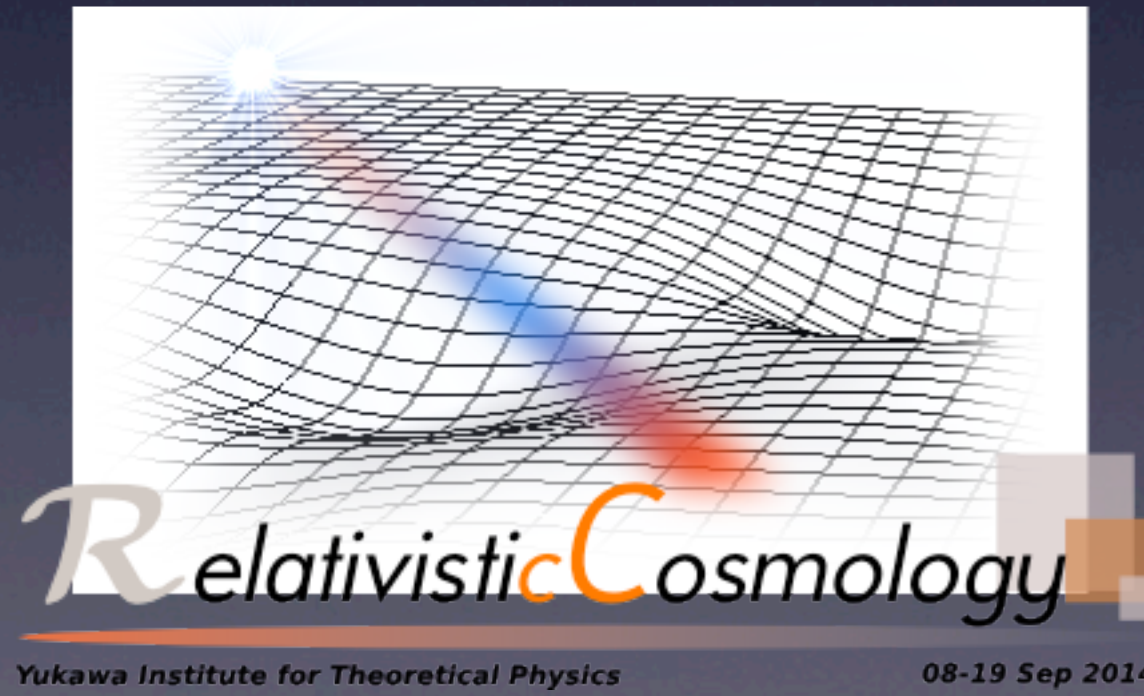
The covariant fluid-flow approach and non-linearity in cosmology

(+ a post-Friedmann framework)

Marco Bruni,
Institute of Cosmology and Gravitation
University of Portsmouth, UK

ICG Portsmouth



Credits

- work with Irene Milillo (Rome, ICG), Daniele Bertacca (ICG, Cape Town) and Andrea Maselli (Rome), in progress
- MB, D. B. Thomas and D. Wands, Physical Review D,89, 044010 (2014) [arXiv:1306.1562]
- D. B. Thomas, MB and D. Wands, in progress
- D. B. Thomas, M. Bruni and D. Wands, Relativistic weak lensing from a fully non-linear cosmological density field, [arXiv:1403.4947]
- MB, J. C. Hidalgo, N. Meures, D. Wands, Astrophysical Journal 785:2 (2014) [arXiv:1307:1478]
- MB, J. C. Hidalgo and D. Wands, Einstein's signature in cosmological large scale structure, ApJ Letters, submitted [arXiv:1405:7006]

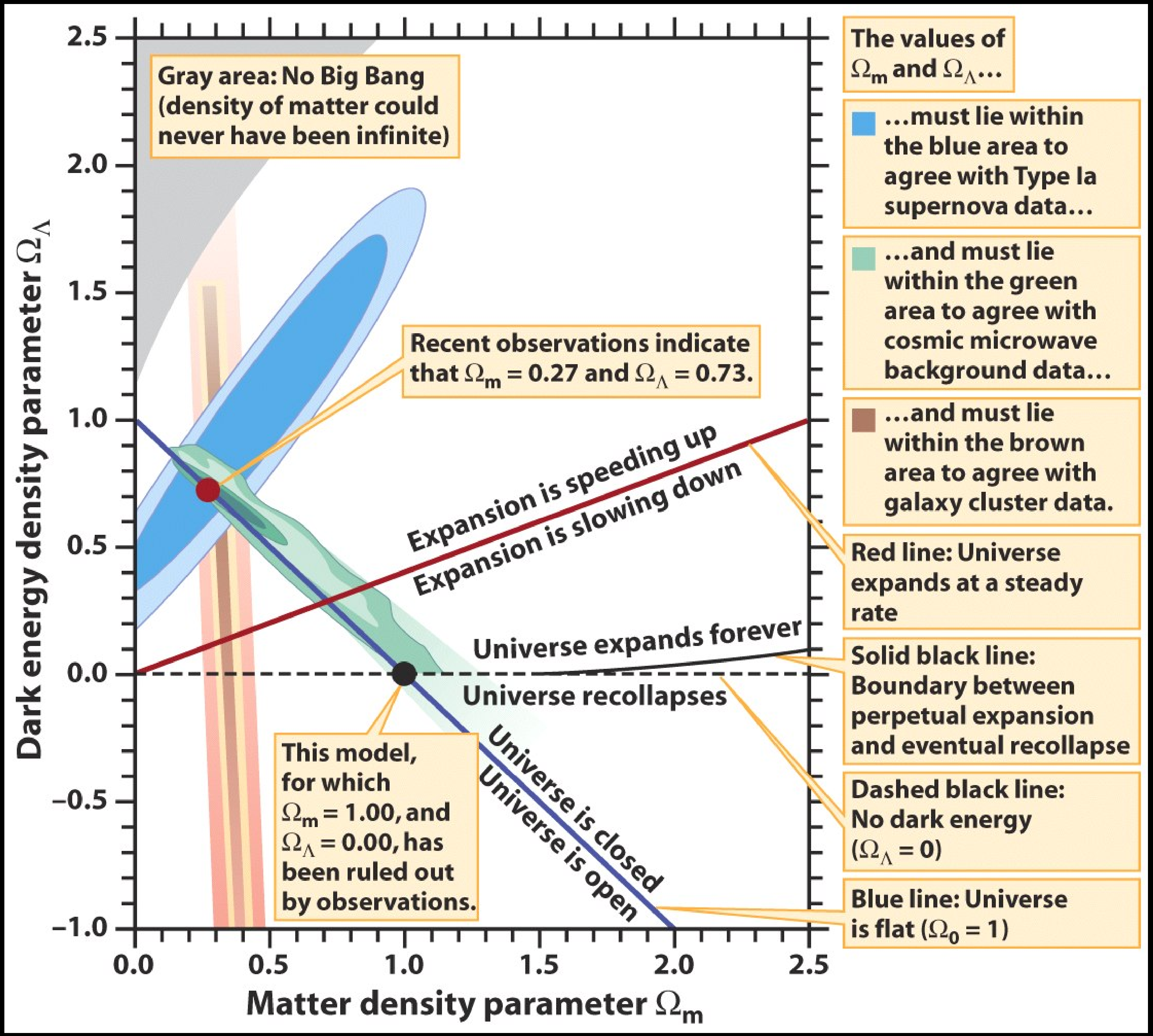
Outline

- the 3 ingredients of standard cosmological studies, the standard ΛCDM model, and some problems
- Relativistic Cosmology, non-linearity, back-reaction
- Newtonian cosmology vs Relativistic covariant fluid-flow approach
- non-linear Post-Friedmann ΛCDM: a new post-Newtonian type approximation scheme for cosmology

Standard ΛCDM Cosmology

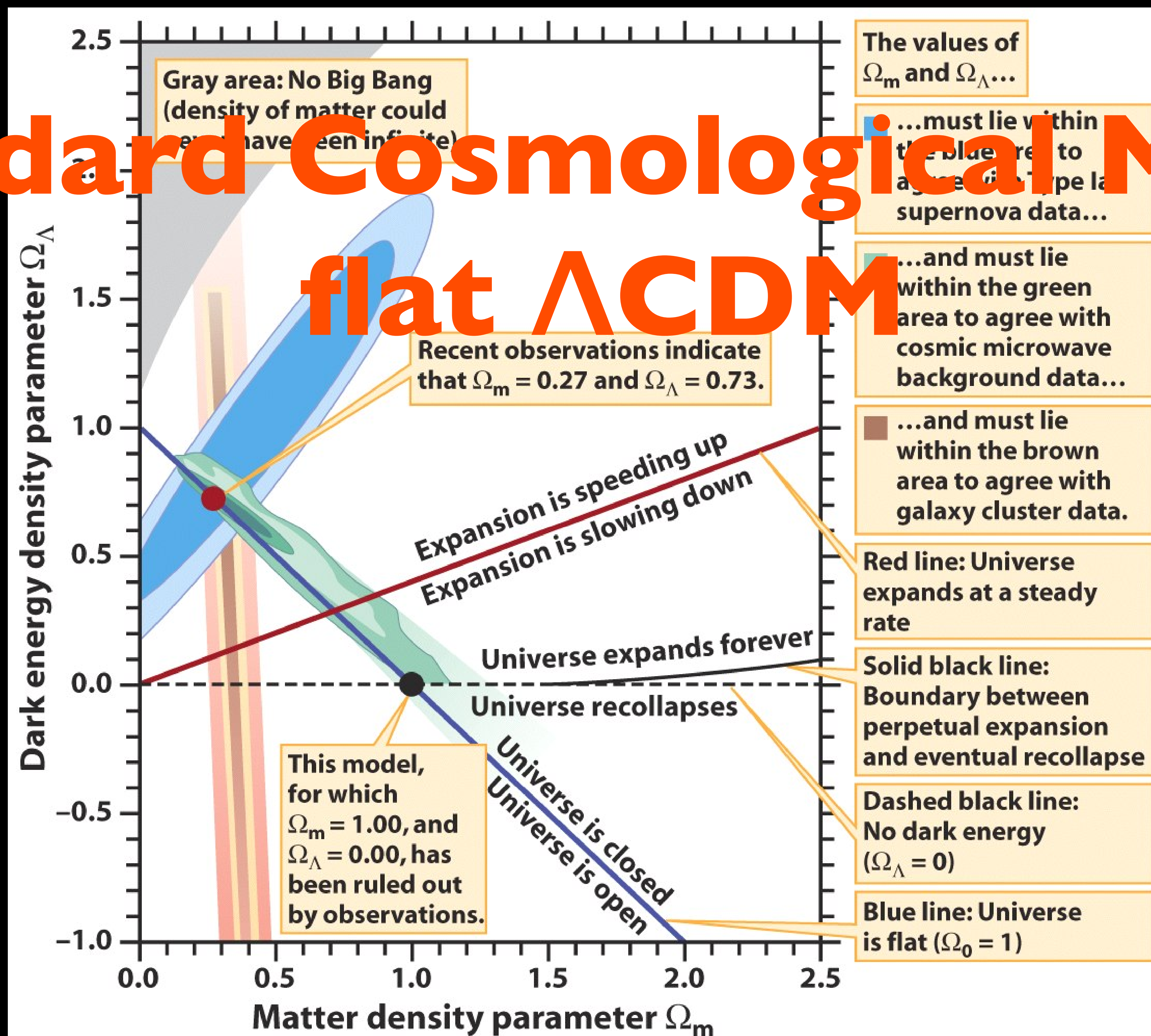
- Recipe for modelling based on 3 main ingredients:
  1. Homogeneous isotropic background, FLRW models
  2. Relativistic Perturbations (e.g. CMB), good for very large scales, mostly I-order (but now also II-order, see CMB talks this morning)
  3. Newtonian study of non-linear structure formation (N-body simulations or approx. techniques, e.g. 2LPT) at small scales (e.g. see Taruya talk yesterday)
- on this basis, well supported by observations, the flat ΛCDM model has emerged as the Standard "Concordance" Model of cosmology.

Standard Cosmology



Standard Cosmology

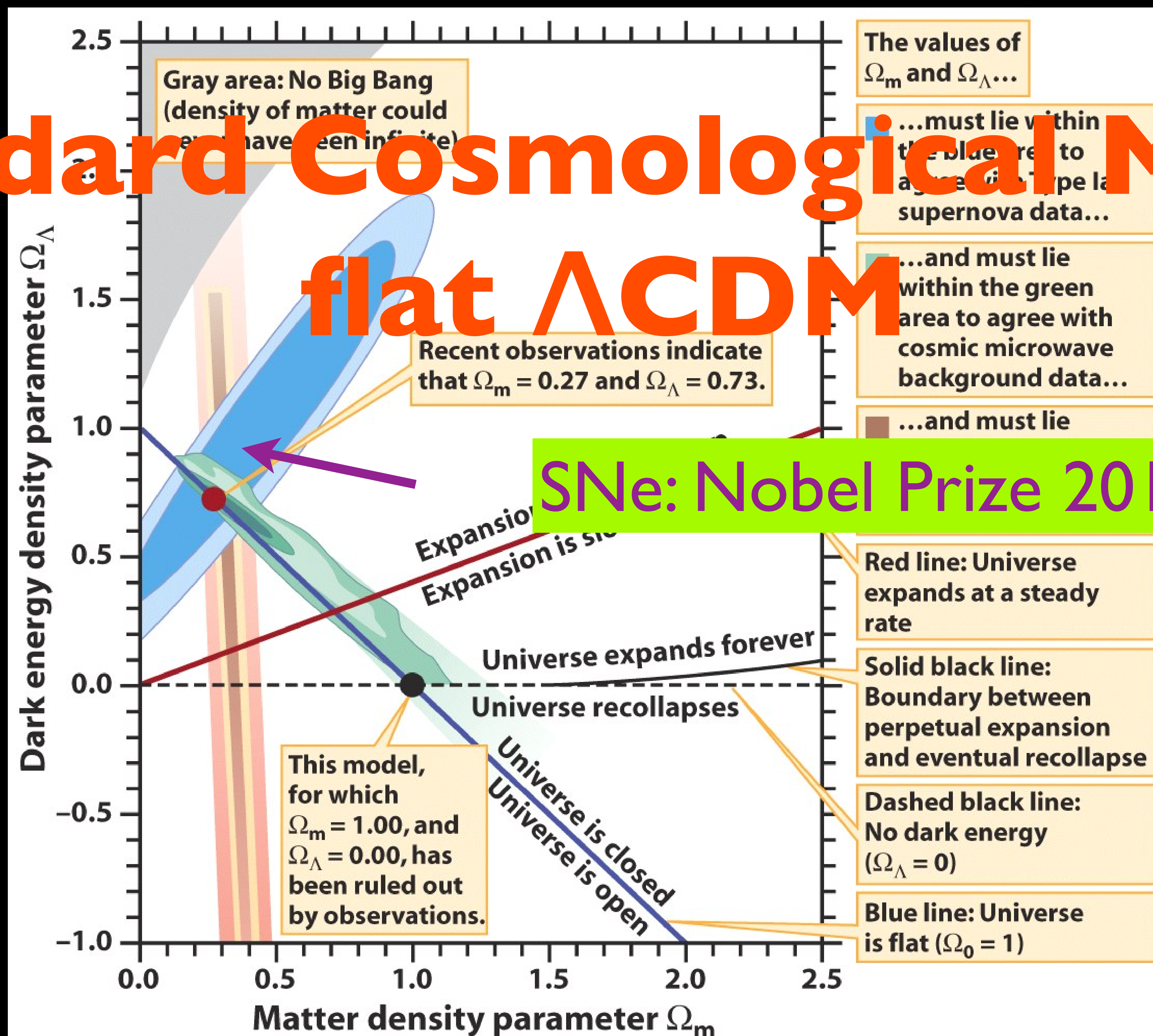
Standard Cosmological Model: flat ΛCDM



Standard Cosmology

Standard Cosmological Model: flat ΛCDM

SNe: Nobel Prize 2011



Questions on ΛCDM

- Recipe for modelling based on 3 main ingredients:
  1. Homogeneous isotropic background, FRW models
  2. Relativistic Perturbations (e.g. CMB)
  3. Newtonian study of non-linear structure formation (numerical simulations or approx. techniques)
- do we really need $\Omega_\Lambda \approx 0.7$? (or some other Dark Energy). Can we explain the observed acceleration differently?
- Is 3 enough? (more data, precision cosmology, observations and simulations covering large fraction of H^{-1}, etc...)
  - We need to bridge the gap between 2 and 3

Alternatives to ΛCDM

ΛCDM is the simplest and very successful model supporting the observations that, assuming the Cosmological Principle, are interpreted as acceleration of the Universe expansion

Going beyond ΛCDM, two main alternatives:

1. Maintain the Cosmological Principle (FLRW background), then either

   a) maintain GR + dark components (CDM+DE or UDM, or interacting CDM+vacuum, see Wands talk on thursday)

   b) modified gravity (f(R), branes, etc...)

Alternatives to ΛCDM

Going beyond ΛCDM, two main alternatives:

2. Maintain GR, drop CP, then either

   a) try to construct an homogeneous isotropic model from averaging, possibly giving acceleration: dynamical back-reaction

   b) consider inhomogeneous models, e.g. LTB (violating the CP) or Szekeres (not necessarily violating the CP): back-reaction on observations

back-reaction

- in essence, back-reaction is typical of non-linear systems, a manifestation of non-linearity
- in cosmology, we may speak of two types of BR[(*)]:
  - Strong BR: proper dynamical BR, i.e. the growth of structure really changes the expansion
    - in perturbation theory BR neglected by construction
    - In essence, in a a Newtonian N-body simulations a big volume is conformally expanded, neglecting back-reaction
  - Weak BR: optical BR, i.e. effects of inhomogeneities on observations (neglected in SNa, but the essence of lensing and ISW)

(*) Kolb, E.W., Marra, V. & Matarrese, S., 2010, GRG 42(6), pp.1399–1412.

the strong BR challenge

- standard flat ΛCDM: $\Omega_M + \Omega_\Lambda = 1, \qquad \Omega_\Lambda^{'} = 3\Omega_\Lambda(1-\Omega_\Lambda)$

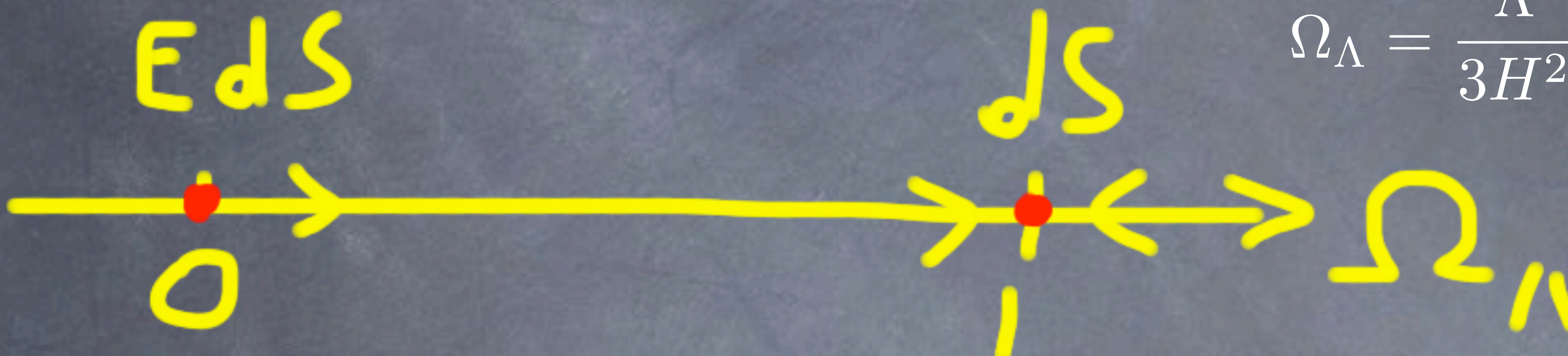
$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$

- BR cosmology: from EdS to an accelerated attractor, an effective de Sitter model or something else

the strong BR challenge

- standard flat ΛCDM: $\Omega_M + \Omega_\Lambda = 1, \qquad \Omega_\Lambda^{'} = 3\Omega_\Lambda(1-\Omega_\Lambda)$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$

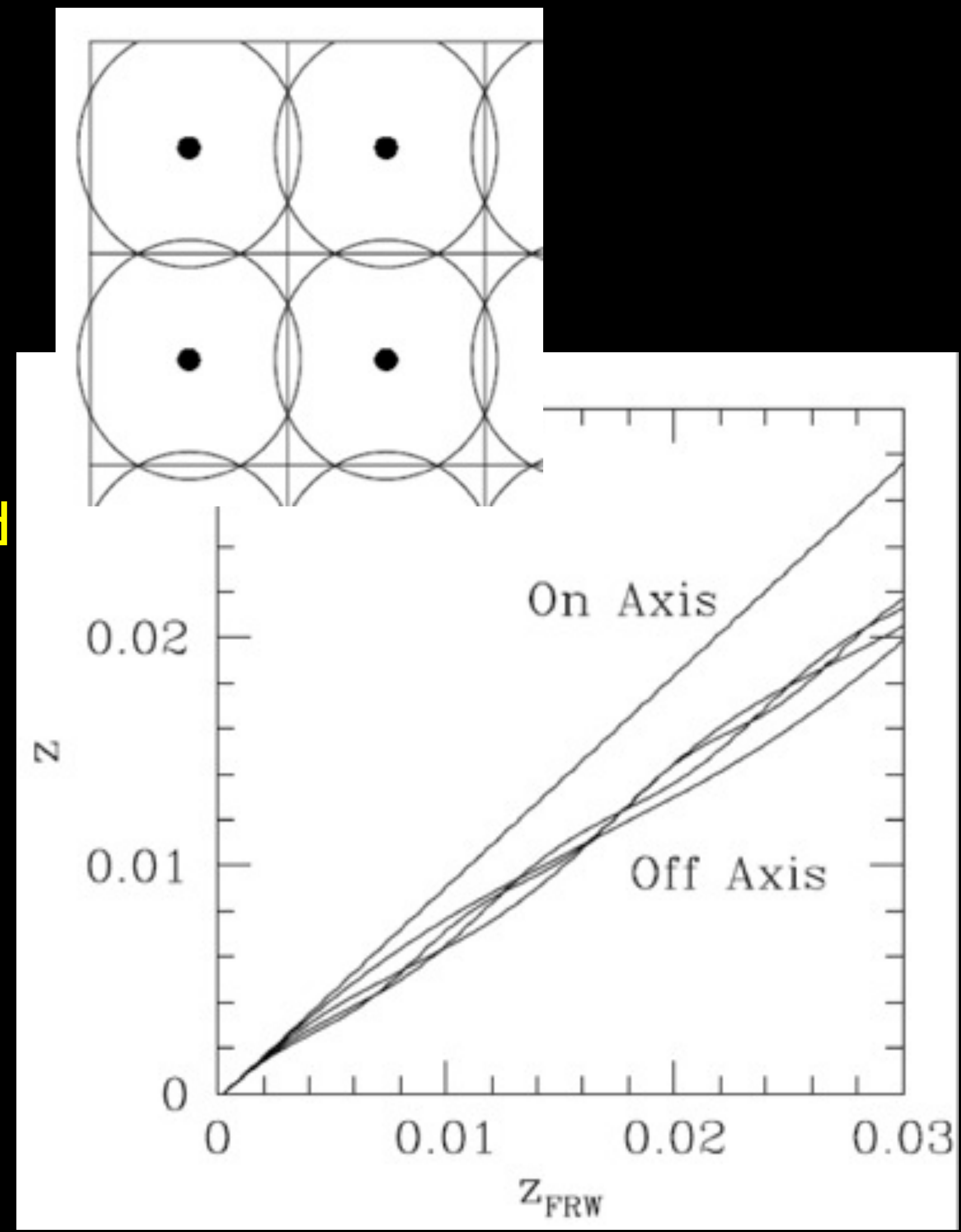


- BR cosmology: from EdS to an accelerated attractor, an effective de Sitter model or something else

Motivations for weak BR

- dynamical (strong) BR may be irrelevant, the overall cosmological dynamics is FLRW, yet effects of inhomogeneities on light propagation may affect redshifts and distances. e.g. Clifton & Ferreira, PRD 80, 10 (2009) [arXiv:0907.4109], based on Lindquist and Wheeler, Rev. Mod. Phys. 29, 432 (1957)

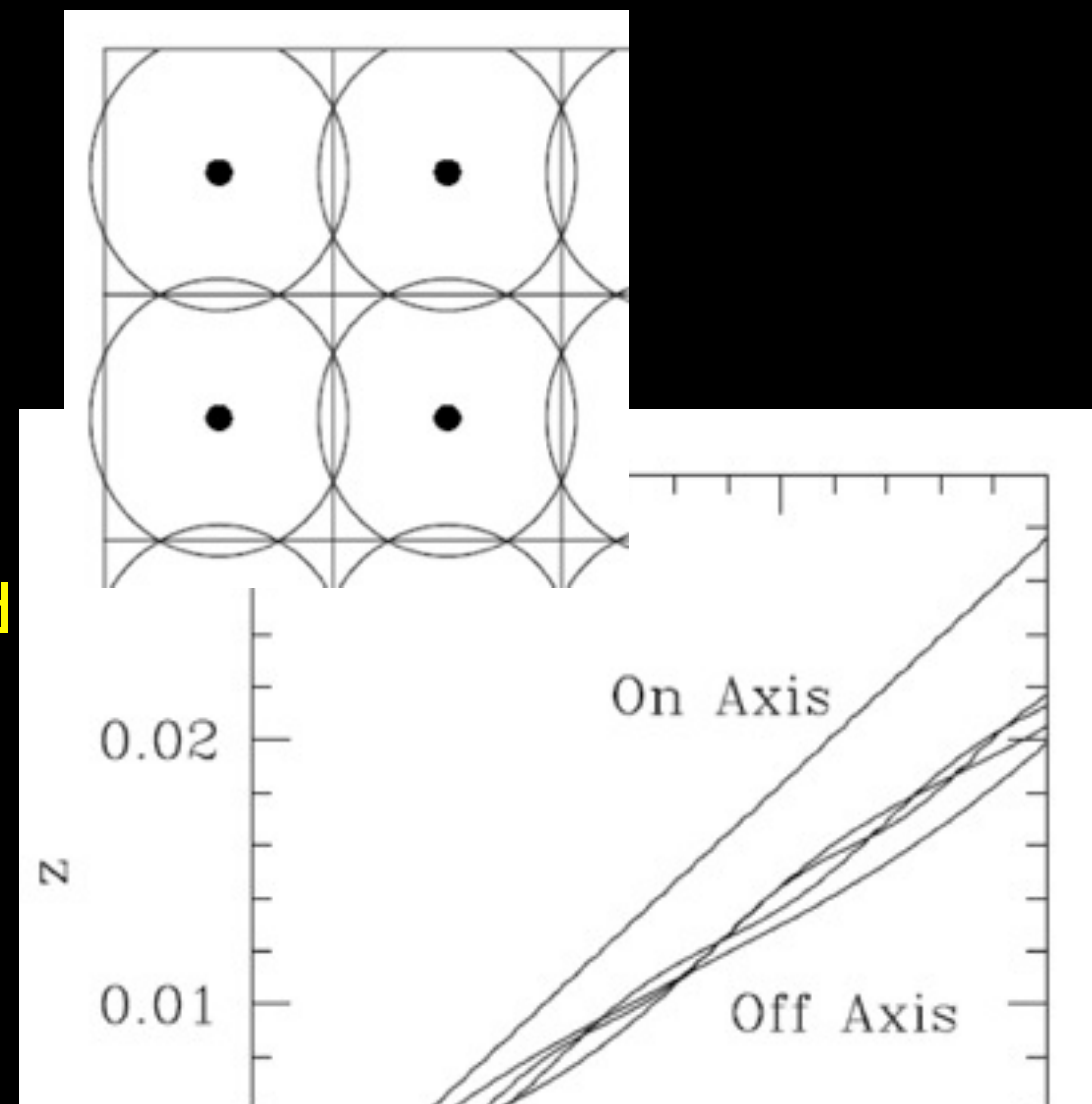
- less radical scenario, based on inhomogeneous Szekeres models (matter continuously distributed and evolving from standard growing mode in ΛCDM) seems to indicate that effects are small (but depends crucially on the "right background"). Meures, N. & MB, PRD, 8 (2011) arXiv:1103.0501 Meures, N. & MB, MN 419 (2012) arXiv:1107.4433



Motivations for weak BR

- dynamical (strong) BR may be irrelevant, the overall cosmological dynamics is FLRW, yet effects of inhomogeneities on light propagation may affect redshifts and distances. e.g. Clifton & Ferreira, PRD 80, 10 (2009) [arXiv:0907.4109], based on Lindquist and Wheeler, Rev. Mod. Phys. 29, 432 (1957)

- less radical scenario, based on inhomogeneous Szekeres models (matter continuously distributed and evolving from standard growing mode in ΛCDM) seems to indicate that effects are small (but depends crucially on the "right background").
Meures, N. & MB, PRD, 8 (2011) arXiv:1103.0501
Meures, N. & MB, MN 419 (2012) arXiv:1107.4433



cf. Clarkson et al. *Interpreting supernovae observations in a lumpy universe* arXiv1109.2484
and
Clarkson et al. "Anti-lensing" arXiv:1207

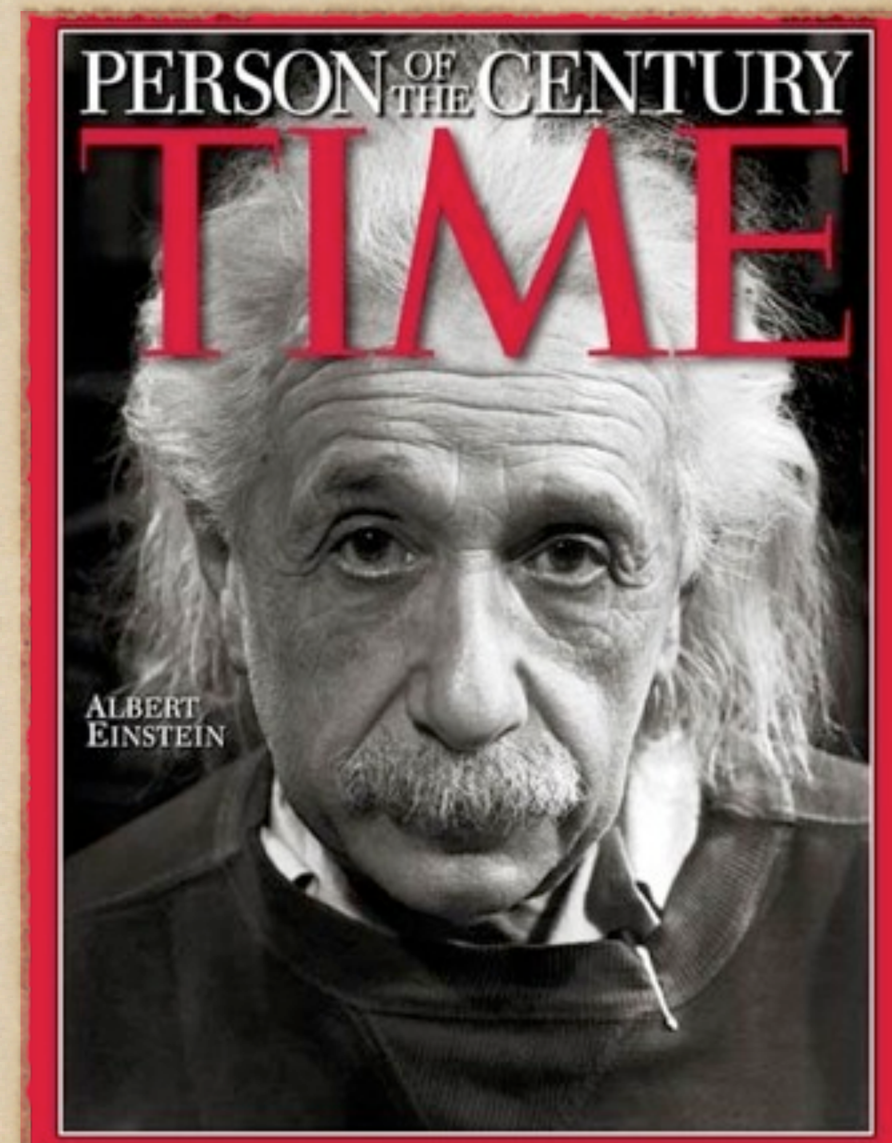
recent and more realistic back-reaction

- More Realistic: either 1) assume a perturbed FLRW background in the initial conditions or 2)-3) go all the way and study II-order effects on observations (in essence assuming that strong back-reaction is negligible)

1. Adamek et al. 2014, arXiv:1408.2741, using N-body simulations, seem to conclude that dynamical backreaction "is a small effect independently of initial conditions" (see refs. therein)
2. Bertacca et al. 2014, arXiv:1406.0319, II-order perturbations on the light-cone and observed galaxy number count
3. Clarkson et al.,arXiv:1405.7860, claim that relativistic corrections remove the tension with local H_0 measurements

Aims of Relativistic Cosmology

- in view of future data, is Newtonian non-linear structure formation good enough?
- GR itself is a successful gravity theory, but we don't know how to average E.E.s
- back-reaction may be relevant: if not dynamically, on light propagation through inhomogeneities (e.g. effects on distances)
- relativistic effects relevant on large scales (e.g. Power Spectrum), possibly on intermediate and small scales because of non-linearity



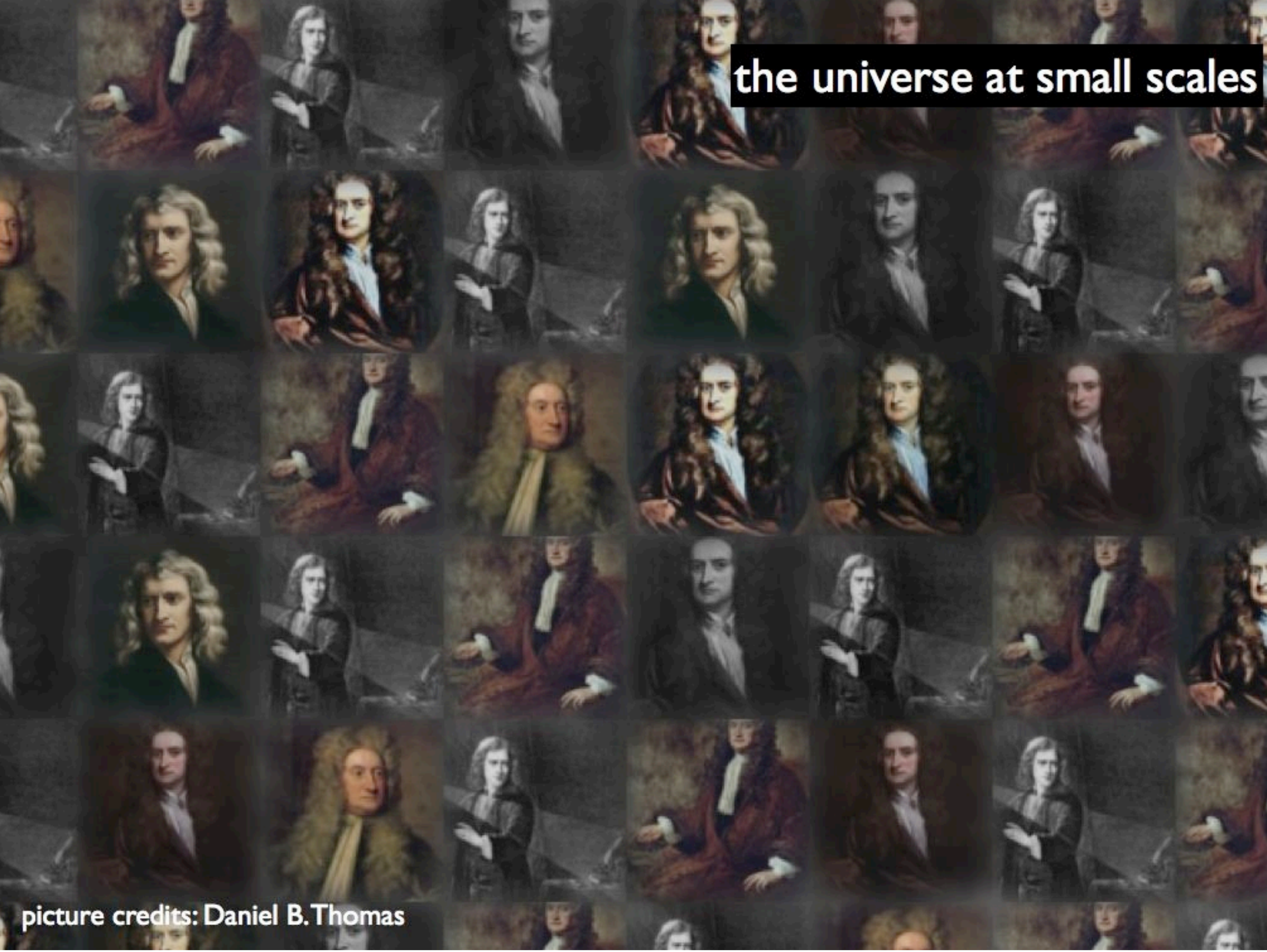
TIME cover, January 2000

standard ΛCDM, General Relativity and non-linearity

- from now on, I assume GR and a flat ΛCDM background
- perturbation theory is only valid for small δ
- clearly, to bridge the gap between Newtonian non-linear structure formation and large scale small inhomogeneities we need to go beyond the standard perturbative approach, considering non-linear density inhomogeneities within a relativistic framework

the universe at large scales: GR

picture credits: Daniel B. Thomas


the universe at small scales
picture credits: Daniel B. Thomas

Newtonian Cosmology

- starting point: Newtonian self-gravitating fluid: described by the continuity, Euler and Poisson equations:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v} \,,$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \phi - \frac{\nabla p}{\rho} \,,$$

$$\nabla^2 \phi = 4\pi G \rho \,.$$

Kinematical variables

- splitting the deformation tensor gives

$$\partial_j v_i = H\delta_{ij} + \sigma_{ij} + \omega_{ij} \,,$$

- trace, trace-less symmetric and anti-symmetric parts are defined as

$$H := \frac{1}{3}\nabla \cdot \mathbf{v} \,, \quad \sigma_{ij} := \partial_{(i}v_{j)} - \delta_{ij}H \,, \quad \omega_{ij} := \partial_{[i}v_{j]} \,.$$

- where H is the expansion scalar, σ_{ij} is the shear and ω_{ij} is the vorticity

Raychaudhuri equation

- defining $2\sigma^2 = \sigma_{ij}\sigma^{ij}, 2\omega^2 = \omega_{ij}\omega^{ij}$
  and using $\partial_j v_i \partial^i v^j = 3H^2 + 2(\sigma^2 - \omega^2)$
  we can take the divergence of the Euler equation to get, also using the Poisson eq.:

$$\dot{\rho} = -3H\rho\,,$$

$$\dot{H} = -H^2 - \frac{2}{3}(\sigma^2 - \omega^2) - \frac{4\pi G}{3}\rho - \nabla \cdot \left(\frac{\nabla p}{\rho}\right)\,,$$

- the latter is the Newtonian version of the Raychaudhuri eq. (see later).

- For a fluid with $\nabla p = \sigma_{ij} = \omega_{ij} = 0$ (homogeneous isotropic) we get Friedmann equations for $p = \Lambda = 0$

famous prof. Raychaudhuri with two young unknown

density and expansion perturbations

- Split each quantity in a background part and a perturbation, assuming $p_\mathrm{b} = 0$

$$\phi = \phi_\mathrm{b} + \delta\phi\,, \quad \mathbf{v} = \mathbf{v}_\mathrm{b} + \delta\mathbf{v}\,, \quad H = H_\mathrm{b} + \delta H\,, \quad \rho = \rho_\mathrm{b} + \delta\rho\,, \quad p = \delta p = v_s^2\delta\rho$$

- defining $v_s^2 = \frac{\partial p}{\partial \rho}$ we can substitute into previous equation and neglect second order terms to get

$$\dot{\delta\rho} = -3H_\mathrm{b}\delta\rho - 3\rho_\mathrm{b}\delta H\,,$$

$$\dot{\delta H} = -2H_\mathrm{b}\delta H - \frac{4\pi G}{3}\delta\rho - \frac{v_s^2}{3}\frac{\nabla^2\delta\rho}{\rho_\mathrm{b}}$$

- system of two coupled first-order (in time) eqs. for $\delta\rho$ and δH; describes scalar perturbations responsible for structure formations

Density perturbations

- It is standard to focus on $\delta := \frac{\delta\rho}{\rho_{\rm b}}$, deriving a second order eq. This is easily obtained by deriving the first eq., noting that this implies $\delta H = -\dot{\delta}/3$
- So far we used physical coordinates r_i; change to co-moving x_i, with $r_i = a(t)x_i$, where $a(t)$ is the scale factor; then $\nabla^2(..) = \nabla^2_{\mathbf{x}}(..)/a^2$; with this we get

$$\ddot{\delta} + 2H_{\rm b}\dot{\delta} - \left(4\pi G\rho_{\rm b} + \frac{v_s^2}{a^2}\nabla^2_{\mathbf{x}}\right)\delta = 0\,.$$

- wave eq. with damping term $H\dot{\delta}$

non-linear Newtonian theory

- Some results of linear theory can be turned into an ansatz for the mildly non-linear regime; let's first look at these results.
- Peculiar velocity V: $\mathbf{r} = a(t)\mathbf{x}$, $\mathbf{v} = \dot{\mathbf{r}} = \dot{a}\mathbf{x} + a\dot{\mathbf{x}} = H\mathbf{r} + \mathbf{V}$
- Consider an EdS background, and re-scale variables using a(t) as time variable, so that

$\tilde{\mathbf{u}} = \mathrm{d}\mathbf{x}/\mathrm{d}a$

$\tilde{\mathbf{u}} = \mathbf{V}/a^2 H$

$\tilde{\varphi} = 2\varphi/3H_0^2 a_0^3$

$$\frac{\partial \delta}{\partial a} = -\nabla_{\mathbf{x}} \cdot (\delta \tilde{\mathbf{u}}) - \nabla_{\mathbf{x}} \tilde{\mathbf{u}} \,,$$

$$\frac{\partial \tilde{\mathbf{u}}}{\partial a} = -\tilde{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \tilde{\mathbf{u}} - \frac{3}{2a}\tilde{\mathbf{u}} - \frac{3}{2a}\nabla_{\mathbf{x}} \tilde{\varphi} \,,$$

$$\nabla_{\mathbf{x}}^2 \tilde{\varphi} = \frac{1}{a}\delta \,.$$

the linear growing mode

- From these exact equations for a p=0 fluid in a EdS background, if we linearize and neglect the decaying mode it can be seen that the growing mode $\delta_+ \propto a$ solution corresponds to:

$$\frac{\partial \delta}{\partial a} = -\nabla_{\mathbf{x}} \cdot \cancel{(\delta \tilde{\mathbf{u}})} - \nabla_{\mathbf{x}} \tilde{\mathbf{u}}\,,$$

$$\frac{\partial \tilde{\mathbf{u}}}{\partial a} = -\tilde{\mathbf{u}} \cdot \cancel{\nabla_{\mathbf{x}}} \tilde{\mathbf{u}} - \frac{3}{2a}\tilde{\mathbf{u}} - \frac{3}{2a}\nabla_{\mathbf{x}}\tilde{\varphi}\,,$$

$$\nabla^2_{\mathbf{x}}\tilde{\varphi} = \frac{1}{a}\delta\,.$$

$$\tilde{\mathbf{u}} = constant$$

- This further implies that the fluid is irrotational and in "free fall" motion (in the re-scaled variables):

$$\tilde{\mathbf{u}} = -\nabla_{\mathbf{x}}\tilde{\varphi} \qquad \tilde{\mathbf{u}}' = 0$$

Remarks on exact Newtonian theory

- Assume irrotational motion for the non-linear fluid and define the re-scaled kinematical variables: $\tilde{\Theta} = \nabla_{\mathbf{x}} \cdot \tilde{\mathbf{u}}$, $\tilde{\sigma}_{ij} = \partial_{(j}\tilde{u}_{i)} - \frac{1}{3}\tilde{\Theta}\delta_{ij}$

- Then the exact system for these is:

$$\frac{\mathrm{d}\delta}{\mathrm{d}a} = -(1+\delta)\tilde{\Theta}\,, \qquad \text{continuity}$$

$$\frac{\mathrm{d}\tilde{\Theta}}{\mathrm{d}a} = -\frac{1}{3}\tilde{\Theta}^2 - 2\tilde{\sigma}^2 - \frac{3}{2a}(\tilde{\Theta} + \nabla^2_{\mathbf{x}}\tilde{\varphi})\,, \qquad \text{Raychaudhuri}$$

$$\frac{\mathrm{d}\tilde{\sigma}_{ij}}{\mathrm{d}a} = \frac{2}{3}\tilde{\sigma}^2\delta_{ij} - \frac{2}{3}\tilde{\Theta}\tilde{\sigma}_{ij} - \tilde{\sigma}_{ik}\tilde{\sigma}_{kj} - \frac{3}{2a}(\tilde{\sigma}_{ij} + \tilde{E}_{ij}) \qquad \text{shear evol.}$$

- the tidal field $\tilde{E}_{ij}$ has no evolution eq.: effect of action-at-the-distance in Newtonian gravity (Poisson is elliptic).

mildly non-linear regime: Zel'dovich approximation

- Using the shear and expansion equations, extrapolate to the non-linear regime the results of linear theory:

$$\tilde{\mathbf{u}} = -\nabla_{\mathbf{x}}\tilde{\varphi} \quad \Rightarrow \quad \tilde{\Theta} = -\nabla^2_{\mathbf{x}}\tilde{\varphi}\,, \quad \tilde{\sigma}_{ij} = -\tilde{E}_{ij}$$

- With this Zel'dovich ansatz, focus on collapse $\tilde{\Theta} < 0$ and use as time $d\tau = -\tilde{\Theta} da$; then we end up with a planar autonomous dynamical system for the two dimensionless rescaled shear vars.:

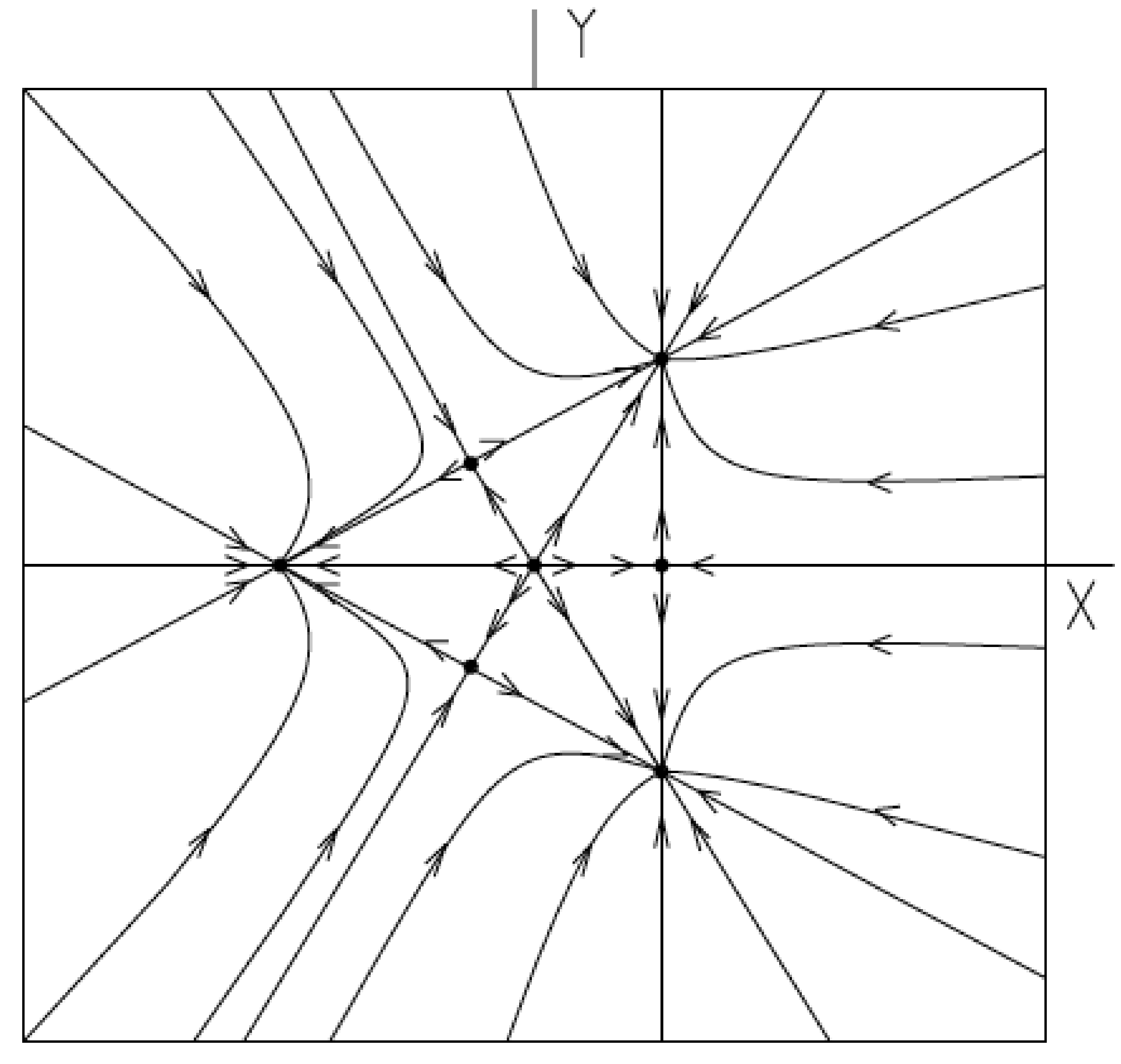
$$\frac{\mathrm{d}\Sigma_+}{\mathrm{d}\tau} = \frac{1}{3}\left(1 - 2\Sigma_+\right)\left[\Sigma_+\left(\Sigma_+ + 1\right) + \Sigma_-^2\right]\,,$$

$$\frac{\mathrm{d}\Sigma_-}{\mathrm{d}\tau} = \frac{1}{3}\left[1 - 2\left(\Sigma_+^2 + \Sigma_-^2\right) + 2\Sigma_+\right]\Sigma_-\,.$$

Pancakes as attractors from stability analysis

- Plotting the evolution and finding fixed points and eigenvalues of the linearized system we get:

Fixed Point	Σ_+	Σ_-	s_+	s_-	Stability
pancakes	-1	0	-1	-1	asymptotically stable node
filaments	$1/2$	0	$-1/2$	$1/2$	saddle
spherical	0	0	$1/3$	$1/3$	unstable node



Covariant fluid flow relativistic cosmology

- We normally use Einstein Field Equations (EFE) to determine a metric
- We can however look at $G_{ab}=8\pi G\, T_{ab}$ as an algebraic relation between G_{ab} and T_{ab}
- We can then use two geometrical identities, the Ricci and Bianchi identities, and transform these into field equations by substituting the Ricci tensor from EFE with the EMT T_{ab}

for this part see e.g.: Ellis & van Elst (1998), gr-qc/9812046 and
Ellis, G. F. R. 1971, "Varenna Lectures",Republished in: Gen.Rel.Grav. 41 581-660 (2009)

Covariant fluid flow cosmology

- If we split the Riemann tensor into the Weyl and Ricci tensors, the Bianchi identities take the form

$$C^{ijkl}{}_{;l} = R^{k[i;j]} - \frac{1}{6} g^{k[i} R^{,j]}$$

- If the then substitute the Ricci tensor with $T_{ab,}$ and also split the Weyl tensor in its electric and magnetic parts E_{ab} and H_{ab}, we obtain a set of "Maxwell-like" equations for E_{ab} and H_{ab}.

- From the Ricci identities for the 4-velocity u^a of the fluid we obtain equations for the kinematical quantities, i.e. the Raychaudhuri, shear and vorticity equations

- These fluid equations are coupled to the "Maxwell-like" equations

Covariant fluid flow cosmology

- For the kinematical quantities we obtain the Raychaudhuri equation, the shear and vorticity equations

$$\dot{\Theta} - \tilde{\nabla}_a \dot{u}^a = -\tfrac{1}{3}\,\Theta^2 + (\dot{u}_a \dot{u}^a) - 2\,\sigma^2 + 2\,\omega^2 - \tfrac{1}{2}\,(\mu + 3p) + \Lambda$$

$$\dot{\sigma}^{\langle ab\rangle} - \tilde{\nabla}^{\langle a}\dot{u}^{b\rangle} = -\tfrac{2}{3}\,\Theta\,\sigma^{ab} + \dot{u}^{\langle a}\,\dot{u}^{b\rangle} - \sigma^{\langle a}{}_{c}\,\sigma^{b\rangle c} - \omega^{\langle a}\,\omega^{b\rangle} - (E^{ab} - \tfrac{1}{2}\,\pi^{ab})$$

$$\dot{\omega}^{\langle a\rangle} - \frac{1}{2}\,\eta^{abc}\,\tilde{\nabla}_b \dot{u}_c = -\frac{2}{3}\,\Theta\,\omega^a + \sigma^a{}_b\,\omega^b\;;$$

- the contracted Bianchi identities give, for a perfect fluid

$$\dot{\mu} = -\,\Theta\,(\mu + p)\,,$$

$$0 = \tilde{\nabla}_a p + (\mu + p)\,\dot{u}_a\,.$$

Covariant fluid flow cosmology

- The Maxwell-like equations are

$$
\begin{aligned}
(\dot{E}^{\langle ab\rangle} + \tfrac{1}{2}\,\dot{\pi}^{\langle ab\rangle}) - (\mathrm{curl}\,H)^{ab} + \tfrac{1}{2}\,\tilde{\nabla}^{\langle a} q^{b\rangle} \;=\;& -\tfrac{1}{2}\,(\mu+p)\,\sigma^{ab} - \Theta\,(E^{ab} + \tfrac{1}{6}\,\pi^{ab}) \\
&+ 3\,\sigma^{\langle a}{}_{c}\,(E^{b\rangle c} - \tfrac{1}{6}\,\pi^{b\rangle c}) - \dot{u}^{\langle a}\,q^{b\rangle} \\
&+ \eta^{cd\langle a}\,[\,2\,\dot{u}_c\,H_d{}^{b\rangle} + \omega_c\,(E_d{}^{b\rangle} + \tfrac{1}{2}\,\pi_d{}^{b\rangle})\,]
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; (C_4)^a \;=\;& \tilde{\nabla}_b(E^{ab} + \tfrac{1}{2}\,\pi^{ab}) - \tfrac{1}{3}\,\tilde{\nabla}^a\mu + \tfrac{1}{3}\,\Theta\,q^a - \tfrac{1}{2}\,\sigma^a{}_b\,q^b - 3\,\omega_b\,H^{ab} \\
&- \eta^{abc}\,[\,\sigma_{bd}\,H_c{}^d - \tfrac{3}{2}\,\omega_b\,q_c\,]\,,
\end{aligned}
$$

$$
\begin{aligned}
\dot{H}^{\langle ab\rangle} + (\mathrm{curl}\,E)^{ab} - \tfrac{1}{2}(\mathrm{curl}\,\pi)^{ab} \;=\;& -\Theta\,H^{ab} + 3\,\sigma^{\langle a}{}_c\,H^{b\rangle c} + \tfrac{3}{2}\,\omega^{\langle a}\,q^{b\rangle} \\
&- \eta^{cd\langle a}\,[\,2\,\dot{u}_c\,E_d{}^{b\rangle} - \tfrac{1}{2}\,\sigma^{b\rangle}{}_c\,q_d - \omega_c\,H_d{}^{b\rangle}\,]\,,
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; (C_5)^a \;=\;& \tilde{\nabla}_b H^{ab} + (\mu+p)\,\omega^a + 3\,\omega_b\,(E^{ab} - \tfrac{1}{6}\,\pi^{ab}) \\
&+ \eta^{abc}\,[\,\tfrac{1}{2}\,\tilde{\nabla}_b q_c + \sigma_{bd}\,(E_c{}^d + \tfrac{1}{2}\,\pi_c{}^d)\,]\,,
\end{aligned}
$$

- there are also few extra constraint equations

- the main point is that the evolution system is closed

Covariant fluid flow cosmology

- provides an excellent framework for non-linear studies, with great similarity to Newtonian theory
- in synchronous comoving (irrotational) gauge (Wands talk this morning):
  - both at II-order (+ large scales) in standard perturbation theory and using a gradient expansion (long-wavelenght approximation) the evolution system is closed by two variables only, δ and θ, and the energy constraints implies that the 3-Ricci curvature is conserved
- MB, J. C. Hidalgo, N. Meures, D. Wands, ApJ 785:2 (2014)
  MB, J. C. Hidalgo and D. Wands, ApJ L, submitted [arXiv:1405:7006]

non-linear post-Friedmann framework

Questions/Motivations

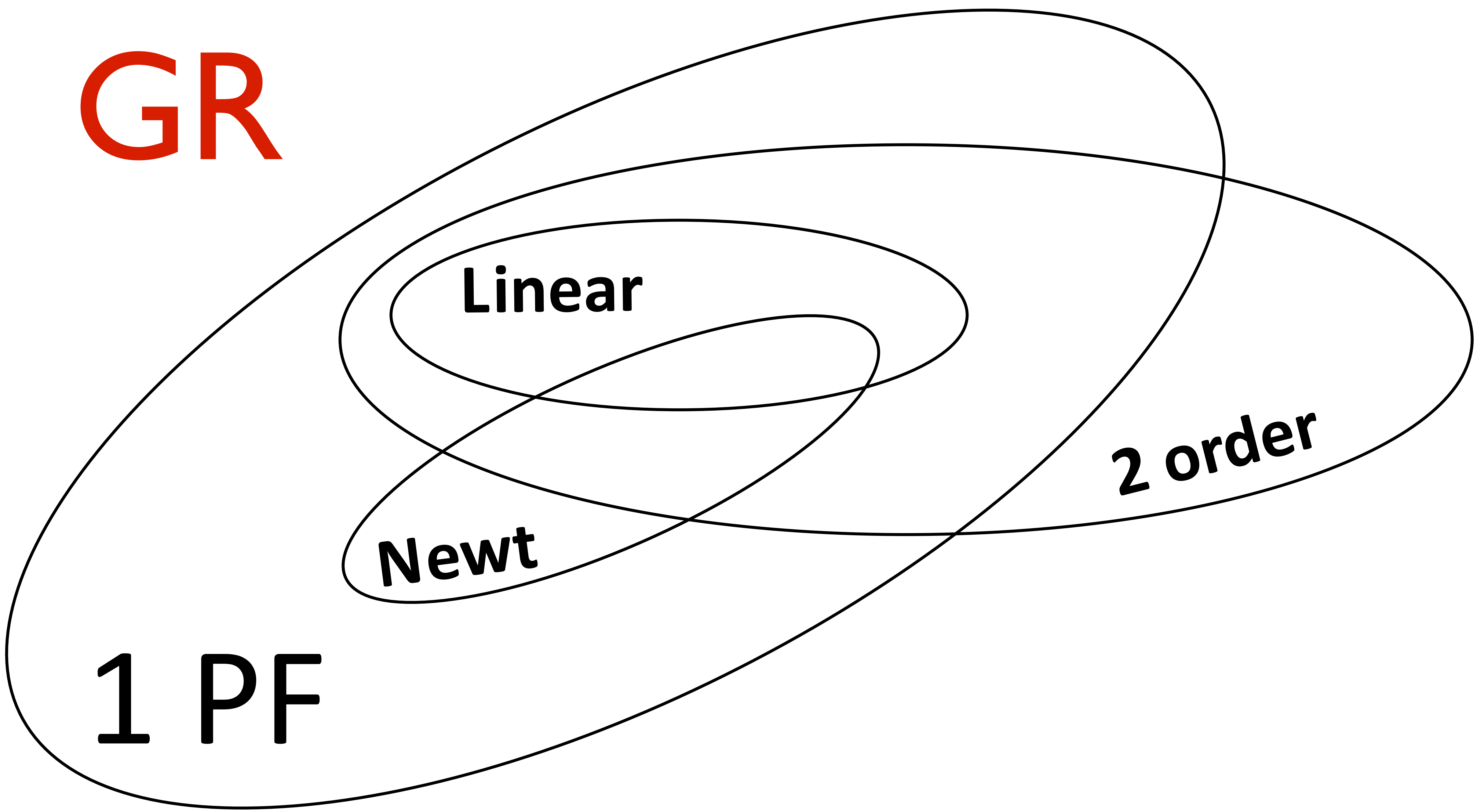
- Is the Newtonian approximation good enough to study non-linear structure formation?
  - surveys and simulations covering large fraction of H^{-1}
  - we are going to have more data: precision cosmology
  - we also need accurate cosmology: not only we want accurate observations, we also need accurate theoretical predictions (e.g.: Euclid target: N-body simulations wih 1% accuracy)
  - what if relativistic corrections are ~ few%?
    - We need to bridge the gap between small scale non-linear Newtonian approximation and large scale relativistic perturbation theory
    - We need a relativistic framework ("dictionary") to interprete N-body simulations [Chisari & Zaldarriaga (2011), Green & Wald (2012)]

non-linear post-Friedmann framework

- current goals:
  - develop a non-linear relativistic approximate framework, incorporating fully non-linear Newtonian theory at small scales and standard relativistic perturbations at large scales ($\sim H^{-1}$ and beyond)
  - extract leading order relativistic corrections from standard N-body simulations
  - ‣ more accurate ΛCDM cosmology

post-Friedmann framework

- spaces of equations (not solutions!)



Post-Newtonian cosmology

- post-Newtonian: expansion in 1/c powers (more later)
- various attempts and studies:
  - Tomita Prog. Theor. Phys. 79 (1988) and 85 (1991)
  - Matarrese & Terranova, MN 283 (1996)
  - Takada & Futamase, MN 306 (1999)
  - Carbone & Matarrese, PRD 71 (2005)
  - Hwang, Noh & Puetzfeld, JCAP 03 (2008)
- even in perturbation theory it is important to distinguish post-Newtonian effects, e.g. in non-Gaussianity and initial conditions. MB, J. C. Hidalgo, N. Meures, D. Wands, ApJ 785:2 (2014) [arXiv:1307:1478], cf. Bartolo et al. CQG 27 (2010) [arXiv:1002.3759]

post-N vs. post-F

- possible assumptions on the 1/c expansion:
  - Newton: field is weak, appears only in g_{00}; small velocities
  - post-Newtonian: next order, in 1/c, add corrections to g_{00} and g_{ij}
  - post-Minkowski (weak field): velocities can be large, time derivatives ~ space derivative
  - post-Friedmann: something in between, using a FLRW background, Hubble flow is not slow but peculiar velocities are small

$$\boxed{\dot{\vec{r}} = H\vec{r} + a\vec{v}}$$

  - post-Friedmann: we don't follow an iterative approach

metric and matter

starting point: the 1-PN cosmological metric
(Chandrasekhar 1965)

$$g_{00} = -\left[1 - \frac{2U}{c^2} + \frac{1}{c^4}(2U^2 - 4\Phi)\right] + O\left(\frac{1}{c^6}\right) ,$$

$$g_{0i} = -\frac{a}{c^3}P_i - \frac{a}{c^5}\tilde{P}_i + O\left(\frac{1}{c^7}\right) ,$$

$$g_{ij} = a^2\left[\left(1 + \frac{2V}{c^2} + \frac{1}{c^4}(2V^2 + 4\Psi)\right)\delta_{ij} + \frac{1}{c^4}h_{ij}\right] + O\left(\frac{1}{c^6}\right)$$

we assume a Newtonian-Poisson gauge: P_i is solenoidal and h_{ij} is TT, at each order 2 scalar DoF in g_{00} and g_{ij}, 2 vector DoF in frame dragging potential P_i and 2 TT DoF in h_{ij} (not GW!)

metric and matter

velocities, matter and the energy momentum tensor

Having in mind the Newtonian cosmology it is natural to define the peculiar velocity as $v^i = adx^i/dt$, obtain

$$u^i = \frac{dx^i}{cd\tau} = \frac{dx^i}{cdt}\frac{dt}{d\tau} = \frac{v^i}{ca}u^0 .$$

$$u^i = \frac{1}{c}\frac{v^i}{a}u^0 ,$$

$$u^0 = 1 + \frac{1}{c^2}\left(U + \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[\frac{1}{2}U^2 + 2\Phi + v^2V + \frac{3}{2}v^2U + \frac{3}{8}v^4 - P_iv^i\right] ,$$

$$u_i = \frac{av_i}{c} + \frac{a}{c^3}\left[-P_i + v_iU + 2v_iV + \frac{1}{2}v_iv^2\right] ,$$

$$u_0 = -1 + \frac{1}{c^2}\left(U - \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[2\Phi - \frac{1}{2}U^2 - \frac{1}{2}v^2U - v^2V - \frac{3}{8}v^4\right] .$$

$$T^\mu{}_\nu = c^2\rho u^\mu u_\nu ,$$

$$T^0{}_0 = -c^2\rho - \rho v^2 - \frac{1}{c^2}\rho\left[2(U+V)v^2 - P^iv_i + v^4\right] ,$$

$$T^0{}_i = c\rho av_i + \frac{1}{c}\rho a\left\{v_i[v^2 + 2(U+V)] - P_i\right\} ,$$

$$T^i{}_j = \rho v^iv_j + \frac{1}{c^2}\rho\left\{v^iv_j[v^2 + 2(U+V)] + v^iP_j\right\} ,$$

$$T^\mu{}_\mu = T = -\rho c^2 .$$

metric and matter

velocities, matter and the energy momentum tensor

Having in mind the Newtonian cosmology it is natural to define the peculiar velocity as $v^i = adx^i/dt$, obtain

$$u^i = \frac{dx^i}{cd\tau} = \frac{dx^i}{cdt}\frac{dt}{d\tau} = \frac{v^i}{ca}u^0 .$$

$$u^i = \frac{1}{c}\frac{v^i}{a}u^0 ,$$

$$u^0 = 1 + \frac{1}{c^2}\left(U + \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[\frac{1}{2}U^2 + 2\Phi + v^2V + \frac{3}{2}v^2U + \frac{3}{8}v^4 - P_iv^i\right] ,$$

$$u_i = \frac{av_i}{c} + \frac{a}{c^3}\left[-P_i + v_iU + 2v_iV + \frac{1}{2}v_iv^2\right] ,$$

$$u_0 = -1 + \frac{1}{c^2}\left(U - \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[2\Phi - \frac{1}{2}U^2 - \frac{1}{2}v^2U - v^2V - \frac{3}{8}v^4\right] .$$

$$T^\mu{}_\nu = c^2\rho u^\mu u_\nu ,$$

$$T^0{}_0 = -c^2\rho - \rho v^2 - \frac{1}{c^2}\rho\left[2(U+V)v^2 - P^iv_i + v^4\right] ,$$

$$T^0{}_i = c\rho a v_i + \frac{1}{c}\rho a\left\{v_i[v^2 + 2(U+V)] - P_i\right\} ,$$

$$T^i{}_j = \rho v^i v_j + \frac{1}{c^2}\rho\left\{v^iv_j[v^2 + 2(U+V)] + v^iP_j\right\} ,$$

$$T^\mu{}_\mu = T = -\rho c^2 .$$

note:
ρ is a non-perturbative quantity

Newtonian ΛCDM, with a bonus

- insert leading order terms in E.M. conservation and Einstein equations
- subtract the background, getting usual Friedmann equations
- introduce usual density contrast by $\rho=\rho_b(1+\delta)$

from E.M. conservation:
Continuity & Euler equations

$$\frac{d\delta}{dt} + \frac{v^i{}_{,i}}{a}(\delta+1) = 0\,,$$
$$\frac{dv_i}{dt} + \frac{\dot{a}}{a}v_i = \frac{1}{a}U_{,i}\,.$$

Poisson

$$G^0{}_0 + \Lambda = \frac{8\pi G}{c^4}T^0{}_0 \;\rightarrow\; \frac{1}{c^2}\frac{1}{a^2}\nabla^2 V = -\frac{4\pi G}{c^2}\rho_b\delta\,,$$

Newtonian ΛCDM, with a bonus

what do we get from the ij and 0i Einstein equations?

$$\text{trace of } G^i{}_j + \Lambda\delta^i{}_j = \frac{8\pi G}{c^4}T^i{}_j \;\to\; \frac{1}{c^2}\frac{2}{a^2}\nabla^2(V-U) = 0\,,$$

zero "Slip"

$$\text{traceless part of } G^i{}_j + \Lambda\delta^i{}_j = \frac{8\pi G}{c^4}T^i{}_j \;\to\; \frac{1}{c^2}\frac{1}{a^2}\left[(V-U)_{,i}{}^{,j} - \frac{1}{3}\nabla^2(V-U)\delta_i^j\right] = 0\,.$$

bonus

$$G^0{}_i = \frac{8\pi G}{c^4}T^0{}_i \;\to\; \frac{1}{c^3}\left[-\frac{1}{2a^2}\nabla^2 P_i + 2\frac{\dot a}{a^2}U_{,i} + \frac{2}{a}\dot V_{,i}\right] = \frac{8\pi G}{c^3}\rho_b(1+\delta)v_i$$

- Newtonian dynamics at leading order, with a bonus: the frame dragging potential P_i is not dynamical at this order, but cannot be set to zero: doing so would forces a constraint on Newtonian dynamics
- result entirely consistent with vector relativistic perturbation theory
- in a relativistic framework, gravitomagnetic effects cannot be set to zero even in the Newtonian regime, cf. Kofman & Pogosyan (1995), ApJ 442:

magnetic Weyl tensor at leading order

$$H_{ij} = \frac{1}{2c^3}\left[P_{\mu,\nu(i}\varepsilon_{j)}{}^{\mu\nu} + 2v_\mu(U+V)_{,\nu(i}\varepsilon_{j)}{}^{\mu\nu}\right]$$

Post-Friedmannian ΛCDM

next to leading order: the 1-PF variables

- resummed scalar potentials

$$\phi_P = -(U + \frac{2}{c^2}\Phi),$$
$$\psi_P = -(V + \frac{2}{c^2}\Psi),$$

- resummed gravitational potential

$$\phi_G = \frac{1}{2}(\psi_P + \phi_P),$$

- resummed “Slip” potential

$$\frac{D_P}{c^2} = \frac{1}{2}(\psi_P - \phi_P);$$

- resummed vector “frame dragging” potential

$$P_i^* = P_i + \frac{1}{c^2}\tilde{P}_i.$$

- Chandrasekhar velocity:

$$v_i^* = v_i - \frac{1}{c^2}P_i \ ,$$

Post-Friedmannian ΛCDM

The 1-PF equations: scalar sector

Continuity & Euler

$$\frac{d\delta}{dt} + \frac{v^{\star i}{}_{,i}}{a}(\delta+1) - \frac{1}{c^2}\left[(\delta+1)\left(3\frac{d\phi_G}{dt} + \frac{v_k^\star \phi_{G,k}}{a} + \frac{\dot a}{a}v^{\star 2}\right)\right] = 0\,.$$

$$\frac{dv_i^\star}{dt} + \frac{\dot a}{a}v_i^\star + \frac{1}{a}\phi_{G,i} + \frac{1}{c^2}\left[\frac{\phi_{G,i}}{a}(4\phi_G + v^{\star 2}) - 3v_i^\star\frac{d\phi_G}{dt} - \frac{D_{P,i}}{a} - \frac{v_i^\star}{a}v_j^\star\phi_{G,j} - \frac{\dot a}{a}v^{\star 2}v_i^\star + \frac{P_{j,i}v^{\star j}}{a}\right] = 0\,.$$

generalized Poisson: a non-linear wave eq. for ϕ_G

$$\frac{1}{c^2}\frac{2}{3a^2}\nabla^2\phi_G + \frac{1}{c^4}\left[\ddot\phi_G + 2\frac{\dot a}{a}\dot\phi_G + 2\frac{\ddot a}{a}\phi_G - \left(\frac{\dot a}{a}\right)^2\phi_G + \frac{2}{3a^2}\nabla^2\phi_G^2 - \frac{3}{2a^2}\phi_{G,i}\phi_G{}^{,i}\right] = \frac{4\pi G}{3}\rho_b\left[\frac{1}{c^2}\delta + \frac{1}{c^4}\rho_b(1+\delta)v^{\star 2}\right]$$

non-dynamical "Slip"

$$\frac{1}{c^4}\frac{1}{3a^2}\nabla^2\nabla^2 D_{PN} = -\frac{1}{c^2}\frac{1}{3a^2}\nabla^2\nabla^2\phi_G - \frac{1}{c^4}\left[\frac{1}{3a^2}\nabla^2\nabla^2\phi_G^2 - \frac{5}{6a^2}\nabla^2(\phi_{G,i}\phi_{G,i})\right]$$
$$+\frac{4\pi G}{3}\rho_b\left\{\frac{1}{c^2}\nabla^2\delta - \frac{1}{c^4}\left[\nabla^2((1+\delta)v^{\star 2})) + \dot a((1+\delta)v_k^\star)^{,k}\right]\right\}\,,$$

Post-Friedmannian ΛCDM

The 1-PF equations: vector and tensor sectors

- the frame dragging vector potential becomes dynamical at this order

- the TT metric tensor h_{ij} is not dynamical at this order, but it is instead determined by a non-linear constraint in terms of the scalar and vector potentials

linearized equations

linearized equations:
standard scalar and vector perturbation equations
in the Poisson gauge

$$\nabla^2\psi_P - \frac{3}{c^2}a^2\left[\frac{\dot a}{a}\dot\psi_P + \left(\frac{\dot a}{a}\right)^2\phi_P\right] = 4\pi G\rho_b a^2\delta\ ,$$

$$-\nabla^2(\psi_P-\phi_P) + \frac{3}{c^2}a^2\left[\frac{\dot a}{a}(\dot\phi_P+3\dot\psi_P) + 2\frac{\ddot a}{a}\phi_P + \left(\frac{\dot a}{a}\right)^2\phi_P + \ddot\psi_P\right] = 0$$

$$\nabla^2\left(\frac{\dot a}{a}\phi_P + \dot\psi_P\right) = -4\pi G a\rho_b\theta\ ,$$

$$\frac{1}{c^2a^2}\frac{2}{3}\nabla^2\nabla^2(\phi_P-\psi_P) = 0\ ,$$

$$\dot\delta + \frac{\theta}{a} - \frac{3}{c^2}\dot\psi_P = 0\ ,$$

$$\dot\theta + \frac{\dot a}{a}\theta + \frac{\nabla^2\phi_P}{a} = 0\ .$$

cf. Ma & Bertschinger, ApJ (1994)

Summary

- dynamical back-reaction most likely small, optical back-reaction important and worth further investigation
- Non-linear GR effects worth investigating in view of future surveys
- PF: at leading Newtonian order in the dynamics, consistency of Einstein equations requires a non-zero gravito-magnetic vector potential
- PF framework provides a straightforward relativistic interpretation of Newtonian simulations: quantities are those of Newton-Poisson gauge
- linearised equations coincide with I-order relativistic perturbation theory in Poisson gauge (probably OK up to II-order, except sub-dominant terms)
- 2 scalar potentials, become 1 in the Newtonian regime and in the linear regime, valid at horizon scales: slip non-zero in relativistic mildly non-linear (intermediate scales?) regime