Geodesic "*curve*"-of-sight formulae for the cosmic microwave background

A unified treatment of redshift, time delay and lensing

Atsushi NARUKO (APC -> TITECH)

with **R. Saito (APC)**, T. Hiramatsu, M. Sasaki (YITP)

based on arXiv : 1409.2464 today !!

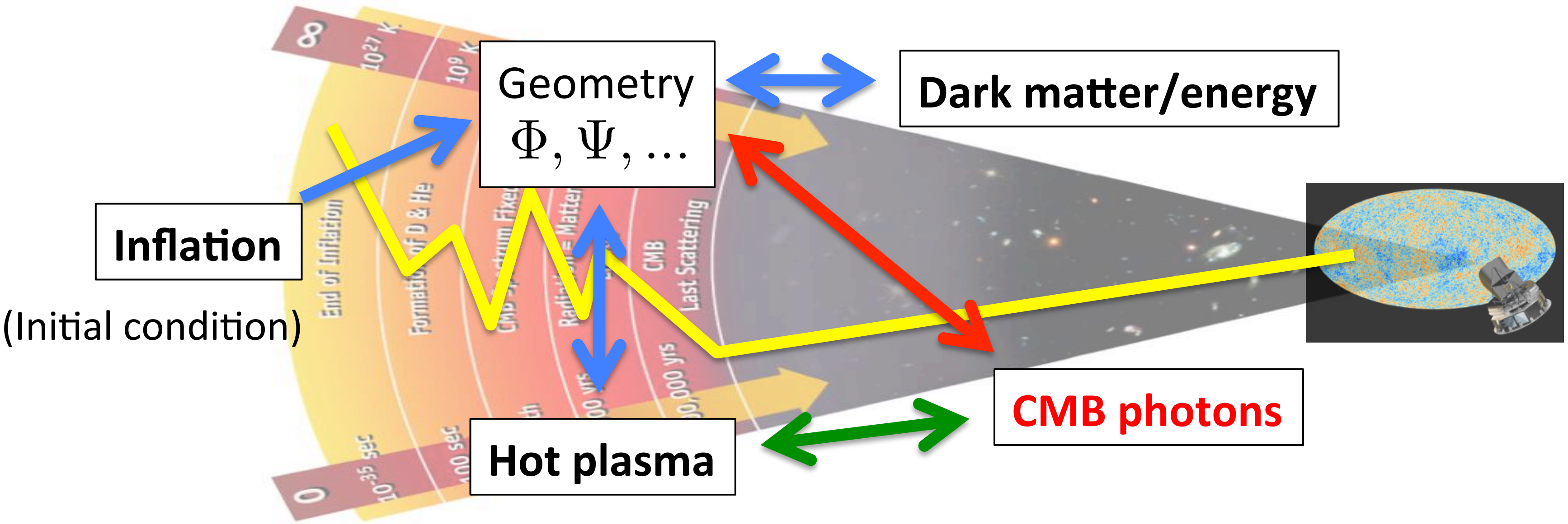
an alternative approach → 1409.2461 by C. Fidler, K. Koyama, G. W. Pettinari

(see also 1304.6929 by AN, C.Pitrou, KK, MS)

Cosmic Microwave Background (CMB)

The CMB can probe the history of the Universe:

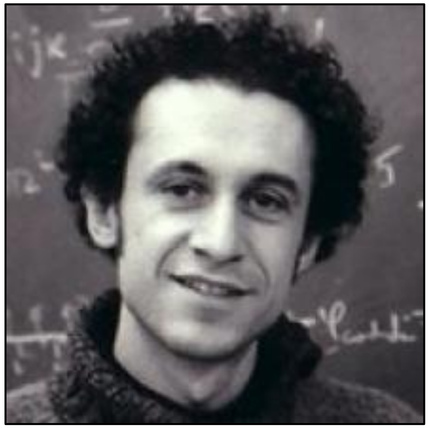
Inflation, thermal history of the early Universe, dark components, …



Recent precise measurements (WMAP, PLANCK …) → **Accurate tool** (beyond linear order)
(Kinetic theory, Cosmological Perturbation Theory)

Second-order calculation of the CMB

…., [Pitrou], [Huang & Vernizzi], [Portsmouth group], [Cambridge group],



UNIVERSITY OF CAMBRIDGE

ICG Portsmouth

YITP
YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Second-order calculation of the CMB

…., [Pitrou], [Huang & Vernizzi], [Portsmouth group], [Cambridge group], **+α !?**

Difficulty in solving the Boltzmann eq. (at 1st order)

To know the intensity of CMB photons at the present time …

→ solve a 7D differential equation, ***Boltzmann equation***,

$$\frac{\mathrm{d}}{\mathrm{d}\eta} f = \left(\frac{\partial}{\partial \eta} + \frac{\mathrm{d}x^i}{\mathrm{d}\eta} \frac{\partial}{\partial x^i} + \frac{\mathrm{d}q}{\mathrm{d}\eta} \frac{\partial}{\partial q} + \frac{\mathrm{d}n^{(i)}}{\mathrm{d}\eta} \frac{\partial}{\partial n^{(i)}} \right) f(\eta\,, \mathbf{x}\,, q\,, \mathbf{n}) = \mathfrak{C}[f]$$

time

position [3]

(comoving) momentum (energy)

direction [2 ← 3]

$$P_\mu P^\mu = 0$$

- <u>q (energy) – dependence</u>

  -- well parameterized by a single parameter: *temperature (or brightness, Δ)*

- <u>x – dependence (or |k|-dependence in Fourier space)</u>

  -- *Fourier mode* evolves independently at linear order

- <u>n - dependence (or ell-dependence after multi-pole expansion)</u>

  -- *Multi-pole moments* couple with each other → most problematic !!

huge number of coupled differential eqs... 😱

$$\left(\frac{\partial}{\partial\eta}+\frac{\mathrm{d}x^i}{\mathrm{d}\eta}\frac{\partial}{\partial x^i}+\frac{\mathrm{d}q}{\mathrm{d}\eta}\frac{\partial}{\partial q}+\frac{\mathrm{d}n^i}{\mathrm{d}\eta}\frac{\partial}{\partial n^i}\right)f(\eta,\mathbf{x},q,\mathbf{n})=\mathfrak{C}[f;\eta,\mathbf{x},q,\mathbf{n}]$$

Brightness equation: $\frac{\mathrm{d}x^i}{\mathrm{d}\eta}=n^{(i)}+\mathcal{O}(\varepsilon)\,,\quad \frac{\mathrm{d}(\ln q)}{\mathrm{d}\eta}=-\dot{\Phi}-\Psi_{,i}\,n^{(i)}\,,\quad \frac{\mathrm{d}n^{(i)}}{\mathrm{d}\eta}=\mathcal{O}(\varepsilon)\,,$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|\Delta=(\partial_\eta+n^{(i)}\partial_i)\Delta=\underline{\mathfrak{C}^\Delta}+\underline{\mathfrak{D}^\Delta}$$

BG

derivative along geodesic

collision

SW and ISW

$$=\dot{\tau}C^\Delta$$

optical depth
$\dot{\tau}\equiv-n_e\sigma_T a$

→ In Fourier space, after the multi-pole expansion :

$$\partial_\eta\Delta_{lm}+k\sum_{l'm'}\mathcal{M}_{l\,l'}^{m\,m'}\Delta_{l'm'}=\mathfrak{C}^\Delta_{lm}+\mathfrak{D}^\Delta_{lm}\,,$$

... To know Δ_l, you need to know Δ_{l+1} and Δ_{l-1}. To know Δ_{l+1}, you need to know Δ_{l+2} ...

need to solve coupled differential equations for a huge number of (η, **x**)-functions Δ_{lm}

Line-of-sight integration method (1st order)

[Seljak & Zalddariaga 96]

✓ Line-of-sight integration method
(integrating along a photon geodesic ≠ time-const line)
which can reduce the computational cost very much !!!

✓ The brightness eq. in an integral form *along a photon geodesic*,

$\equiv g_v(\eta')$: visibility function

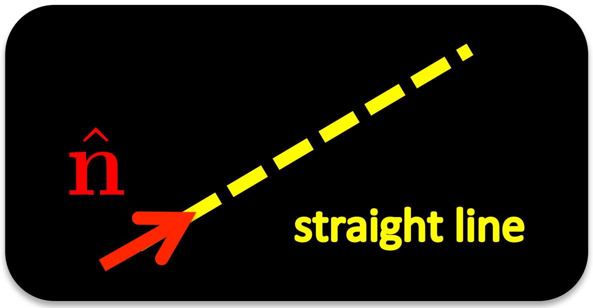
$$\Delta(\eta_0, k^i, n_0^{(i)}) = \int_0^{\eta_0} \mathrm{d}\eta' e^{i\mathbf{k}\cdot\mathbf{n}_0(\eta'-\eta_0)} \left[-\dot{\tau} e^{-\tau} (\Delta - C^{\Delta}) + e^{-\tau} \mathfrak{D}^{\Delta} \right] ,$$

$$\text{at } (\eta' , \bar{x}^i(\eta') , \bar{n}^{(i)}(\eta'))$$

☝ fluctuations are evaluated at a point on a BG. geodesic (straight line),

$$\bar{x}^i(\eta) \equiv x_0^i + n_0^{(i)}(\eta - \eta_0)$$

$$\bar{n}^{(i)}(\eta) \equiv n_0^{(i)}$$



line-of-sight formula

$$\Delta(\eta_0, k^i, n_0^{(i)}) = \int_0^{\eta_0} d\eta' e^{i\mathbf{k}\cdot\mathbf{n}_0(\eta'-\eta_0)} \left[-\dot{\tau}e^{-\tau}(\Delta - C^\Delta) + e^{-\tau}\mathfrak{D}^\Delta\right],$$

- **Thomson collision term**

$$\Delta - C^\Delta = -4\Delta_{00} + 16i[v_e]\mu - \frac{2}{5}(\Delta_{20} - \sqrt{6}E_{20})P_2(\mu)$$

- **SW/ISW term**

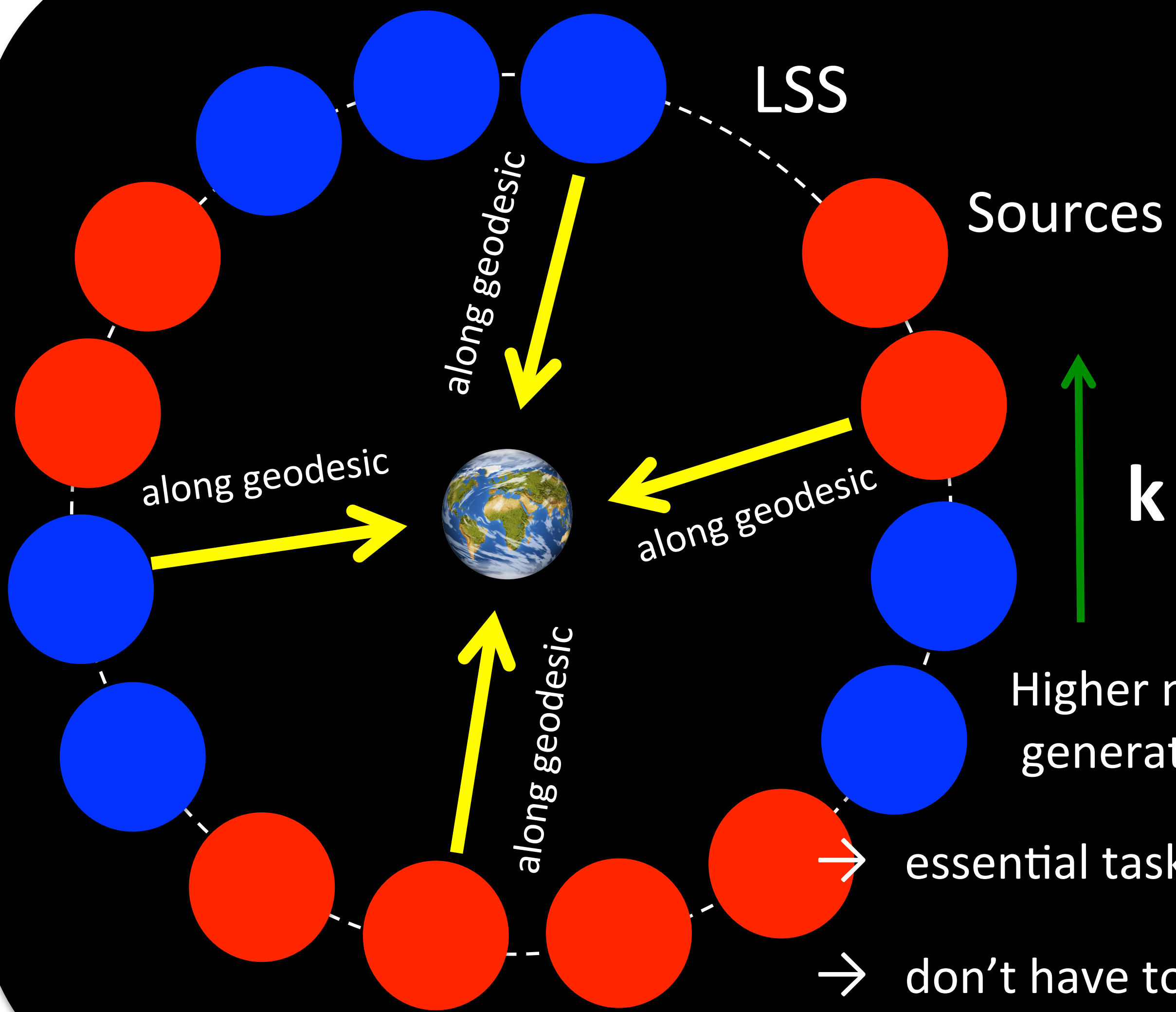
$$\mathfrak{D}^\Delta = -4(\dot{\Phi} + i\mu\Psi); \qquad \mu \equiv \mathbf{k}\cdot\mathbf{n}_0$$

✓ Infinite # of Δ_{lm} are determined only by 6 functions: Φ, Ψ, Δ_{00}, Δ_{20}, E_{20}, v_e.

✓ Information of the observed higher-order multi-pole moments are encoded in the known function, **$\exp[ikn_0(\eta'-\eta_0)]$** .
(the spherical Bessel function $j_l[kn_0(\eta'-\eta_0)]$ after the multi-pole expansion.)

It is ***not*** necessary to solve coupled differential equations for a huge number of functions Δ_{lm}.

Intuitive picture



Sources [monopole]

Higher multi-pole moments are generated during propagation.

→ essential task is to solve the geodesic eq.

→ don't have to solve the Boltzmann eq. !!

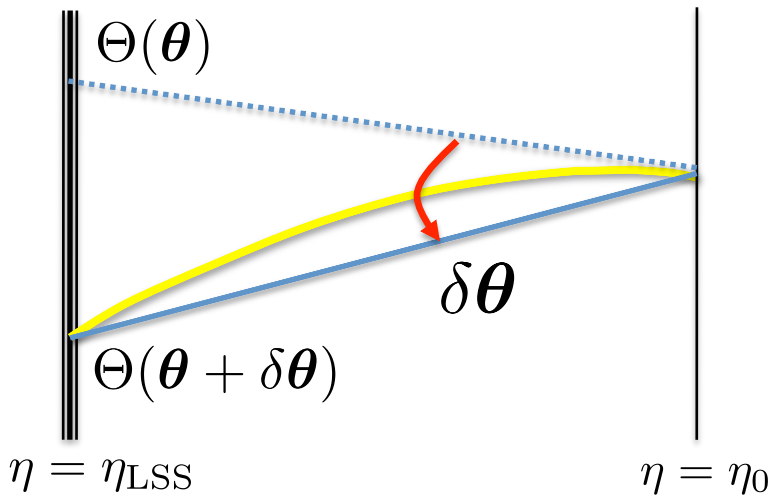
Extension to higher orders

✓ At non-linear order,
the CMB photons propagate through the *perturbed* spacetime.

→ **various (non-linear) effects will be expected (time delay, lensing, redshift ...)**

$$\frac{\mathrm{d}f}{\mathrm{d}\eta} = \partial_\eta f + \left(n^{(i)} + \frac{\mathrm{d}\,\delta x^i}{\mathrm{d}\eta}\right)\partial_i f + \frac{\mathrm{d}q}{\mathrm{d}\eta}^{(1)+(2)} \partial_q f + \frac{\mathrm{d}n^{(i)}}{\mathrm{d}\eta}\partial_{n^{(i)}} f$$

Standard approach → Remapping : $\Theta_{\rm lensed}(\boldsymbol{\theta}) = \Theta_{\rm unlensed}(\boldsymbol{\theta} + \delta\boldsymbol{\theta})$



$$\frac{\mathrm{d}n^{(i)}}{\mathrm{d}\eta} = -S^{(i)(j)}(\Psi - \Phi)_{,j} + \cdots$$

extend the LOS formula including
various non-linear gravitational effects
such as redshift, time delay, lensing ???

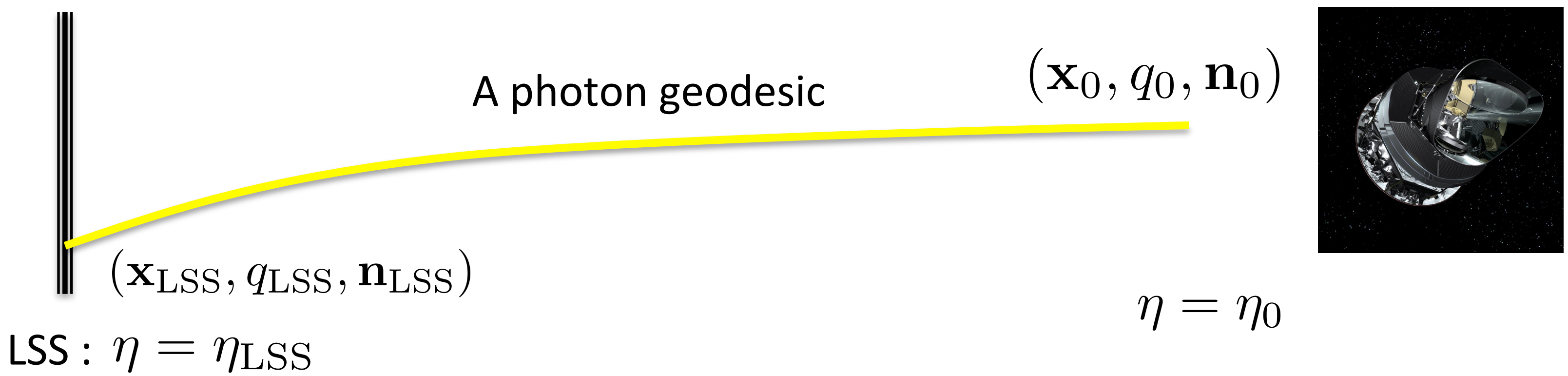
Geodesic “*curve*”-of-sight formulae

-- A unified treatment of redshift, time delay and lensing --

Liouville's theorem

Liouville's theorem states that
the distribution function along geodesics is conserved,
even when the geodesics are curved by the metric perturbations :

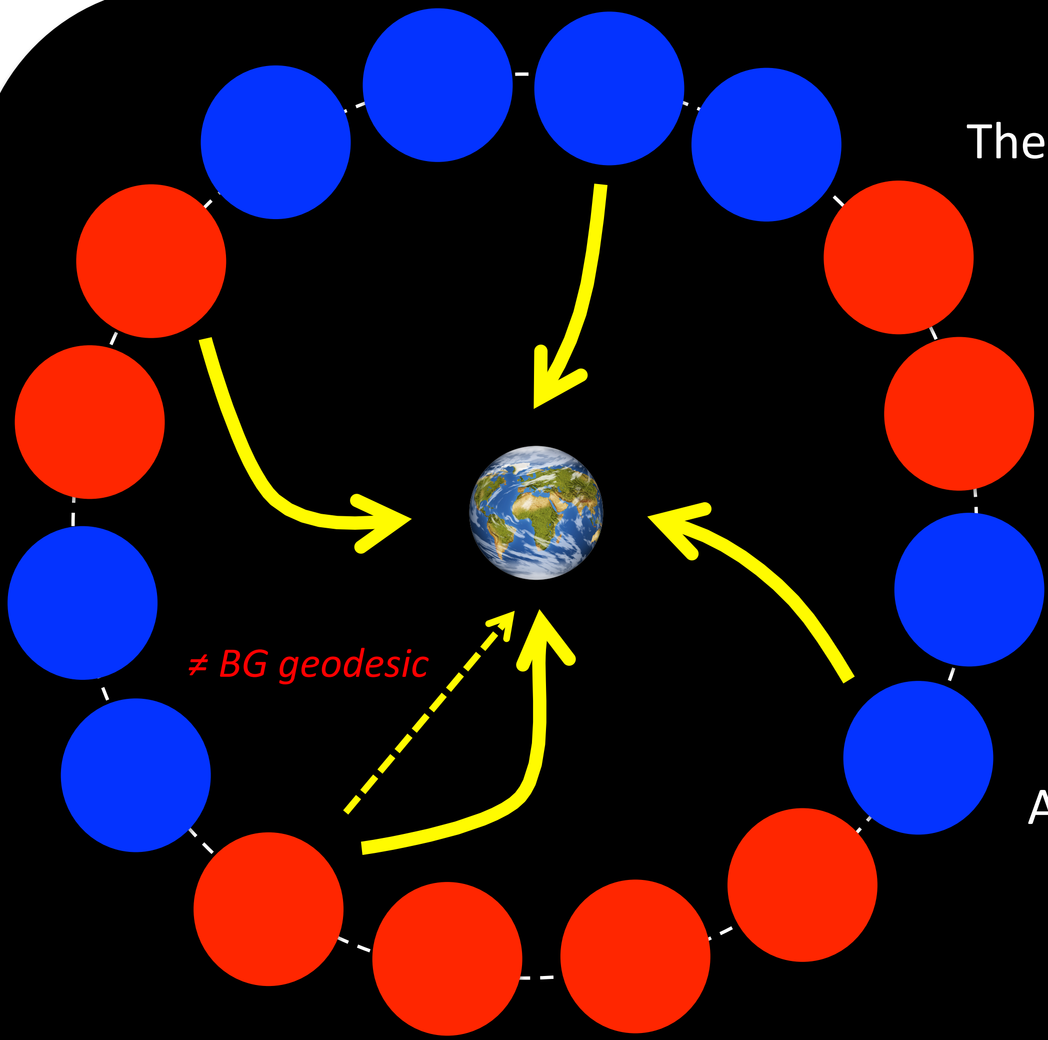
$$f(\eta_0, \mathbf{x}_0, q, \mathbf{n}_0) = f(\eta_{\rm LSS}, \mathbf{x}_{\rm LSS}, q_{\rm LSS}, \mathbf{n}_{\rm LSS})$$



✓ The observed fluctuations = the projection of fluctuations on the LSS !!!

→ the projection should be done along the *perturbed geodesics*.
(curved photon trajectories)

Intuitive picture (with the gravitational effects)



The projection should be done along

perturbed (≠ BG) geodesics

A geodesic is governed by gravitational perturbations.

⬇

Additionally, information on the evolution of Φ and Ψ is necessary.

Again, it is ***not*** necessary to solve coupled differential equations for a huge number of functions.

New approach to the gravitational effects

✓ We rewrite the Boltzmann equation
in an integral form *along a perturbed geodesic*,

$$f(\eta_0, \mathbf{x}_0, q_0, \mathbf{n}_0) = \int_0^{\eta_0} \mathrm{d}\eta' \underbrace{g_v(\eta')[f - C]}_{\equiv \mathfrak{S}}{}_{\mathrm{at}\ (\eta', \mathbf{x}(\eta'), q(\eta'), \mathbf{n}(\eta'))}$$

a point on a perturbed geodesic

✓ The gravitational effects appear as deviations in the mapping
between the phase-space coordinates of the observer and sources.

→ ***Generalization of the remapping approach to CMB lensing***

(No approximation and it includes all gravitational effects)

(Cf. Huang & Vernizzi 13, Su & Lim 14, Fidler, Koyama, & Pettinari 14)

Mapping formula

The gravitational effects can be extracted as, after expanding $\mathcal{Q} = \mathcal{Q}_0 + \delta\mathcal{Q}$,

$$\delta f(\eta_0, \mathbf{x}_0, q_0, \mathbf{n}_0) = \int_0^{\eta_0} \mathrm{d}\eta' \left[\mathfrak{S}(\eta', \underline{\mathbf{x}(\eta'), q(\eta'), \mathbf{n}(\eta')}) - \mathfrak{S}(\eta', \underline{\bar{\mathbf{x}}(\eta'), \bar{q}(\eta'), \bar{\mathbf{n}}(\eta')})\right]$$

Perturbed Background

$$= \int_0^{\eta_0} \mathrm{d}\eta' \left[\frac{\partial\mathfrak{S}}{\partial q}\,\delta q(\eta') + \frac{\partial\mathfrak{S}}{\partial x^i}\,\delta x^i(\eta') + \frac{\partial\mathfrak{S}}{\partial n^i}\,\delta n^i(\eta') + \frac{1}{2}\frac{\partial^2\mathfrak{S}}{\partial q^2}\,\delta q(\eta')^2 + \cdots\right]$$

$(\delta\mathbf{x}, \delta q, \delta\mathbf{n})$ are written in terms of the line-of-sight integral of the gravitational potentials.

The distribution function can be estimated from the source function $\mathfrak{S}$ and the gravitational potentials [still a finite number of functions].

(This property is satisfied at all orders)

Deviations in geodesics

✓ The deviations in a geodesic $(\delta\mathbf{x}, \delta q, \delta\mathbf{n})$ can be evaluated by solving the geodesic eqs. with $(\delta\mathbf{x}, \delta q, \delta\mathbf{n}) = (\mathbf{0}, 0, \mathbf{0})$ at $\eta = \eta_0$

- **Displacement & Deflection**

$$\delta x^i(\eta') = -\int_{\eta'}^{\eta_0} \mathrm{d}\eta_1 \left[D^i(\eta_1, \bar{x}^i, \bar{n}^{(i)}) - (\eta_1 - \eta')D^{n^{(i)}}(\eta_1, \bar{x}^i, \bar{n}^{(i)})\right]$$

$$\delta n^{(i)}(\eta') = -\int_{\eta'}^{\eta_0} \mathrm{d}\eta_1 D^{n^{(i)}}(\eta_1, \bar{x}^i, \bar{n}^{(i)})$$

$$(\eta, \mathbf{x}, \mathbf{n}) \equiv (\Psi - \Phi)n^i,$$

$$D^{n^{(i)}}(\eta, \mathbf{x}, \mathbf{n}) \equiv -\nabla_{\perp}^i(\Psi - \Phi)$$

- **Redshift**

$$\delta \ln q(\ldots \mathbf{x}) - \delta x(\eta')\partial_i D^{\mathrm{SW}}(\eta', \bar{\mathbf{x}}) +$$

$$-\int_{\eta'}^{\eta_0} \mathrm{d}\eta_1 \left[D^{\mathrm{ISW}}(\eta_1, \bar{\mathbf{x}}) + \delta x^i(\eta_1)\partial_i D^{\mathrm{ISW}}\right]$$

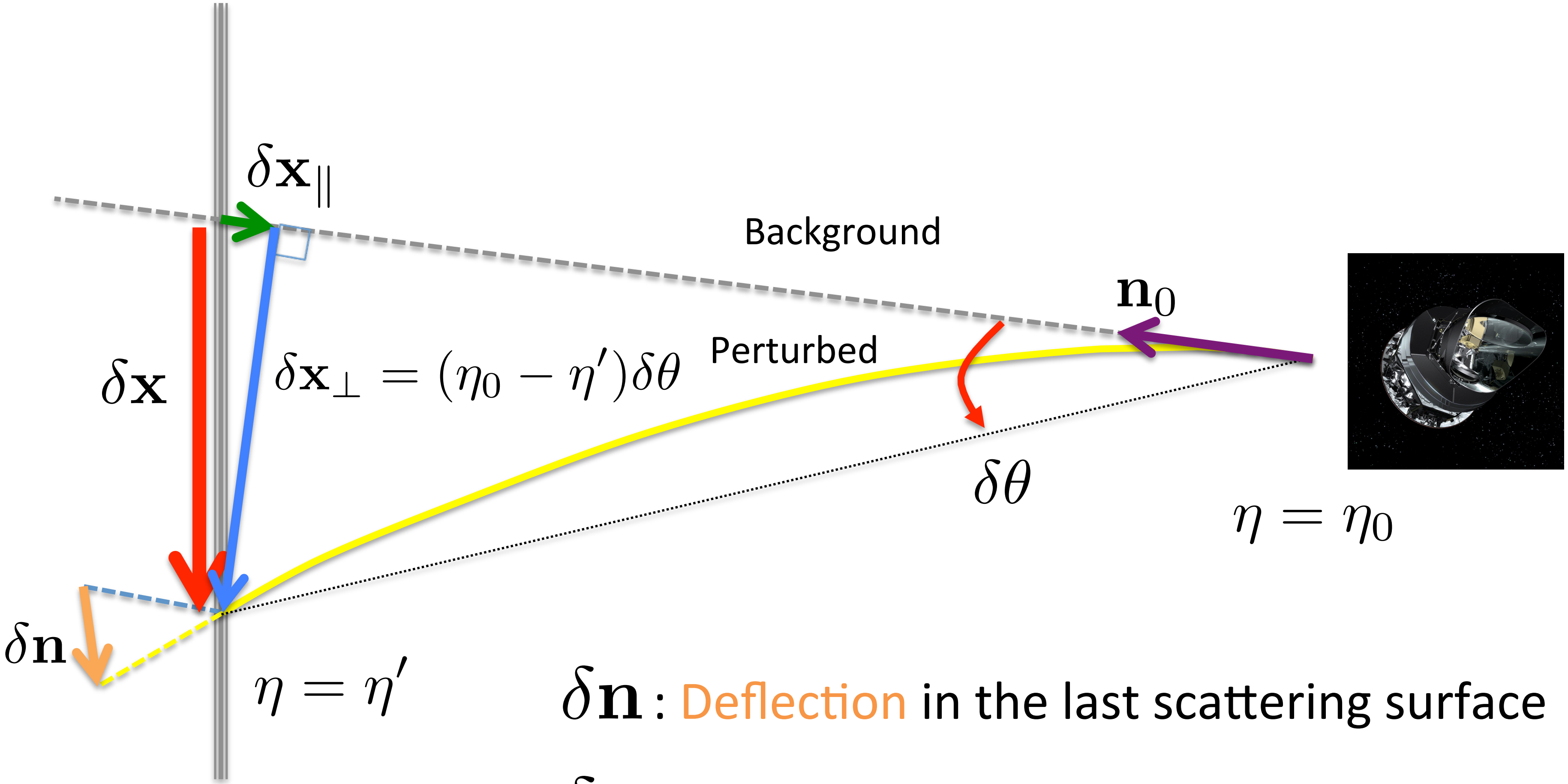
where

$$D^{\mathrm{SW}}(\eta, \mathbf{x}) \equiv \Psi,$$ *Sachs-Wolfe effect*

$$D^{\mathrm{ISW}}(\eta, \mathbf{x}) \equiv (\Psi - \Phi)^{\cdot}$$ *Integrated Sachs-Wolfe effect (Rees-Sciama effect)*

They are written in terms of the gravitational potentials.

Geometrical meanings



$\delta\mathbf{n}$: Deflection in the last scattering surface

$\delta\mathbf{x}$: Lensing + Time delay

$$f = \int_0^{\eta_0} d\eta' \left[\underbrace{\mathfrak{S}}_{\text{I}} + \underbrace{\frac{\partial \mathfrak{S}}{\partial q} \delta q(\eta')}_{\text{II}} + \underbrace{\frac{\partial \mathfrak{S}}{\partial x^i} \delta x^i(\eta')}_{\text{III}} + \underbrace{\frac{\partial \mathfrak{S}}{\partial n^i} \delta n^i(\eta')}_{\text{IV}} + \underbrace{\frac{1}{2}\frac{\partial^2 \mathfrak{S}}{\partial q^2} \delta q(\eta')^2}_{\text{V}} + \cdots \right]$$

Each term represents the contributions from…

I : Intrinsic (non-gravitational)

II : Redshift + Source × Redshift + Redshift × Displacements

e.g. Rees-Sciama effect   Sachs-Wolfe effect + Integrated Sachs-Wolfe effect

III : Source × Displacements

= Lensing + Time delay

IV : Source × Deflections

V : Redshift × Redshift

All possible second-order effects are included.

Line-of-sight formula at second order

✓ Let us pick up a term, the source × lensing term, for example.

$$\delta f \supset \int_0^{\eta_0} \mathrm{d}\eta' \frac{\partial \mathfrak{S}}{\partial x^i}\, \delta x^i_\perp(\eta') = \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3} \int_0^{\eta_0} \mathrm{d}\eta'\, k_1^i\, \mathfrak{S}^{(1\mathrm{st})}(\eta', \mathbf{k}_1)\, T_i(\eta', \mathbf{k}_1)$$

where

$$T_i(\eta, \mathbf{k}) \equiv \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \int_\eta^{\eta_0} \mathrm{d}\eta' k'_{\perp i}(\eta' - \eta)(\Psi - \Phi)\, e^{i[\mathbf{k}(\eta_0 - \eta) + \mathbf{k}'(\eta_0 - \eta')]\cdot \mathbf{n}_O}$$

✓ The effect of the propagation (geometrical effect) is separated as the term T_i.

✓ The geometrical term is no longer a known function (≠ Bessel function), but it is written in terms of *the linear gravitational potentials (Φ and Ψ)*, which depends on $\mathbf{n}_0$ (or multi-pole) through a known function.

It is ***not*** necessary to solve coupled differential eqs for an infinite # of functions.

Computation of bi-spectrum (technical)

The computation of the bispectrum is simplified.

Angle-averaged bispectrum (Lensing × 1st order × 1st order):

$$B_{l_1 l_2 l_3}[\Delta] = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 1 & -1 & 0 \end{pmatrix} b_{l_1 l_2 l_3}[\Delta] + 2 \text{ sym.}$$

with

$$b_{l_1 l_2 l_3} = 2[1 + (-1)^{l_1+l_2+l_3}] \int_0^{\eta_0} \mathrm{d}\eta' b^S_{l_1}(\eta') b^T_{l_2}(\eta')$$

where

$$b^S_{l_1}(\eta') = \frac{2}{\pi}\sqrt{\frac{l_1(l_1+1)}{2}} \int k_1^2 \mathrm{d}k_1 P_\phi(k_1) \mathcal{T}^{(I)}_{l_1}(k_1) \frac{S\left(\eta', k_1, \frac{i}{k_1}\frac{\mathrm{d}}{\mathrm{d}\eta'}\right)}{k_1(\eta_0 - \eta')} j_{l_1}[k_1(\eta_0 - \eta')]$$

Power spectrum ($P_\phi(k_1)$) First-order transfer function ($\mathcal{T}^{(I)}_{l_1}(k_1)$)

$$b^T_{l_2}(\eta') = \frac{2}{\pi}\sqrt{\frac{l_2(l_2+1)}{2}} \int k_2^2 \mathrm{d}k_2 P_\phi(k_2) \mathcal{T}^{(I)}_{l_2}(k_2) \int_{\eta'}^{\eta_0} \mathrm{d}\eta_1 \frac{\eta_1 - \eta'}{\eta_0 - \eta_1} (\Psi^{(I)}_{k_2} - \Phi^{(I)}_{k_2}) j_{l_2}[k_2(\eta_0 - \eta_1)]$$

$$\propto (\eta_0 - \eta')\psi_{l_2}(\eta', k_2)$$

Lensing potential

Its multi-pole dependence is simple.

Summary +

- ✓ We have derived **Geodesic *curve*-of-sight formulae**, which determines an *infinite* number of multi-pole moments of distribution function from a *finite* number of functions.
- ✓ These formulae enable us to treat all non-linear gravitational effects on the same footing (redshift, time delay, lensing and deflection) making clear the geometrical meaning of each contribution.
- ✓ They will reduce the computational cost of non-linear CMB calculations.
- ✓ Extensions to the spectral distorsion & polarization.
- ✓ The current status of our numerical code -> Hiramatsu-san's talk.