi)^6}\ \delta(\boldsymbol{k_1} + \boldsymbol{k_2} + \boldsymbol{k_3})$$
$$\times \left[\, 2\, \mathcal{T}^{(1)}_{\ell_1 m_1}(\boldsymbol{k_1})\, \mathcal{T}^{(1)}_{\ell_2 m_2}(\boldsymbol{k_2})\, \mathcal{T}^{(2)}_{\ell_3 m_3}(-\boldsymbol{k_1}, -\boldsymbol{k_2},\, \boldsymbol{k_3})\, P_\Phi(-\boldsymbol{k_1})\, P_\Phi(-\boldsymbol{k_2}) \ +\ 2 \text{ perm.} \,\right]$$

Bispectrum

- Statistical isotropy

$$\langle \Delta_{\ell_1 m_1} \Delta_{\ell_2 m_2} \Delta_{\ell_3 m_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1 \ell_2 \ell_3}[\Delta]$$

- Integration over angles

$$B^{\rm intr}_{\ell_1 \ell_2 \ell_3}[\Delta] = \sum_{m=-\infty}^{\infty} \sum_{L_3=|\ell_3-|m||}^{\ell_3+|m|} \sum_{L_1=|\ell_1-|m||}^{\ell_1+|m|} 8\, i^{L_1+\ell_2+L_3}\, 4\pi\, (2L_1+1)(2\ell_2+1)(2L_3+1)$$

$$\times \begin{pmatrix} L_1 & \ell_2 & L_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & L_1 & |m| \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_3 & L_3 & |m| \\ m & 0 & -m \end{pmatrix} \begin{Bmatrix} \ell_1 & \ell_3 & \ell_2 \\ L_3 & L_1 & |m| \end{Bmatrix} \int \frac{\mathrm{d}\,k_1\, \mathrm{d}\,k_2\, \mathrm{d}\,k_3\, \mathrm{d}\,r}{(2\,\pi)^6} (k_1\, k_2\, k_3\, r)^2$$

$$\times\ \widetilde{\mathcal{T}}^{(1)}_{\ell_1\,0}(k_1)\, \widetilde{\mathcal{T}}^{(1)}_{\ell_2\,0}(k_2)\, 2\, \overline{\mathcal{T}}^{(2)}_{\ell_3 m}(k_1, k_2, k_3)\, P_\Phi(k_1)\, P_\Phi(k_2)\, j_{L_1}(rk_1)\, j_{\ell_2}(rk_2)\, j_{L_3}(rk_3)\ +\ 2 \text{ perm.}$$

$$B^{\rm intr}_{\ell_1 \ell_2 \ell_3} = \sum_m B^{\{m\}}_{\ell_1 \ell_2 \ell_3}$$

scalar contribution (m=0) gives dominant contributions

Observed temperature

- Temperature perturbations

$$f(\eta,\mathbf{x},\mathbf{q}) = \left[\exp\left(\frac{q}{T\,(1+\Theta(\eta,\mathbf{x},\mathbf{q}))}\right)-1\right]^{-1} \qquad 1+\Delta(\eta,\mathbf{x},\mathbf{n}) = \frac{\int dq\, q^3\, f(\eta,\mathbf{x},q\mathbf{n})}{\int dq\, q^3\, f^{(0)}(q)}$$

- Spectral distortion

Pitrou et.al. 0912.3655; Naruko et.al. 1304.6929; Renaux-Petel et.al. 1312.4448; Pitrou et.al. 1402.0968

at second order, "temperature" depends on momentum

$$I\left(q,n^{(i)}\right) \simeq I_{\rm BB}\left(\frac{q}{aT}\right) + y\left(n^{(i)}\right)q^{-3}\frac{\partial}{\partial\ln q}\left[q^3\frac{\partial}{\partial\ln q}I_{\rm BB}\left(\frac{q}{aT}\right)\right]$$

- Bolometric and observed temperature

$$B \equiv \int \mathrm{d}q\; q^3 I = \frac{2\pi^4 a^4}{15}T^4(1+4y) = \bar{B}(1+\Theta)^4(1+4y) \qquad \Delta = 4\Theta + 6\Theta^2 + 4y,$$

$$B^{\rm intr}_{\ell_1\ell_2\ell_3}[\Theta] = \hat{B}^{\rm intr}_{\ell_1\ell_2\ell_3}[\Delta] - 3\,h_{\ell_1\ell_2\ell_3}\left(C_{\ell_1}C_{\ell_2} + C_{\ell_2}C_{\ell_3} + C_{\ell_3}C_{\ell_1}\right)$$

A second order Boltzmann code @ Portsmouth

- **G.W.Pettinari, C. Fidler,** R. Crittenden, K.Koyama, D.Wands
  JCAP 1304 003 (2013) 1302.0832
- **C. Fidler, G.W. Pettinari,** M. Beneke, R. Crittenden, K.Koyama, D.Wands
  JCAP 1407 011 (2014) 1401.3296;
- **G.W.Pettinari, C. Fidler,** R. Crittenden, K.Koyama, A. Lewis, D.Wands
  PRL submitted 1406.2981
- **G.W. Pettinari,** PhD thesis (Portsmouth) 1405.2280

SONG (Second Order Non-Gaussianity)

- We have written SONG to compute
  - → Second order non-Gaussianity
  - → Second order B-polarisation
- induced from
  - → Scattering sources
  - → Metric sources
  - → Redshift terms
  - → Including scalar, vector and tensor sources
- Inherits structure from CLASS
- Numerically optimised and parallelised
- Numerical stability tested
- All results double checked by independent computation based on Green functions

SONG

1) We solve the full differential system including photons, neutrinos, dark matter and baryons from the deep radiation era, till last scattering
2) After last scattering higher multipoles are exited by free streaming complicating the differential system Instead we solve a simplified hierarchy until today, neglecting ultra-relativistic species.
3) We build the line-of-sight sources for photon perturbations using the result of the differential systems
4) The line-of-sight integration is used to evolve the photon perturbations until today
5) We compute the bispectrum and compare the shape against the templates for local, equilateral and orthogonal shapes

Differential system

- We include photon temperature and polarisation, neutrinos, dark matter and baryons ➔ about 100 equations per (k_1, k_2, k) configuration

- We solve the system for about 1.000.000 configurations in (k_1, k_2, k)

Computation time: ~ 2 hours

Full-sky bispectrum

Involved 4D integral on 6 highly oscillatory functions

Computation time: ~ 1 hour

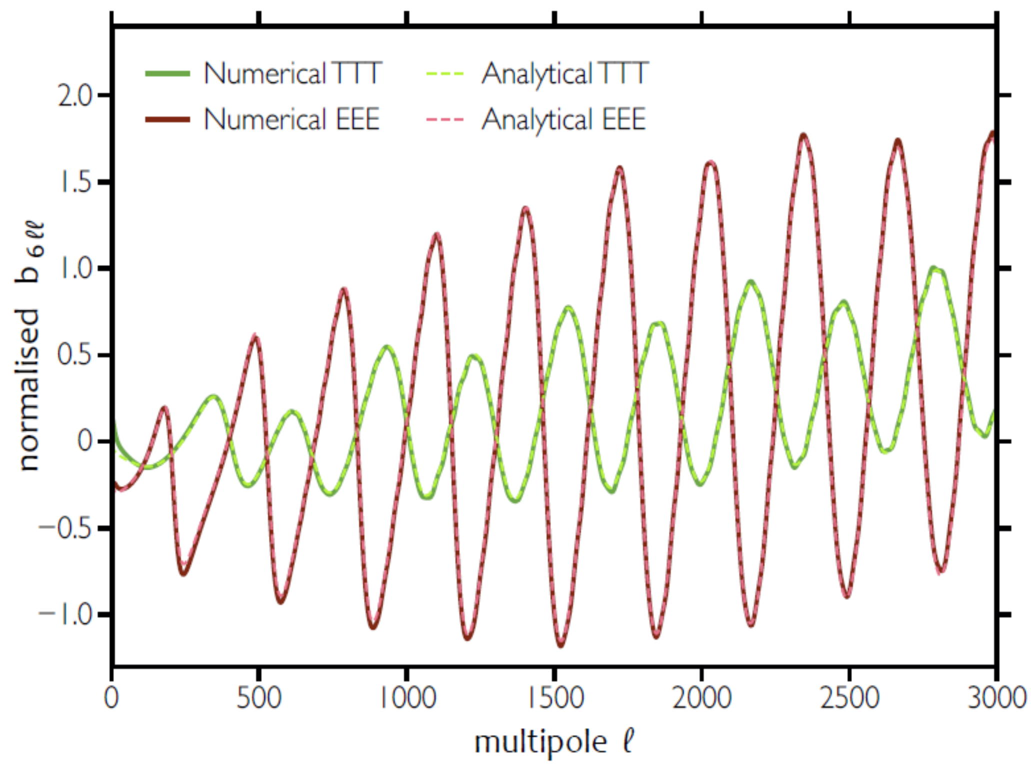
Results

- Comparison with analytic results

Ricci focusing + redshift modulation

$$b^{\mathrm{sq},XYZ}_{\ell_1\ell_2\ell_3} = -\frac{1}{2}\,C^{X\zeta}_{\ell_1}\left[\frac{\mathrm{d}\,(\ell_2{}^2\,C^{YZ}_{\ell_2})}{\ell_2\,\mathrm{d}\,\ell_2} + \frac{\mathrm{d}\,(\ell_3{}^2\,C^{YZ}_{\ell_3})}{\ell_3\,\mathrm{d}\,\ell_3}\right. \\ \left. + C^{XT}_{\ell_1}\left[\delta_{ZT}\,C^{YT}_{\ell_2} + \delta_{YT}\,C^{ZT}_{\ell_3}\right]\right],$$

$b_{6\ell\ell}$



Very good agreements

for squeezed configurations

both for intensity and polarisation

Fisher matrix

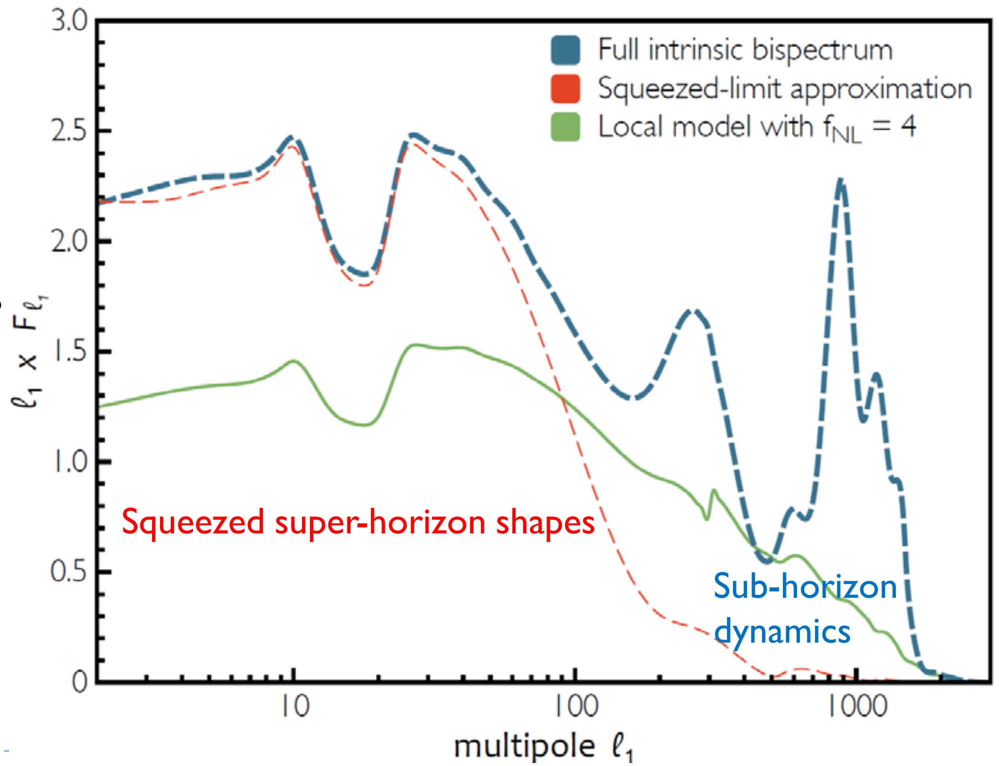
- Fisher matrix

$$F^{(i),(j)} = \sum_{ABC,XYZ} \sum_{2\le\ell_1\le\ell_2\le\ell_3}^{\ell_{\max}} \frac{1}{\Delta_{\ell_1\ell_2\ell_3}} B^{(i),ABC}_{\ell_1\ell_2\ell_3} \left(\widetilde{C}^{-1}_{\rm tot}\right)^{AX}_{\ell_1} \left(\widetilde{C}^{-1}_{\rm tot}\right)^{BY}_{\ell_2} \left(\widetilde{C}^{-1}_{\rm tot}\right)^{CZ}_{\ell_3} B^{(j),XYZ}_{\ell_1\ell_2\ell_3}$$

Differential contributions to the Fisher matrix

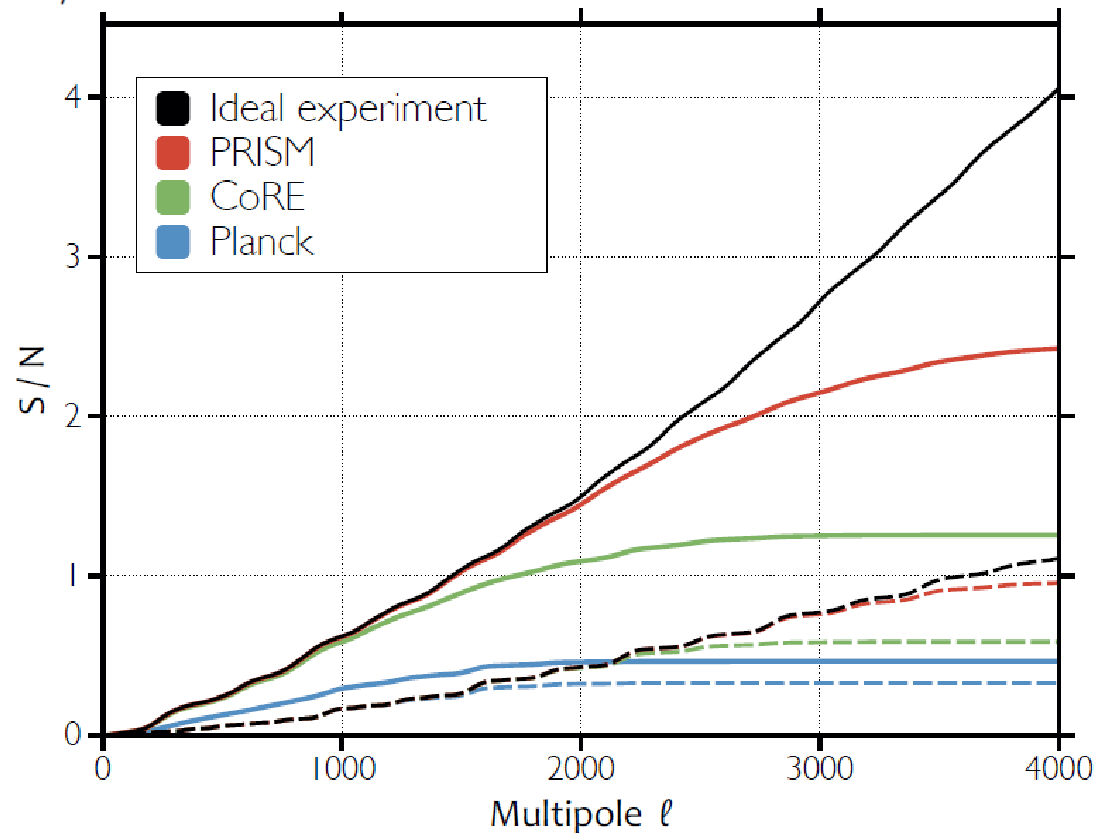
For small ℓ_1 analytic approximations work well

Large multipole contributions can be calculated only with 2nd order Boltzmann code



Signal to noise ratio

- Signal to noise ratio $S/N = \sqrt{F^{\mathrm{intr,intr}}}$



dashed: T only

solid: T, E, B (8 spectra)

Inclusion of polarisation significantly enhances our ability to detect the intrinsic bispectrum

$2.1\sigma, 1.3\sigma, 0.46\sigma$

PRISM, COrE and Planck

The bias to local type non-Gaussianity remains small

$f_{\mathrm{NL}}^{\mathrm{intr}} = 0.58, 0.45, 0.37 \quad \text{PRISM, COrE and Planck}$

Polarisation

- Temperature v polarisation

Temperature:

Sourced by density and velocity

Acoustic peaks are blurred

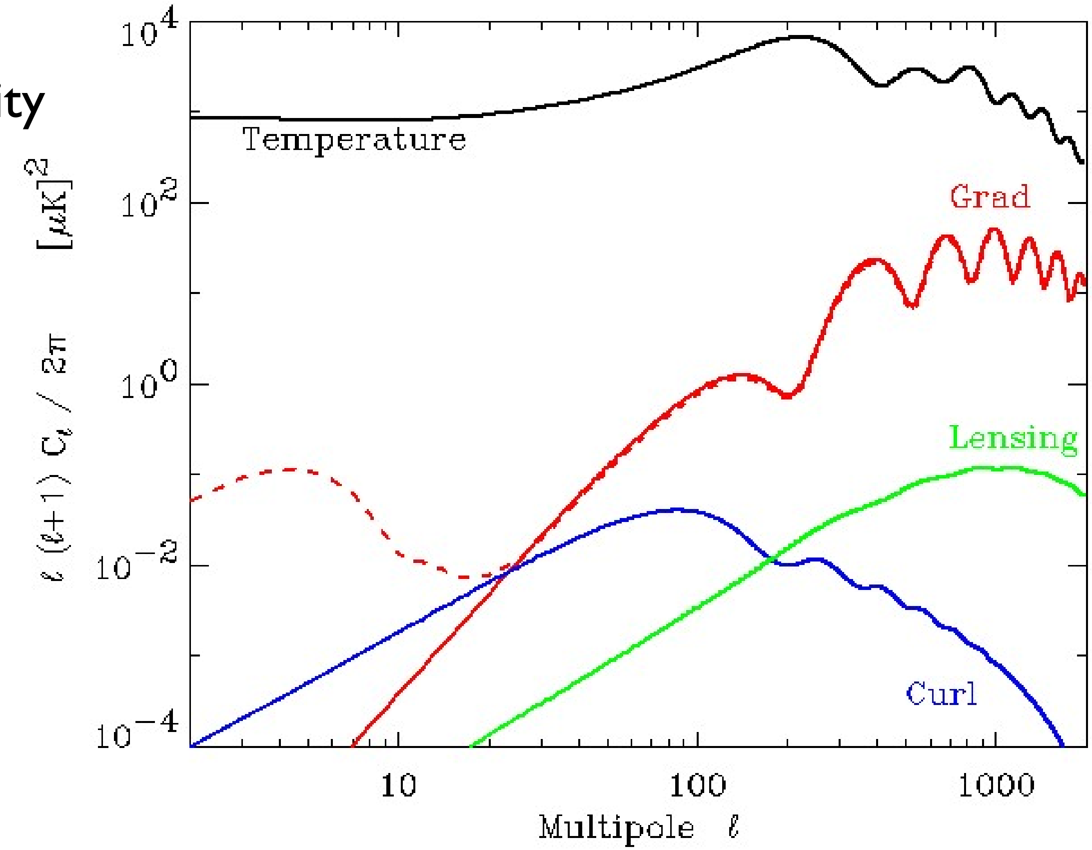
Polarisaiton:

Sourced only by scattering

the peakes are sharper, giving

larger Ricci focusing terms

$$b_{l_1 l_2 l_3} \approx -C_{l_1}^{T\zeta_0^*} \frac{1}{l^2} \frac{\mathrm{d}}{\mathrm{d}\ln l}(l^2 \tilde{C}_l)$$

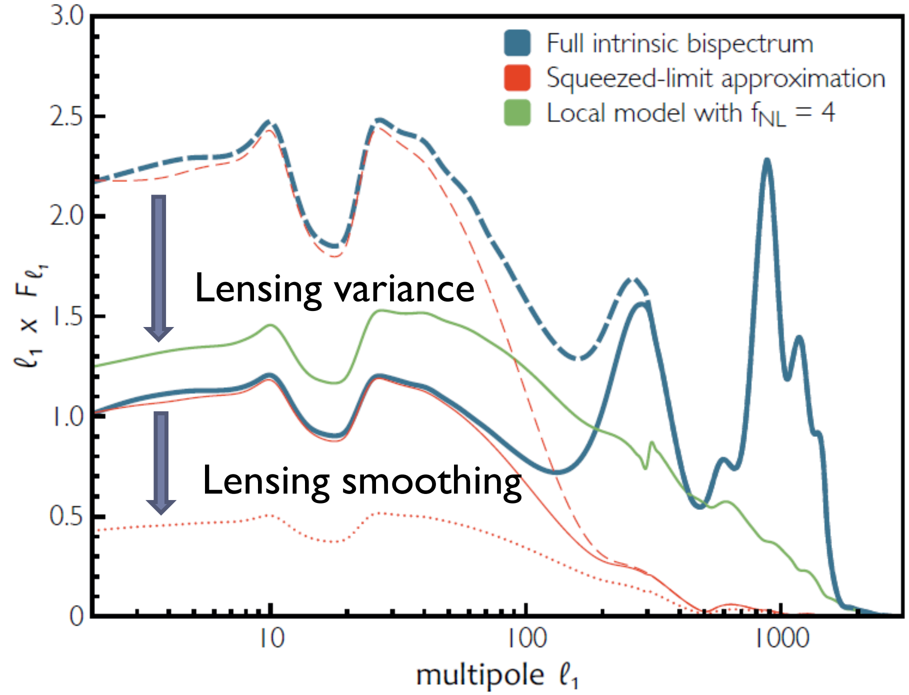


Lensing

- Lensing variance

$$F^{i,j} = \sum_{2\leq l_1\leq l_2\leq l_3}^{\ell_{\max}} \frac{B^i_{\ell_1\ell_2\ell_3}\, B^j_{\ell_1\ell_2\ell_3}}{C_{\ell_1} C_{\ell_2} C_{\ell_3}\, \Delta_{\ell_1\ell_2\ell_3}}\,,$$

$$C_\ell^{YZ} \longrightarrow \widetilde{C}_\ell^{YZ} \quad \text{(lensed)}$$



Lensing variance reduces siganl for squeezed configurations

- Effects on the squeezed limit bispectrum

$$b^{\mathrm{sq},XYZ}_{\ell_1\ell_2\ell_3} = -\frac{1}{2}\, C^{X\zeta}_{\ell_1} \left[\frac{\mathrm{d}\,({\ell_2}^2\, C^{YZ}_{\ell_2})}{\ell_2\, \mathrm{d}\,\ell_2} + \frac{\mathrm{d}\,({\ell_3}^2\, C^{YZ}_{\ell_3})}{\ell_3\, \mathrm{d}\,\ell_3} \right]$$

$$+\, C^{XT}_{\ell_1} \left[\delta_{ZT}\, C^{YT}_{\ell_2} + \delta_{YT}\, C^{ZT}_{\ell_3} \right]\,,$$

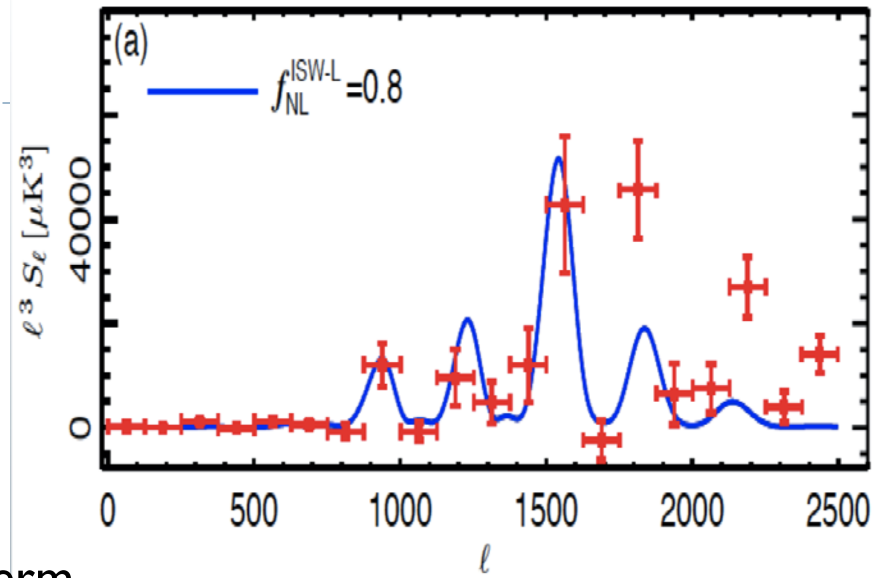
$$C_\ell^{YZ} \longrightarrow \widetilde{C}_\ell^{YZ} \quad \text{(lensed)}$$

lensing smears acoustic peaks, further reducing the signal

Lensing

- Secondary contribution

Weyl lensing in the squeezed limit
(the ISW-lensing bispectrum)



$$b_{l_1 l_2 l_3} \approx C_{l_1}^{T\kappa} \left[\frac{1}{l^2} \frac{\mathrm{d}(l^2 \tilde{C}_l)}{d \ln l} + \cos 2\phi_{l_1 l} \frac{\mathrm{d}\tilde{C}_l}{\mathrm{d} \ln l} \right]$$

isotropic part is similar to Ricci focusing term
(this is the reasons why lensing variance reduces the signal-to-noise)

However, the full intrinsic bispectrum is sufficiently different (a correlation is 0.6%) so we can distinguish between the two contributions

This bispectrum was detected by Planck at 4 sigma level!

Full effects of lensing on the intrinsic bispectrum are still unknown
(in the S/N calculation, we only included the lensing variance)

Recap

- SONG

  A code to solve the second order Boltzman-Einstein system

  including all physical effects

  A bispectum module gives an efficient computation of the full-sky bispectra

- Full intrinsic bispecta including polarisation

  have been calculated by SONG

  future experiments may be able to detect the intrinsic non-Gaussianity

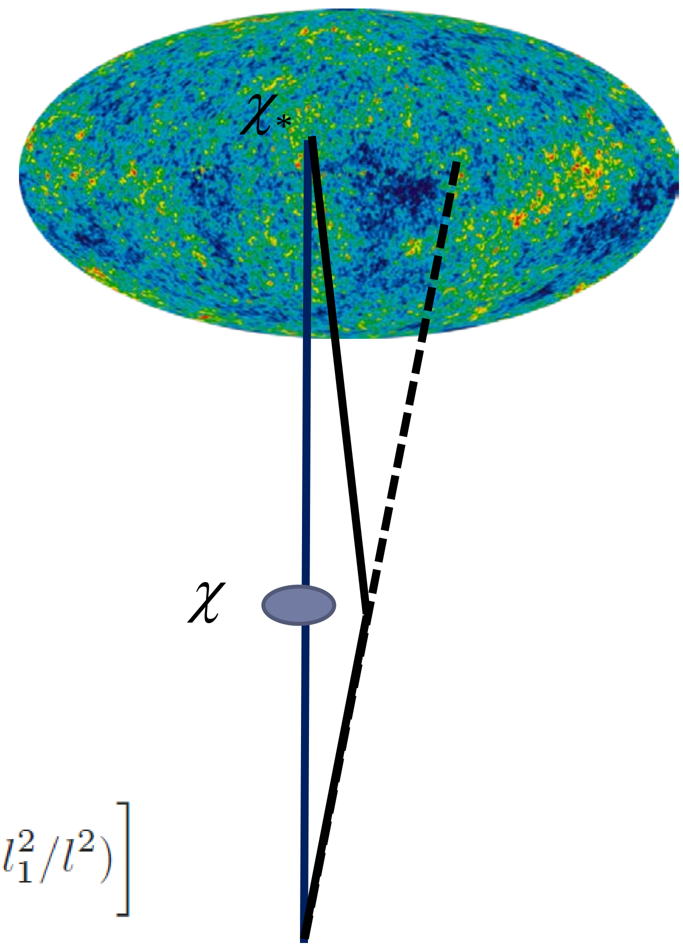
- … except late-time propagation effects (lensing and time-delay)

Remapping approach

- Lensing

$$\tilde{\Theta}(\hat{\mathbf{n}}) = \Theta(\hat{\mathbf{n}} + \nabla\phi)$$
$$= \Theta(\hat{\mathbf{n}}) + \nabla_i \phi(\hat{\mathbf{n}}) \nabla^i \Theta(\hat{\mathbf{n}})$$

$$\phi(\widehat{\mathbf{n}}) = -2 \int_0^{\chi_*} d\chi \left(\frac{\chi_* - \chi}{\chi \chi_*} \right) \Psi(\chi \widehat{\mathbf{n}}, \eta_0 - \chi)$$



- Bispectrum

$$b_{l_1 l_2 l_3} \approx -C^{T\phi}_{l_1} \left[(\mathbf{l}_1 \cdot \mathbf{l}_2) \tilde{C}^{TT}_{l_2} + (\mathbf{l}_1 \cdot \mathbf{l}_3) \tilde{C}^{TT}_{l_3} \right]$$

$$\approx l_1^2 C^{T\psi}_{l_1 \phi} \frac{1}{2} \left[\cos 2\phi_{l_1 l} \frac{\mathrm{d}\tilde{C}^{TT}_l}{\mathrm{d}\ln l} + \frac{1}{l^2} \frac{\mathrm{d}(l^2 \tilde{C}^{TT}_l)}{\mathrm{d}\ln l} + \mathcal{O}(l_1^2/l^2) \right]$$

in the squeezed limit

A new line-of-sight approach to the non-linear Cosmic Microwave Background

Christian Fidler,[1] Kazuya Koyama,[1] and Guido W. Pettinari[2]

[1] *Institute of Cosmology and Gravitation,*

*University of Portsmouth, Portsmouth PO1 3FX, UK**

[2] *Department of Physics & Astronomy,*

University of Sussex, Brighton BN1 9QH, UK

Geodesic *curve*-of-sight formulae for the cosmic microwave background: a unified treatment of redshift, time delay, and lensing

Ryo Saito[1], Atsushi Naruko[2], Takashi Hiramatsu[3], and Misao Sasaki[3]

[1] *APC (CNRS-Université Paris 7), 10 rue Alice Domon et Léonie Duquet, 75205 Paris, France,*

[2] *Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan*

[3] *Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

Unified treatment of propagation effects

- Boltzmann equation

$$\mathcal{D}f + \tau f = 0$$

$$\mathcal{D} = \frac{\partial}{\partial \eta} + n^i \frac{\partial}{\partial x^i}$$

$$\tau = \frac{dp}{d\eta}\frac{\partial}{\partial p} + \frac{dn^i}{d\eta}\frac{\partial}{\partial n^i} + \left(\frac{dx^i}{d\eta} - n^i\right)\frac{\partial}{\partial x^i}$$

redshifts lensing time-delay

- Transformation operator

$$f = \mathcal{J}\tilde{f}$$

$$\mathcal{J}D\tilde{f} + [\mathcal{D}, \mathcal{J}]\tilde{f} + \tau\mathcal{J}\tilde{f} = 0$$

transformed distribution function propagates in homegeneous spacetime

$$\mathcal{D}\tilde{f} = 0 \qquad [\mathcal{D}, \mathcal{J}] + \tau\mathcal{J} = 0$$

Transformation operator

- A differential basis

$$\tau = \tau_p \frac{\partial}{\partial p} + \tau_n^i \frac{\partial}{\partial n^i} + \tau_x^i \frac{\partial}{\partial x^i} = \tau_p \frac{\partial}{\partial p} + \tau_n^i \bar{D}_i + (\tau_x^i - \eta \tau_n^i) \frac{\partial}{\partial x^i}$$

$$\bar{D}_i = \tfrac{\partial}{\partial n^i} + \eta \tfrac{\partial}{\partial x^i}$$

$$\partial_a = \{\tfrac{\partial}{\partial p}, \tfrac{\partial}{\partial x^i}, \bar{D}_i\} \qquad [\partial_a , \mathcal{D}] = 0$$

$$[\mathcal{D}, \tau] = \mathcal{D}(\tau_a)\partial_a$$

- Transformation operator

expand the operator using the differential basis

$$\mathcal{J}_{a\ldots c}\partial_{a\ldots c} = \mathcal{J} + \mathcal{J}_a \partial_a + \mathcal{J}_{ab}\partial_{ab} + \ldots \ .$$

Equations for transformation operator

- Differential equations

$$\mathcal{D}(\mathcal{J}_{a\ldots c})\partial_{a\ldots c} = \tau_d \partial_d \Big(\mathcal{J}_{e\ldots g} \partial_{e\ldots g} \Big)$$

- Perturbation in the transport operator

first order

$$\Big[\mathcal{D}(\mathcal{J}^{(1)}_a)\Big]\partial_a = \tau_a \partial_a$$

second order

$$\Big[\mathcal{D}(\mathcal{J}^{(2)}_a)\Big]\partial_a = \Big[\tau_d(\partial_d J^{(1)}_a)\Big]\partial_a$$

$$\Big[\mathcal{D}(\mathcal{J}^{(2)}_{ab})\Big]\partial_a\partial_b = \Big[\tau_a J^{(1)}_b\Big]\partial_a\partial_b$$

Solutions

- Differential operator

$$\partial_d = \partial_d^{\rightarrow} + \partial_d^{\uparrow} \qquad \partial_d^{\rightarrow}(\mathcal{A}_{a\ldots c}\partial_{a\ldots c}) = \partial_d(\mathcal{A}_{a\ldots c})\partial_{a\ldots c}$$

$$\partial_d^{\uparrow}(\mathcal{A}_{a\ldots c}\partial_{a\ldots c}) = \mathcal{A}_{a\ldots c}\partial_{a\ldots cd}$$

$$\mathcal{D}(\mathcal{J}^{(n)}_{a\ldots c})\partial_{a\ldots c} = \tau_d(\partial_d^{\rightarrow} + \partial_d^{\uparrow})\mathcal{J}^{(n-1)}_{e\ldots g}\partial_{e\ldots g}$$

- Integration

$$\mathcal{J}^{(n)}_{a\ldots c}(\eta_0)\partial_{a\ldots c} = \int_{\eta_{ini}}^{\eta_0} d\eta_1 e^{-n^i\frac{\partial}{\partial x^i}(\eta_0-\eta_1)}\tau_d(\eta_1)(\partial_d^{\rightarrow}(\eta_1) + \partial_d^{\uparrow}(\eta_0))\mathcal{J}^{(n-1)}_{e\ldots g}(\eta_1)\partial_{e\ldots g}$$

$$\mathcal{J}^{(n)}(\eta_0) = \int_{\eta_{ini}}^{\eta_0} d\eta_1 e^{-n^i\frac{\partial}{\partial x^i}(\eta_0-\eta_1)}\tau_d(\eta_1)(\partial_d^{\rightarrow}(\eta_1) + \partial_d^{\uparrow}(\eta_0))\mathcal{J}^{(n-1)}(\eta_1)$$

"Lens-Lens coupling"

- Recursive solutions

$$\mathcal{J}^{(n)}(\eta_0) = \int_{\eta_{ini}}^{\eta_0} d\eta_1 e^{-n^i \frac{\partial}{\partial x^i}(\eta_0-\eta_1)} \tau_{a_1}(\eta_1)(\partial^{\rightarrow}_{a_1}(\eta_1) + \partial^{\uparrow}_{a_1}(\eta_0))$$

$$\int_{\eta_{ini}}^{\eta_1} d\eta_2 e^{-n^i \frac{\partial}{\partial x^i}(\eta_1-\eta_2)} \tau_{a_2}(\eta_2)(\partial^{\rightarrow}_{a_2}(\eta_2) + \partial^{\uparrow}_{a_2}(\eta_0))$$

$$\vdots$$

$$\int_{\eta_{ini}}^{\eta_{n-1}} d\eta_n e^{-n^i \frac{\partial}{\partial x^i}(\eta_{n-1}-\eta_n)} \tau_{a_n}(\eta_n)\partial_{a_n}(\eta_0) \ .$$

applied to lensing, this describes lens-lens coupling

Collision

- Collision term

$$D\tilde{f} = \mathcal{J}^{-1} C(\mathcal{J}\tilde{f}) \quad C(f) = -\dot{\kappa} f + \chi(f)$$

- Line of sight integration

$$f(\eta_0) = \mathcal{J}(\eta_0) \int_{\eta_{ini}}^{\eta_0} d\eta e^{-n^i \frac{\partial}{\partial x^i}(\eta_0 - \eta) - \kappa} \mathcal{J}^{-1}(\eta) \chi(f(\eta))$$

Encodes all spacetime effects before today from initial conditions

This part can be calculated non-linear Newtonian approximations for example

Cancels propagation effects prior to the source

This part can be calculated using the 2^{nd} order cosmological perturbation theory

This formalism achieves a split between early time (primary) and late time (secondary) effects

cf. Saito, Naruko, Hiramatsu, Sasaki 2014

Initial conditions

- Initial condition $\mathcal{J}(\eta_{ini}) = 1$ right after inflation

$$[D, \mathcal{J}] + \tau\mathcal{J} = -\dot{\kappa}(\mathcal{J} - 1)$$

$\mathcal{J}$ deviates from one only after recombination

$$f(\eta_0) = \mathcal{J}(\eta_0) \int_{\eta_{ini}}^{\eta_0} d\eta e^{-n^i \frac{\partial}{\partial x^i}(\eta_0-\eta)-\kappa} \mathcal{J}^{-1}(\eta)(\chi(f(\eta)) + \dot{\kappa}_{rec}(1 - \mathcal{J}^{-1})f(\eta))$$

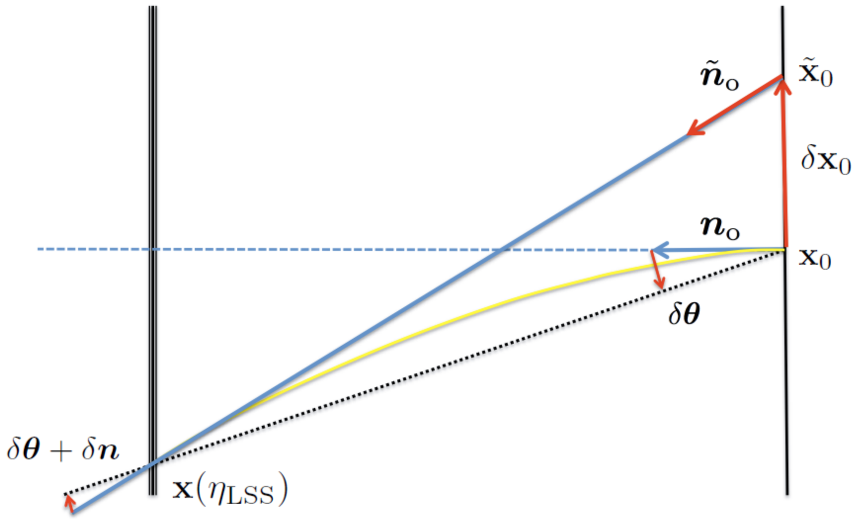
$$\approx \mathcal{J}(\eta_0) \int_{\eta_{ini}}^{\eta_0} d\eta e^{-n^i \frac{\partial}{\partial x^i}(\eta_0-\eta)-\kappa} \chi(f(\eta)) \ ,$$

at the first order in the perturbation theory

Alternative initial conditions

- Initial conditions for $\mathcal{J}$ can be taken freely

$$\mathcal{J}(\eta_0) = 1 \qquad f(\eta_0) = \cancel{\mathcal{J}(\eta_0)} \int_{\eta_{ini}}^{\eta_0} d\eta\, e^{-n^i \frac{\partial}{\partial x^i}(\eta_0-\eta)-\kappa} \mathcal{J}^{-1}(\eta)\chi(f(\eta))$$



This encodes all spacetime effects from today to the source

Saito, Naruko, Hiramatsu, Sasaki 2014

this formula agrees with the approach performing the line of sight integration along the full geodesic

Applications

- **Redshift term** $ds^2 = a(\eta)^2\Big(-(1+2A)d\eta^2 + (1+2D)\delta_{ij}dx^i dx^j\Big)$

$$\mathcal{J}_p(\eta_0, k_0) \;=\; \left(\int_0^{\eta_0} d\eta_1 e^{-in^i k_0(\eta_0-\eta_1)-\kappa(\eta_0,\eta_1)}\tau_p(\eta_1,k_0)\right)\frac{\partial}{\partial p} \qquad \tau_p \;=\; -p\frac{\partial}{\partial x^i}n^i A - p\dot{D},$$

$$\mathcal{J}_p^{(1)}(\eta_0) = 1 - \theta_{\rm ISW}\,p\frac{\partial}{\partial p} \qquad \theta_{\rm ISW}(\eta) = \int_0^{\eta} d\eta_1\Big(\dot{\kappa}A + (\dot{D}-\dot{A})\Big)e^{-in^i k_0(\eta-\eta_1)-\kappa(\eta,\eta_1)}$$

$$f(\eta_0) \;=\; \mathcal{J}_p(\eta_0)\int_{\eta_{ini}}^{\eta_0} d\eta e^{-n^i\frac{\partial}{\partial x^i}(\eta_0-\eta)-\kappa}\chi(f(\eta))$$

$$\;=\; \mathcal{J}_p(\eta_0)f_{coll}(\eta_0) = (1-\theta_{\rm ISW}\,p\frac{\partial}{\partial p})f_{coll} \approx f^{(1)}_{\rm coll} \;-\; \theta_{\rm ISW}\,p\,\frac{\partial}{\partial p}\,(f^{(0)} + f^{(1)}_{\rm coll})$$

At the second order $\tilde{\Delta} = \Delta - \Delta_{\rm ISW}\Delta$ is enough to remove the redshift term (this is better for polarisation)

Lensing

- Lensing $\quad \tau_n^i = \sigma^{ij}\frac{\partial}{\partial x^j}(D-A) \qquad \sigma^{ij} = \delta^{ij} - n^i n^j$

$$\mathcal{J}^{(1)}(\eta_0,k_0) = \left(\int_0^{\eta_0} d\eta_1 e^{-in^ik_0(\eta_0-\eta_1)-\kappa(\eta_0,\eta_1)}\tau_n^i(\eta_1,k_0)\right)\left(\frac{\partial}{\partial n^i}+\eta_0\frac{\partial}{\partial x^i}\right) - \left(\int_0^{\eta_0} d\eta_1 e^{-in^ik_0(\eta_0-\eta_1)-\kappa(\eta_0,\eta_1)}\eta_1\tau_n^i(\eta_1,k_0)\right)\frac{\partial}{\partial x^i}\,.$$

- Assuming all sources are located at LSS $\quad \frac{\partial}{\partial x^i} = \frac{1}{(\eta_{rec}-\eta_0)}\frac{\partial}{\partial n^i}$

$$\mathcal{J}^{(1)}(\eta_0,k_0) = \frac{\partial}{\partial n_i^\perp}\left(\int_0^{\eta_0} d\eta_1 \frac{\eta_{rec}-\eta_1}{(\eta_0-\eta_{rec})(\eta_0-\eta_1)} e^{-in^ik_0(\eta_0-\eta_1)}(D-A)\right)\frac{\partial}{\partial n_\perp^i}$$

Remapping formula

- Remapping approach

$$f(\eta_0) \;=\; \mathcal{J}(\eta_0)\int_{\eta_{ini}}^{\eta_0} d\eta e^{-n^i\frac{\partial}{\partial x^i}(\eta_0-\eta)-\kappa}\chi(f(\eta))$$

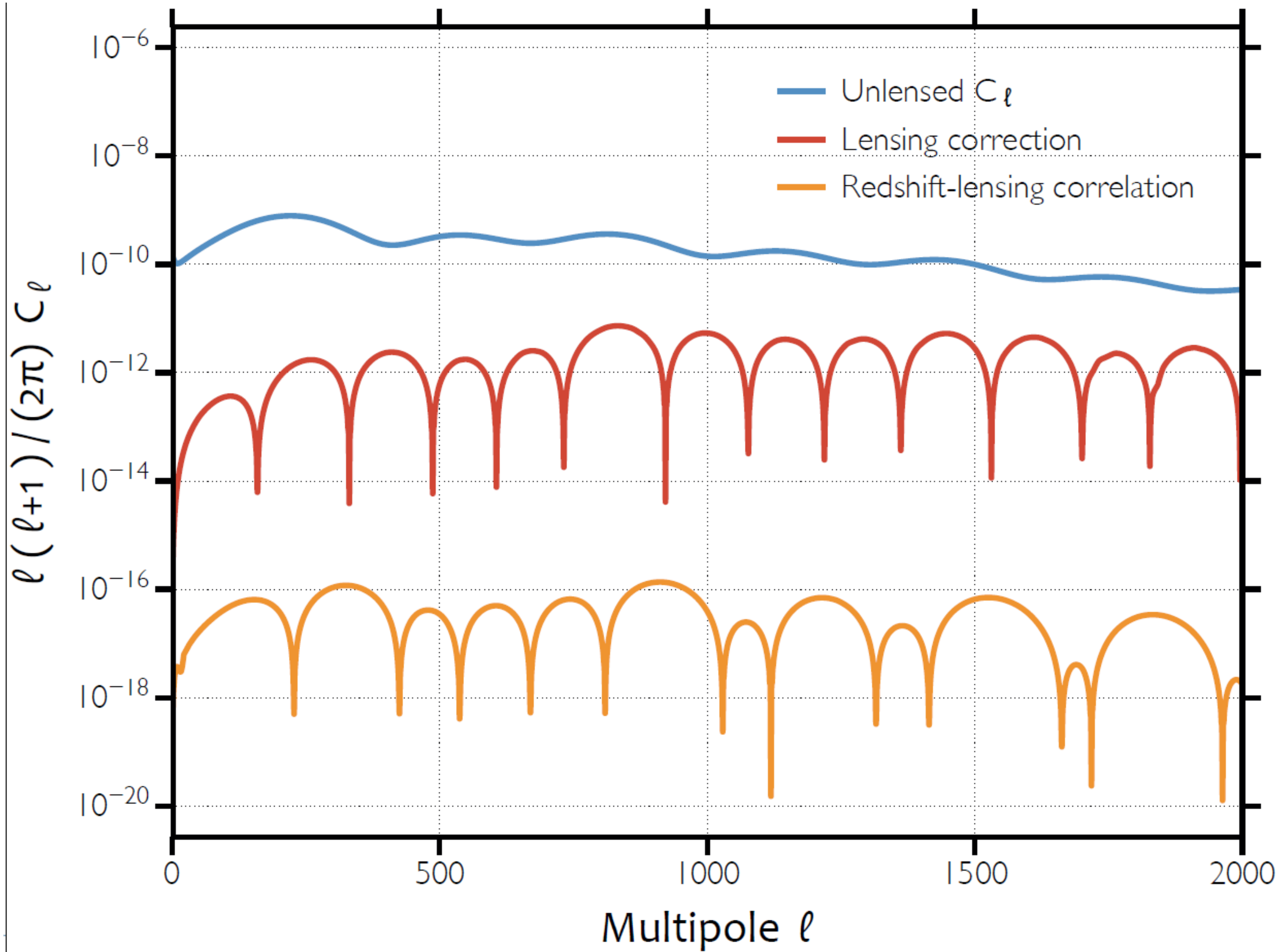
$$= \mathcal{J}(\eta_0)f_{coll}(\eta_0) = f_{coll} + \left(\frac{\partial}{\partial n_i^{\perp}}\psi(n^i)\right)\left(\frac{\partial}{\partial n_{\perp}^i}f_{coll}\right)$$

$$\psi(n^i) = \int \frac{d^3k}{(2\pi)^3}\int_0^{\eta_0} d\eta_1 \frac{\eta_{rec}-\eta_1}{(\eta_0-\eta_{rec})(\eta_0-\eta_1)}e^{-in^ik(\eta_0-\eta_1)}\Big(D(\eta_1,k)-A(\eta_1,k)\Big)$$

- Final formula

$$f(\eta_0) \;=\; f^{(0)} + f_{coll}^{(1)} - \theta_{\rm ISW}\,p\frac{\partial}{\partial p}(f^{(0)}+f_{coll}^{(1)}) + \left(\frac{\partial}{\partial n_i^{\perp}}\psi^{(1)}(n^i)\right)\left(\frac{\partial}{\partial n_{\perp}^i}f_{coll}^{(1)}\right)$$

Redshift-lensing correction



Conclusion

- Our universe is non-Gaussian, so are the CMB anisotropies
- We have now a tool to calculate the intrinsic bispectrum including all physical effects

  (2^{nd} order Boltzmann code + new line of sight approach)