Hamiltonian structure of scalar-tensor theories beyond Horndeski

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Based on arXiv:1408.0670 with Chunshan Lin, Ryo Namba and Rio Saitou
Why alternative gravity theories?

- Dark Energy
- Dark Matter
- Inflation
- Big Bang
  - "Singularity"
- Afterglow Light Pattern 400,000 yrs.
- Development of Galaxies, Planets, etc.
- 1st Stars about 400 million yrs.
- Big Bang Expansion 13.7 billion years

http://map.gsfc.nasa.gov/
• Experimentally, we do not know gravity at short and large scales.

• Cosmology & Compact objects (e.g. BH) as Blackbody radiation & Hydrogen atom
A motivation for IR modification

- Gravity at long distances
  - Flattening galaxy rotation curves
  - Extra gravity
  - Dimming supernovae
  - Accelerating universe

- Usual explanation: new forms of matter (DARK MATTER) and energy (DARK ENERGY).
Dark component in the solar system?

Precession of perihelion observed in 1800’s…

which people tried to explain with a “dark planet”, Vulcan,

But the right answer wasn’t “dark planet”, it was “change gravity” from Newton to GR.
Zoo of gravity theories

- Horava-Lifshitz
  - projectable/non-projectable
  - with/without U(1)
- Einstein-Aether
- TeVeS
- Ghost condensate
- Higgs phase of gravity
- Supertring
  - I, IIA, IIB, hetero O(32), hetero E8xE8
- Sugra
- KK
- Braneworld
- GR
- Horndeski
- Galileon
- Scalar-tensor
- GLPV
- Multi-metric
- Nonlinear massive gravity
- f(R)
- f(T)
- f(G)
- DGP
Checkpoints

- What are the physical d.o.f.?
- How they interact?
- What is the regime of validity?
Checkpoints

- What are the physical d.o.f.?
- How they interact?
- What is the regime of validity?

If two or more theories give the same answers to the there questions above then they are the same even if they look different.
EFT of inflation/DE

- Time diffeo is broken by the background but spatial diffeo is preserved
- All terms that respect spatial diffeo must be included in the EFT action
- Derivative & perturbative expansions
- Diffeo can be recovered by introducing NG boson
Simplest: ghost condensation
Arkani-Hamed, Cheng, Luty and Mukohyama, JHEP 0405:074, 2004

Backgrounds characterized by

\[ \langle \partial_{\mu} \phi \rangle \neq 0 \text{ and timelike} \]

\[ \diamond \text{Background metric is maximally symmetric, either Minkowski or dS.} \]
Gauge choice: $\phi(t, \bar{x}) = t$. $\pi \equiv \delta \phi = 0$
(Unitary gauge)

Residual symmetry: $\bar{x} \rightarrow \bar{x}'(t, \bar{x})$

Write down most general action invariant under this residual symmetry.
( Action for $\pi$: undo unitary gauge!)

Start with flat background $g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}$

$$\delta h_{\mu \nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Under residual $\xi^i$

$$\delta h_{00} = 0, \delta h_{0i} = \partial_0 \xi_i, \delta h_{ij} = \partial_i \xi_j + \partial_j \xi_i$$
Action invariant under $\xi^i$

$$\begin{cases}
(h_{00})^2 & \text{OK} \\
(h_{0i})^2 & \text{OK} \\
K^2, K^{ij} K_{ij} & \text{OK}
\end{cases}$$

$$K_{ij} = \frac{1}{2} \left( \partial_0 h_{ij} - \partial_j h_{0i} - \partial_i h_{0j} \right)$$

$$L_{\text{eff}} = L_{EH} + M^4 \left\{ (h_{00})^2 - \frac{\alpha_1}{M^2} K^2 - \frac{\alpha_2}{M^2} K^{ij} K_{ij} + \cdots \right\}$$

Action for $\pi$

$$\xi^0 = \pi \quad \begin{cases} 
  h_{00} \rightarrow h_{00} - 2\partial_0 \pi \\
  K_{ij} \rightarrow K_{ij} + \partial_i \partial_j \pi
\end{cases}$$

$$L_{\text{eff}} = L_{EH} + M^4 \left\{ \left( h_{00} - 2\tilde{\pi} \right)^2 - \frac{\alpha_1}{M^2} \left( K + \tilde{\nabla}^2 \pi \right)^2 \\
  - \frac{\alpha_2}{M^2} \left( K^{ij} + \tilde{\nabla}^i \tilde{\nabla}^j \pi \right) \left( K_{ij} + \tilde{\nabla}_i \tilde{\nabla}_j \pi \right) + \cdots \right\}$$

Beginning at quadratic order, since we are assuming flat space is good background.
$E \to rE$

$dt \to r^{-1} dt$

$dx \to r^{-1/2} dx$

$\pi \to r^{1/4} \pi$

\[
\int dtd^3x \left[ \frac{1}{2} \dot{\pi}^2 - \frac{\alpha (\nabla^2 \pi)^2}{\bar{M}^2} + \cdots \right]
\]

Make invariant

Leading nonlinear operator in infrared

has scaling dimension $1/4$. \textbf{(Barely) irrelevant}

$\implies$ Good low-$E$ effective theory.
• The structure of low-E EFT is determined by the symmetry breaking pattern!
• Would end up with the same EFT, independently from starting Lagrangian.
• Can make robust predictions!
Extension to FLRW background

Creminelli, Luty, Nicolis, Senatore 2006
Cheung, Creminelli, Fitzpatrick, Kaplan, Senatore 2007

• Action invariant under $x^i \rightarrow x^i(t,x)$

• Ingredients
  
g_{\mu \nu}, g^{\mu \nu}, R_{\mu \nu \rho \sigma} \nabla_\mu
  
• 1\textsuperscript{st} derivative of $t$
  
\[
\partial_\mu t = \delta_\mu^0 \quad n_\mu = \frac{\partial_\mu t}{\sqrt{-g_{\mu \nu} \partial_\mu t \partial_\nu t}} = \frac{\delta_\mu^0}{\sqrt{-g^{00}}}
\]

\[
g^{00}
\]

• 2\textsuperscript{nd} derivative of $t$
  
\[
K_{\mu \nu} \equiv h^\rho_\mu \nabla_\rho n_\nu
\]
Unitary gauge action

\[ I = \int d^4x \sqrt{-g} L(t, \delta^0_\mu, K_{\mu\nu}, g_{\mu\nu}, g^{\mu\nu}, \nabla_\mu, R_{\mu\nu\rho\sigma}) \]

**derivative & perturbative expansions**

\[ I = M_{Pl}^2 \int dx^4 \sqrt{-g} \left[ \frac{1}{2} R + c_1(t) + c_2(t) g^{00} \right.
\]
\[ + L^{(2)}(\tilde{\delta}g^{00}, \tilde{\delta}K_{\mu\nu}, \tilde{\delta}R_{\mu\nu\rho\sigma}; t, g_{\mu\nu}, g^{\mu\nu}, \nabla_\mu) \]
\[ L^{(2)} = \lambda_1(t)(\tilde{\delta}g^{00})^2 + \lambda_2(t)(\tilde{\delta}g^{00})^3 + \lambda_3(t)\tilde{\delta}g^{00}\tilde{\delta}K^\mu_\mu \]
\[ + \lambda_4(t)(\tilde{\delta}K^\mu_\mu)^2 + \lambda_5(t)\tilde{\delta}K^\mu_\nu\tilde{\delta}K^\nu_\mu + \cdots \]
NG boson

- Undo unitary gauge \( t \rightarrow \tilde{t} = t - \pi(\tilde{t}, \bar{x}) \)

\[
H(t) \rightarrow H(t + \pi), \quad \dot{H}(t) \rightarrow \dot{H}(t + \pi),
\]

\[
\lambda_i(t) \rightarrow \lambda_i(t + \pi), \quad a(t) \rightarrow a(t + \pi),
\]

\[
\delta_\mu^0 \rightarrow (1 + \dot{\pi})\delta_\mu^0 + \delta_\mu^i \partial_i \pi,
\]

- NG boson in decoupling (subhorizon) limit

\[
I_\pi = M_{Pl}^2 \int dt d^3 \bar{x} a^3 \left\{-\frac{\dot{H}}{c_s^2} \left(\dot{\pi}^2 - c_s^2 \frac{(\partial_i \pi)^2}{a^2}\right)\right\}
\]

\[
-\dot{H} \left(\frac{1}{c_s^2} - 1\right) \left(\frac{c_3}{c_s^2} \dot{\pi}^3 - \dot{\pi} \frac{(\partial_i \pi)^2}{a^2}\right) + O(\pi^4, \bar{\epsilon}^2) + L^{(2)}_{\delta K, \delta R}
\]

\[
\frac{1}{c_s^2} = 1 - \frac{4\lambda_1}{\dot{H}}, \quad c_3 = c_s^2 - \frac{8c_s^2\lambda_2}{-\dot{H}} \left(\frac{1}{c_s^2} - 1\right)^{-1}
\]
GLPV theory

\[ ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \]

\[ S = \int d^3 x dt N \sqrt{h} \sum_{n=2}^{5} \sum L_n \]

\[ L_2 = A_2(t, N) \quad L_3 = A_3(t, N) K \]

\[ L_4 = A_4(t, N) K_2 + B_4(t, N) R \]

\[ L_5 = A_5(t, N) K_3 + B_5(t, N) K^{ij} G_{ij} \]

\[ K = K^i_i \quad K_2 = K^2 - K^i_j K^j_i \]

\[ K_3 = K^3 - 3K K^{ij} K_{ij} + 2K^i_j K^j_k K^k_i \]
Questions

• Covariant version of GLPV theory is more general than Horndeski theory
• Horndeski theory is the most general scalar tensor theory in 4 dim with $2^{nd}$ order EOMs
• Theories with higher derivative EOMs usually suffer from Ostrogradski’s ghost
• Does GLPV include more than 3 dof?

Hamiltonian analysis for the case with $A_5 = 0$
Lin, Mukohyama, Namba, Saitou 2014
Primary constraints

- \( N_i \) and \( N \) are non-dynamical

\[
\begin{align*}
\pi_i & = 0 \\
\pi_N & = 0
\end{align*}
\]

- No more primary constraints

\[
\pi^{ij} = \frac{\sqrt{h}}{2} \left[ A_3 h^{ij} + 2 A_4 (h^{ij} K - K^{ij}) + B_5 G^{ij} \right]
\]

\[
K_{ij} = -\frac{1}{A_4} \left[ \frac{1}{\sqrt{h}} \left( \pi_{ij} - \frac{1}{2} h_{ij} \pi \right) + \frac{A_3}{4} h_{ij} - \frac{B_5}{2} \left( R_{ij} - \frac{1}{4} R h_{ij} \right) \right]
\]
Secondary constraints

• Hamiltonian

\[ \mathcal{H} = -N\sqrt{\hbar} \left[ \frac{1}{A_4} \left( \frac{\pi^i_j \pi^j_i}{h} - \frac{\pi^2}{2h} \right) + \frac{A_3 \pi}{2\sqrt{\hbar} A_4} - \frac{3A_3^2}{8A_4} + A_2 + B_4 R \right. \\
\left. - \frac{B_5}{A_4 \sqrt{\hbar}} \left( \pi^{ij} R_{ij} - \frac{1}{4} \pi R \right) + \frac{A_3 B_5}{8A_4} R + \frac{B_5^2}{4A_4} \left( R^{ij} R_{ij} - \frac{3}{8} R^2 \right) \right] \]

• Secondary constraints

\[ \{\pi_i(x), H\}_P = -\mathcal{H}_i - \int d^3y \left[ \frac{\delta \lambda^j(y)}{\delta N^i(x)} \pi_j(y) + \frac{\delta \lambda_N(y)}{\delta N^i(x)} \pi_N(y) \right] \approx -\mathcal{H}_i \]

\[ \{\pi_N(x), H\}_P = -\frac{\partial \mathcal{H}}{\partial N} - \int d^3y \left[ \frac{\delta \lambda^j(y)}{\delta N(x)} \pi_j(y) + \frac{\delta \lambda_N(y)}{\delta N(x)} \pi_N(y) \right] \approx -\frac{\partial \mathcal{H}}{\partial N} \]

\[ \mathcal{H}_i \equiv -2\sqrt{\hbar}D_j \left( \frac{\pi^j_i}{\sqrt{\hbar}} \right) \approx 0 \]

\[ C \equiv -\frac{\partial \mathcal{H}}{\partial N} \approx 0 \]
$\mathcal{H}_i \approx 0$ is not first class

• Usual algebra

$$\{\mathcal{H}[f], \mathcal{H}[g]\}_P \approx \mathcal{H}[\{f, g\}] \approx 0 \quad \text{for } \forall f^i, \forall g^i$$

$$\mathcal{H}[f] \equiv \int d^3x f^i(x) \mathcal{H}_i(x) \quad \quad [f, g]^i \equiv f^j \partial_j g^i - g^j \partial_j f^i$$

• However,

$$\{\mathcal{H}[f], C[\phi]\}_P \approx - \int d^3x \partial^2 \mathcal{H} \partial N^2 \phi f^i \partial_i N$$

does not vanish weakly…

• This is actually expected, since the scalar dof should contribute to $T_{0i}$.
$\mathcal{H}^{\text{tot}}_i = \mathcal{H}_i + \pi_N \partial_i N$ is first class

\begin{align*}
\{\mathcal{H}^{\text{tot}}_\text{tot}[f], \pi_N[\varphi]\}_P & \approx \pi_N[f \partial \varphi] \approx 0 , \\
\{\mathcal{H}^{\text{tot}}_\text{tot}[f], \mathcal{C}[\varphi]\}_P & \approx \mathcal{C}[f \partial \varphi] \approx 0 , \\
\{\mathcal{H}^{\text{tot}}_\text{tot}[f], \mathcal{H}^{\text{tot}}_\text{tot}[g]\}_P & \approx \mathcal{H}^{\text{tot}}_\text{tot}[[f, g]] \approx 0 , \\
\{\pi_i(x), \mathcal{H}^{\text{tot}}_j(y)\}_P & = 0
\end{align*}

$\mathcal{H}^{\text{tot}}_\text{tot}[f] \equiv \int d^3x f^i(x) \mathcal{H}^{\text{tot}}_i(x)$ $\pi_N[\varphi] \equiv \int d^3x \varphi(x) \pi_N(x)$

• $\pi_i \approx 0$ is also first class

\begin{align*}
\{\pi_i(x), \pi_j(y)\}_P & = 0 , \\
\{\pi_i(x), \pi_N(y)\}_P & = 0 , \\
\{\pi_i(x), \mathcal{C}(y)\}_P & = 0
\end{align*}

• No further secondary constraints

$\{\mathcal{H}^{\text{tot}}_\text{tot}[f], H\}_P \approx \mathcal{H}[[f, N]] + \pi_N[f \partial \lambda_N] \approx 0$
$\pi_N \approx 0 \ & \mathcal{C} \approx 0$ are second class

- The determinant

$$\det \left( \begin{array}{cc}
\{\pi_N(x), \pi_N(y)\}_P & \{\pi_N(x), \mathcal{C}(y)\}_P \\
\{\mathcal{C}(x), \pi_N(y)\}_P & \{\mathcal{C}(x), \mathcal{C}(y)\}_P
\end{array} \right)$$

does not vanish since

$$\{\pi_N(x), \pi_N(y)\}_P = 0$$
$$\{\mathcal{C}(x), \pi_N(y)\}_P = -\frac{\partial^2 \mathcal{H}}{\partial N^2} \delta^3(x-y)$$

- The consistency conditions

$$\{\pi_N(x), H_{\text{tot}}\}_P \approx 0$$
$$\frac{\partial}{\partial t} \mathcal{C}(x) + \{\mathcal{C}(x), H_{\text{tot}}\}_P \approx 0$$

just determines $\lambda_N$ & $\lambda_C$.

$$H_{\text{tot}} = \int d^3x \left[ \mathcal{H} + N^i \mathcal{H}_i + n^i \mathcal{H}_{i\text{tot}}^\text{tot} + \lambda^i \pi_i + \lambda_N \pi_N + \lambda_C \mathcal{C} \right]$$
Number of degrees of freedom

- 20-dim phase space \((N, N^i, h_{ij}, \pi_N, \pi_i, \pi^{ij})\)
- 6 first class constraints
  each of them reduce phase space dim by 2
- 2 second class constraints
  each of them reduce phase space dim by 1
- Physical phase space dim: \(20 - 6 \times 2 - 2 = 6\)
- This corresponds to 3 degrees of freedom
- 2 in tensor sector + 1 in scalar sector
Gauge fixing

• Gauge fixing function for $\pi_i \approx 0 \& \mathcal{H}_i^{\text{tot}} \approx 0$

\[ G^i(x) \approx 0 \quad F^i(x) \approx 0 \]

\[ \det \left( \begin{array}{cc} \{ \pi_N(x), \pi_N(y) \} & \{ \pi_N(x), C(y) \} \\ \{ C(x), \pi_N(y) \} & \{ C(x), C(y) \} \end{array} \right) \neq 0 \]

• We end up with 14 second class constraints in 20-dim phase space $\rightarrow 6$-dim $\rightarrow 3$ dof

• Total Hamiltonian

\[ H'_{\text{tot}} = \int d^3x \left[ \mathcal{H} + N^i \mathcal{H}_i + n^i \mathcal{H}_i^{\text{tot}} + \lambda^i \pi_i + \lambda^G_i G^i + \lambda^F_i F^i + \lambda_N \pi_N + \lambda_C C \right] \]
Simple gauge fixing

- Gauge fixing function for \( \pi_i \approx 0 \) & \( \mathcal{H}_i^{\text{tot}} \approx 0 \)
  \[
  G^i = N^i \quad F^i = F^i(N, h_{ij}, \pi_N, \pi^{kl}; t)
  \]
  \[
  \det\left(\{\mathcal{H}_i^{\text{tot}}(x), F^j(y)\}\right)_P \neq 0
  \]
- \( (N^i, \pi_i) \) fully eliminated
  \[
  \{G^i(x), H'_\text{tot}\}_P \approx 0
  \]
  \[
  \frac{\partial}{\partial t} F^i(x) + \{F^i(x), H'_\text{tot}\}_P \approx 0
  \]
  \[
  \lambda^i = 0
  \]
  \[
  \lambda^G_i = -\mathcal{H}_i
  \]
- Instead of \( N^i \), \( n^i \) plays the role of the shift
  \[
  H'_\text{tot} = \int d^3x \left[ \mathcal{H} + n^i \mathcal{H}_i^{\text{tot}} + \lambda^F_i F^i + \lambda_N \pi_N + \lambda_C C \right]
  \]
Conclusion

• We performed the Hamiltonian analysis for GLPV theory with $A_5=0$ and confirmed that the number of degrees of freedom is 3.
• How general is this conclusion for theories in which the absence of higher time derivatives is guaranteed only in the unitary gauge?
• How to prove the same statement without fixing the gauge? (in progress with Ryo&Rio)
• Applications? (inflation, dark energy, Vainshtein, emergent time, etc.)
preliminary results on
New matter coupling in
massive gravity
&
new quasidilaton theory

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Effective metric

• Recent proposal of an effective metric in massive gravity \( \text{de Rham, Heisenberg, Ribeiro 2014} \)
  \[
g_{\mu\nu}^{\text{eff}} = \alpha^2 g_{\mu\nu} + 2\alpha\beta g_{\mu\rho} \left( \sqrt{g^{-1} f} \right)^\rho_\nu + \beta^2 f_{\mu\nu}
\]

• Claim: BD ghost shows up only above the cutoff scale of the theory and thus can (and should) be integrated out, i.e. we don’t worry about it

• BD ghost does not show up in linear perturbations around FLRW background. \( \text{Gumrukcuoglu, Heisenberg, Mukohyama arXiv:1409???} \)
Quasidilaton  D’Amico, Gabadadze, Hui, Pirtskhalava, 2012

• New nonlinear instability [DeFelice, Gumrukcuoglu, Mukohyama 2012] → (i) new backgrounds, or (ii) extended theories

• Quasidilaton: scalar $\sigma$ with global symmetry:

$$\sigma \rightarrow \sigma + \sigma_0 \quad \phi^a \rightarrow e^{-\sigma_0/M_{P1}} \phi^a$$

• Action

$$S = \frac{M_{P1}^2}{2} \int d^4x \sqrt{-g} \left[ R - 2\Lambda - \frac{\omega}{M_{P1}^2} \partial_{\mu}\sigma \partial^{\mu}\sigma \right.$$

$$\left. + 2m_g^2 (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4) \right]$$

$$\mathcal{K}^{\mu}_{\nu} = \delta^{\mu}_{\nu} - e^\sigma/M_{P1} \left( \frac{1}{\sqrt{g^{-1} f}} \right)^\mu_{\nu} f_{\mu\nu} = \eta_{ab} \partial_{\mu} \phi^a \partial_{\nu} \phi^b$$

• Scaling solution = self-accelerating de Sitter ($H = \text{const} > 0$ with $\Lambda = 0$)
Extension of quasidilaton


- Self-accelerating solution in the original quasidilaton theory has ghost instability [Gumrukcuoglu, Hinterbichler, Lin, Mukohyama, Trodden 2013; D’Amico, Gabadadze, Hui, Pirtskhalava 2013]
- Simple extension: \[ f_{\mu\nu} \rightarrow \tilde{f}_{\mu\nu} \]
  \[ \tilde{f}_{\mu\nu} \equiv f_{\mu\nu} - \frac{\alpha_\sigma}{M_{Pl}^2 m_g^2} e^{-2\sigma/M_{Pl}} \partial_\mu \sigma \partial_\nu \sigma \]
- Self-accelerating solution is stable within 5 d.o.f. if
  \[ 0 < \omega < 6 \]
  \[ X^2 < \frac{\alpha_\sigma H^2}{m_g^2} < r^2 X^2 \]
  \[ M_{GW}^2 \equiv \frac{(r - 1) X^3 m_g^2}{X - 1} + \frac{\omega H^2 (r X + r - 2)}{(X - 1)(r - 1)} > 0 \]
  \[ X \equiv \frac{e^{\sigma/M_{Pl}}}{a} \]
  \[ r \equiv \frac{n}{N} a \]
New quasidilaton theory?

Mukohyama, arXiv: 14?????

\[ I_{\text{newQD}}[g_{\mu\nu}, f_{\mu\nu}, \sigma] = M_{\text{Pl}}^2 m_g^2 \int d^4 x \sqrt{-g} \left[ \mathcal{L}_2(\bar{\kappa}) + \alpha_3 \mathcal{L}_3(\bar{\kappa}) + \alpha_4 \mathcal{L}_4(\bar{\kappa}) \right] \]

\[ - \frac{\omega}{2} \int d^4 x \sqrt{-g_{\text{eff}}} \ g_{\text{eff}}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \]

\[ g_{\text{eff}}^{\mu\nu} = g_{\mu\nu} + 2\beta e^{\sigma/M_{\text{Pl}}} g_{\mu\rho} \left( \sqrt{g^{-1} f} \right)^\rho_\nu + \beta^2 e^{2\sigma/M_{\text{Pl}}} f_{\mu\nu} \]

- Quasidilaton kinetic term is now defined on the effective metric \( \rightarrow \) new parameter \( \beta \)

- Self-accelerating de Sitter solution is stable in a range of parameters with \( \alpha_\sigma = 0 \) if \( \beta \) is non-zero
No conclusion yet...

- New matter coupling opens up new possibilities
- How heavy is the would-be BD ghost?
- UV sensitivity of quasidilaton theory? Can it be ameliorated by the new coupling?
- ...
- many questions