Spatially covariant gravity and unifying framework for scalar-tensor theories

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*Based on [**1406.0822**] and [**1409.xxxx**]*

Modifying gravity?

Inflation, dark energy and dark matter have been strong motivations for alternative gravity theories beyond Einstein’s general relativity.

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→ How to introduce these extra degrees of freedom?

k-essence

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to add gravity with extra field(s), covariantly.

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GR:

$$\mathcal{L} = \sqrt{-g}\left[\frac{R}{2} + \Lambda \qquad\qquad \right]$$

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***k*-essence:** $$\mathcal{L} = \sqrt{-g}\left[\frac{R}{2} + K\left(\phi, X\right)\right]$$ $$X = -\frac{1}{2}\left(\nabla\phi\right)^2$$

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Higher derivatives in the Lagrangian → Extra mode(s)?

Horndeski theory (galileons)

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$$\mathcal{L}_5 = G_5\left(X,\phi\right)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{6}\frac{\partial G_5}{\partial X}\left[\left(\Box\phi\right)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right].$$

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[Hinterbichler, Trodden & Wesley 1008.1305]

$$\mathcal{L} = \sqrt{-g}\left(\Lambda + c_1 K + c_2 R + c_3 \mathcal{K}_{\rm GB}\right) \qquad \mathcal{K}_{\rm GB} = -\frac{2}{3}K^3_{\mu\nu} + K K^2_{\mu\nu} - \frac{1}{3}K^3 - 2G_{\mu\nu}K^{\mu\nu}$$

$$\Big\downarrow \; g_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu\pi\partial_\nu\pi}{1+(\partial\pi)^2}$$

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Additional degree(s) of freedom may arise when symmetries are reduced.

- Massive gravity: 2t+2v+1s
- Massive vector: 2v+1s *[Tasinato's talk]*
- Scalar-tensor theory: 2t+1s?

EFT (effective field theory) of inflation

Cosmological background breaks the full symmetries of GR, by choosing a preferred time direction or spatial sclices, on which

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$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}R + \Lambda\,(t) + f_1\,(t)\,\delta N + f_2\,(t)\,\delta N^2 + \cdots$$
$$+ g_1\,(t)\,\delta K^\mu_\mu + g_2\,(t)\,\left(\delta K^\mu_\mu\right)^2 + g_3\,(t)\,\delta K_{\mu\nu}\delta K^{\mu\nu} + \cdots\Big]$$

[Cheung, Creminelli, Fitzpatrick, Kaplan, and Senatore, JHEP 0803, 014 (2008)]

Hořava gravity

GR in the ADM formalism:

$$S^{(\mathrm{GR})} = \frac{1}{2}\int d^4x N\sqrt{h}\left(K_{ij}K^{ij} - \quad K^2 + {}^{(3)}R\right)$$

→ respect full spacetime diffeomorphism $\quad t \to \tilde{t}\left(t, x^i\right), \qquad x^i \to \tilde{x}^i\left(t, x^i\right)$

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$$S^{(\mathrm{Horava})} = \frac{1}{2}\int d^4x N\sqrt{h}\left(K_{ij}K^{ij} - \lambda K^2 + \mathcal{V}\left[h_{ij}, {}^{(3)}R_{ij}, D_i\right]\right)$$

[P. Horava, Phys.Rev. D79, 084008 (2009)]

→ time-dependent spatial diffeomorphism
→ space-independent time reparametrization

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Healthy extensions:

$$S^{(\text{Healthy Ext.})} = \frac{1}{2}\int d^4x N\sqrt{h}\left(c_1 a_i a^i + c_2\left(a_i a^i\right)^2 + c_3 R_{ij} a^i a^j + \cdots\right)$$

$$a_i = \partial_i \ln N$$

[Blas, Pujolas & Sibiryakov, JHEP 0910, 029 (2009)]

→ *N* enters the Hamiltonian "nonlinearly"!

Horndeski in ADM form and beyond

Fixing the unitary (uniform scalar field) gauge: $\phi\left(t,\vec{x}\right)\equiv\phi_0\left(t\right)\equiv t$

$$\nabla_\mu\phi=-\frac{1}{N}\delta^0_\mu$$

$$\nabla_\mu\nabla_\nu\phi=-\delta^0_\mu\delta^0_\nu\frac{1}{N^2}\left(\partial_t\ln N-N^i\nabla_i\ln N\right)+\frac{2}{N}\delta^0_{(\mu}\delta^i_{\nu)}\partial_i\ln N-\frac{1}{N}\delta^i_\mu\delta^j_\nu K_{ij}$$

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Horndeski in the ADM form:

[Gleyzes, Langlois, Piazza & Vernizzi, arXiv:1304.4840]

$$\begin{aligned}
\mathcal{L}^{\text{Horndeski}} \simeq\ & G_2 + \frac{1}{N^2}\frac{\partial F_3}{\partial \phi} \\
& + \left[G_4 - \frac{1}{2N^2}\frac{\partial\left(G_5 - F_5\right)}{\partial \phi}\right] {}^{(3)}R \\
& + \left[\left(\frac{\partial F_3}{\partial N} - 2\frac{1}{N}\frac{\partial G_4}{\partial \phi}\right) h_{ij} - \frac{1}{N} F_5\, {}^{(3)}G_{ij}\right] K^{ij} \\
& - \left(\frac{\partial\left(N G_4\right)}{\partial N} + \frac{1}{2N^2}\frac{\partial G_5}{\partial \phi}\right)\left(K^2 - K_{ij}K^{ij}\right) \\
& - \frac{1}{6}\frac{\partial G_5}{\partial N}\left(K^3 - 3K K_{ij}K^{ij} + 2K^i_j K^j_k K^k_i\right)
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functions of (t, N)

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GLPV model (deformed Horndeski):

*[**G**leyzes, **L**anglois, **P**iazza & **V**ernizzi, arXiv:1404.6495]*

$$\begin{aligned}\mathcal{L}^{\rm GLPV} = \; & A_2\left(t,N\right) \\ & + \left[\; B_4\left(t,N\right) \;\right] {}^{(3)}R \\ & + \left[\left(\; A_3\left(t,N\right) \;\right) h_{ij} + B_5\left(t,N\right)\; {}^{(3)}G_{ij}\right] K^{ij} \\ & + \left(\; A_4\left(t,N\right) \;\right)\left(K^2 - K_{ij}K^{ij}\right) \\ & + A_5\left(t,N\right)\; \left(K^3 - 3KK_{ij}K^{ij} + 2K^i_j K^j_k K^k_i\right)\end{aligned}$$

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DGP

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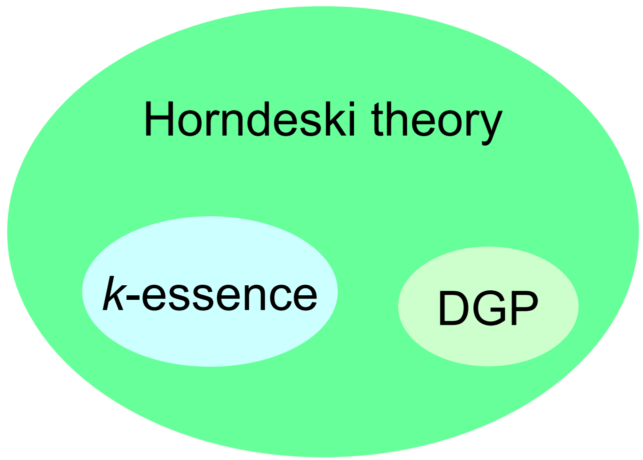

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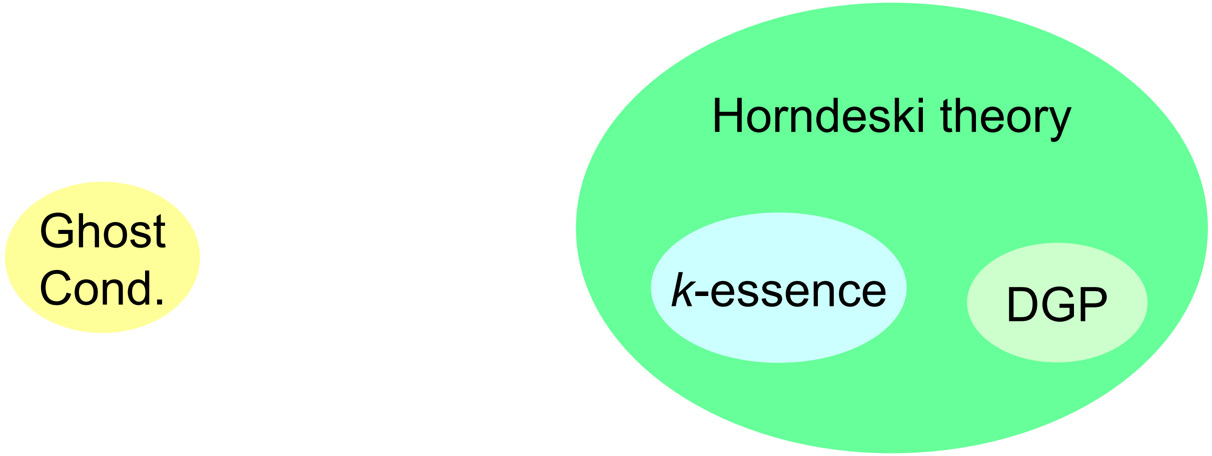

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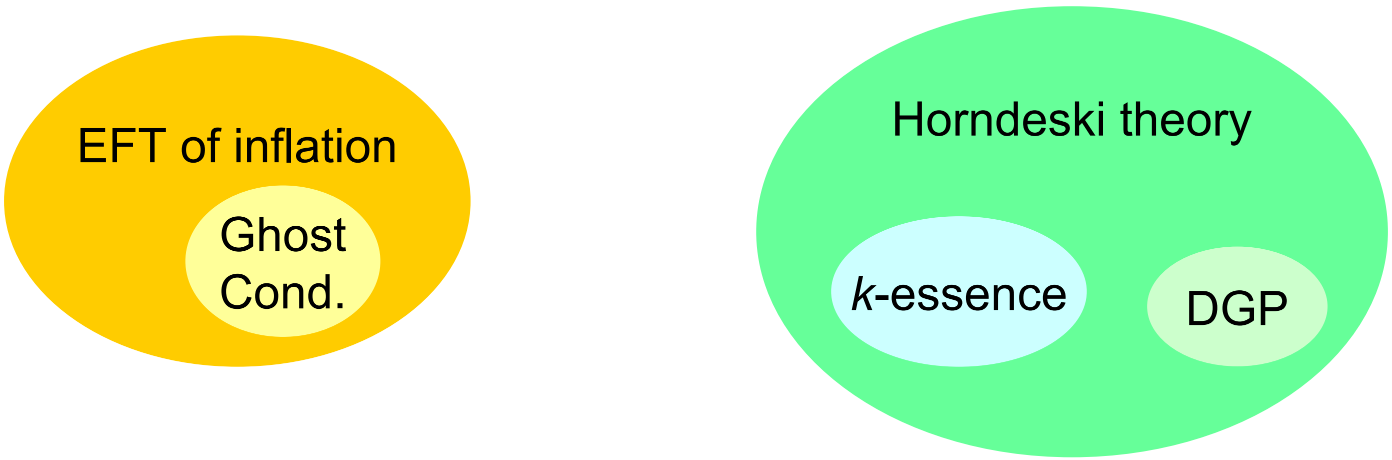

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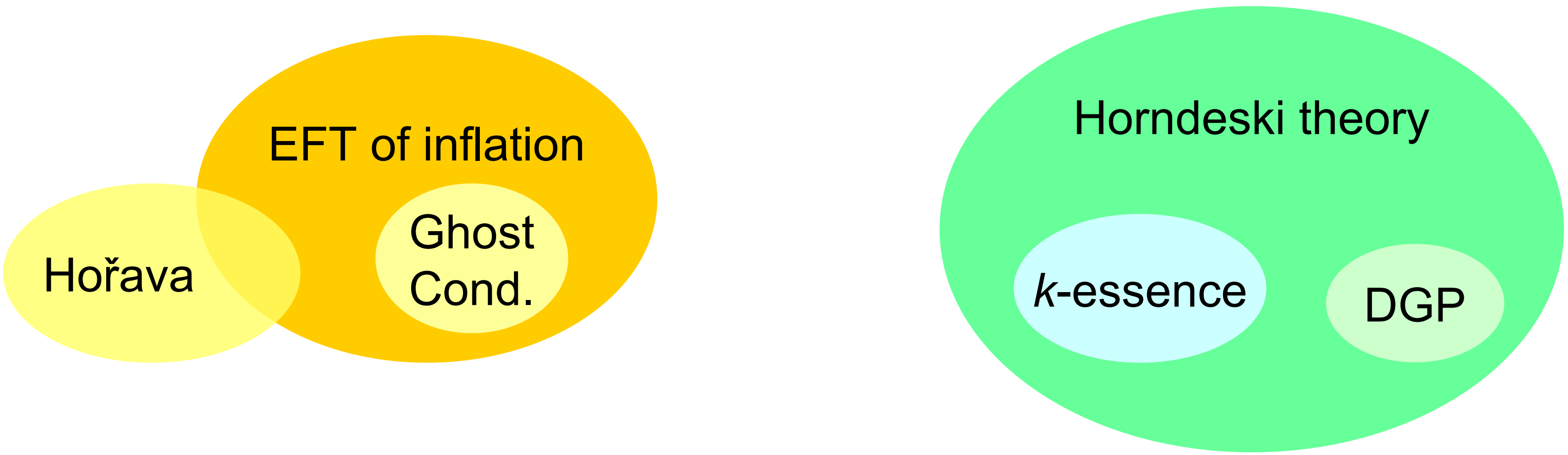

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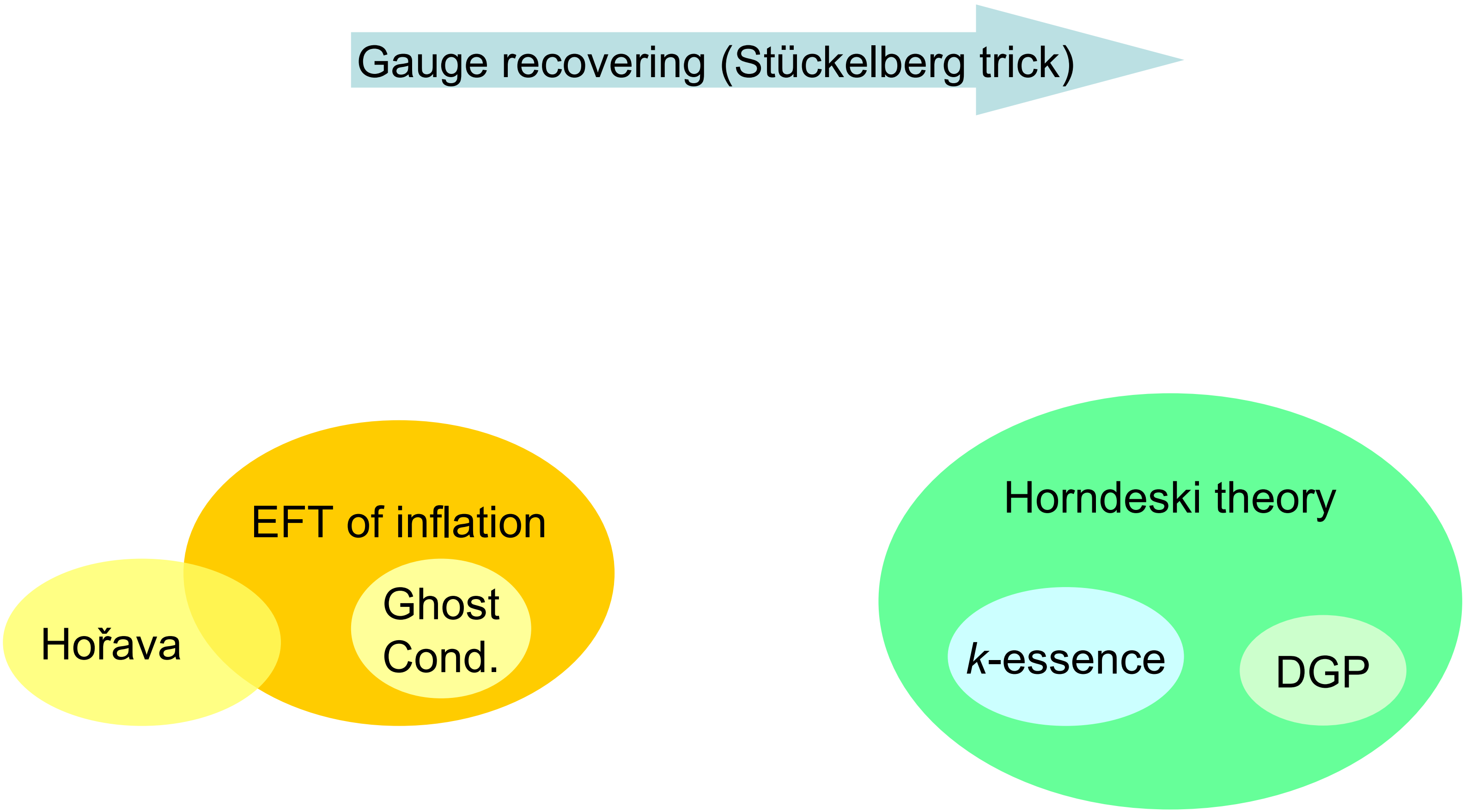

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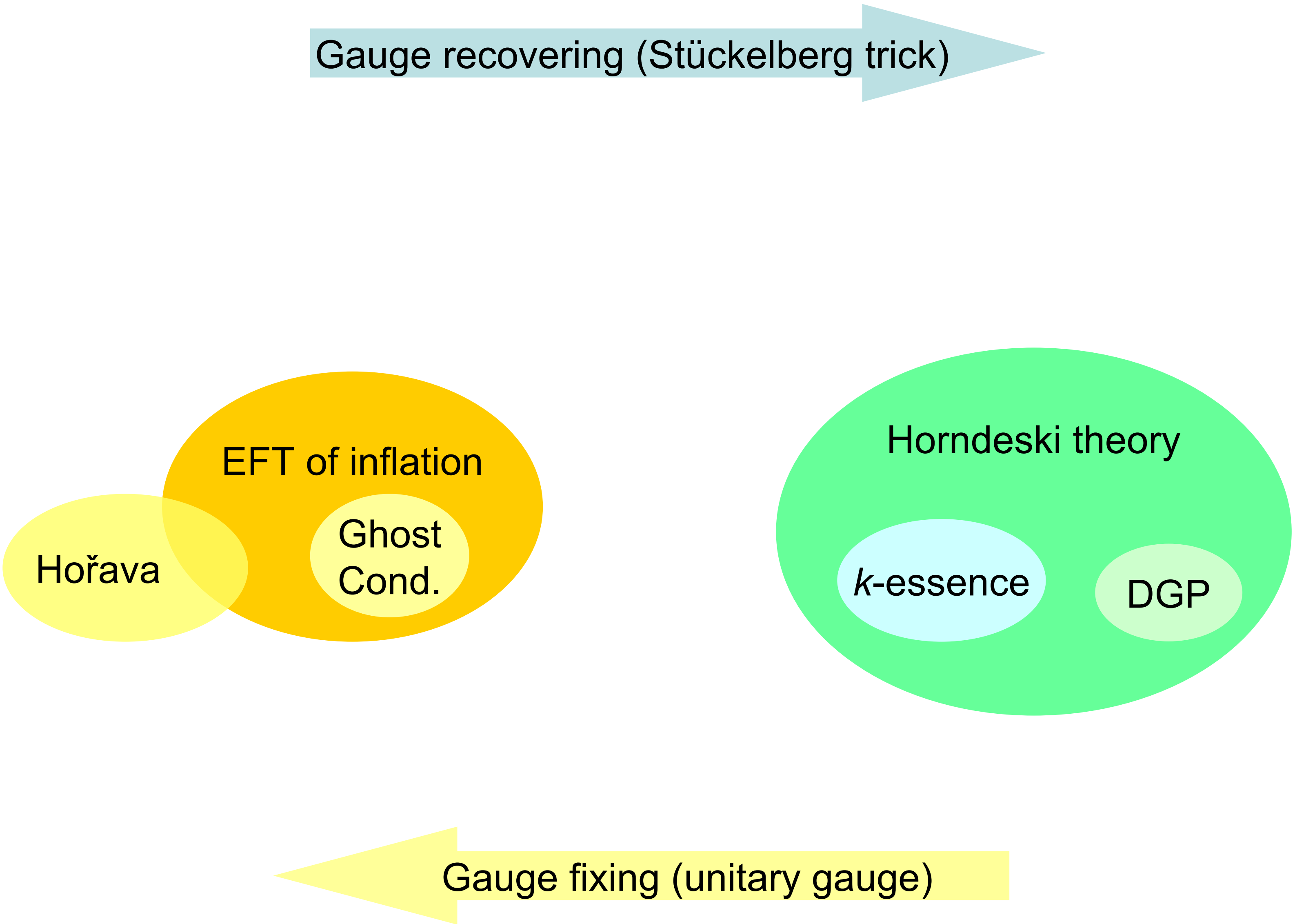

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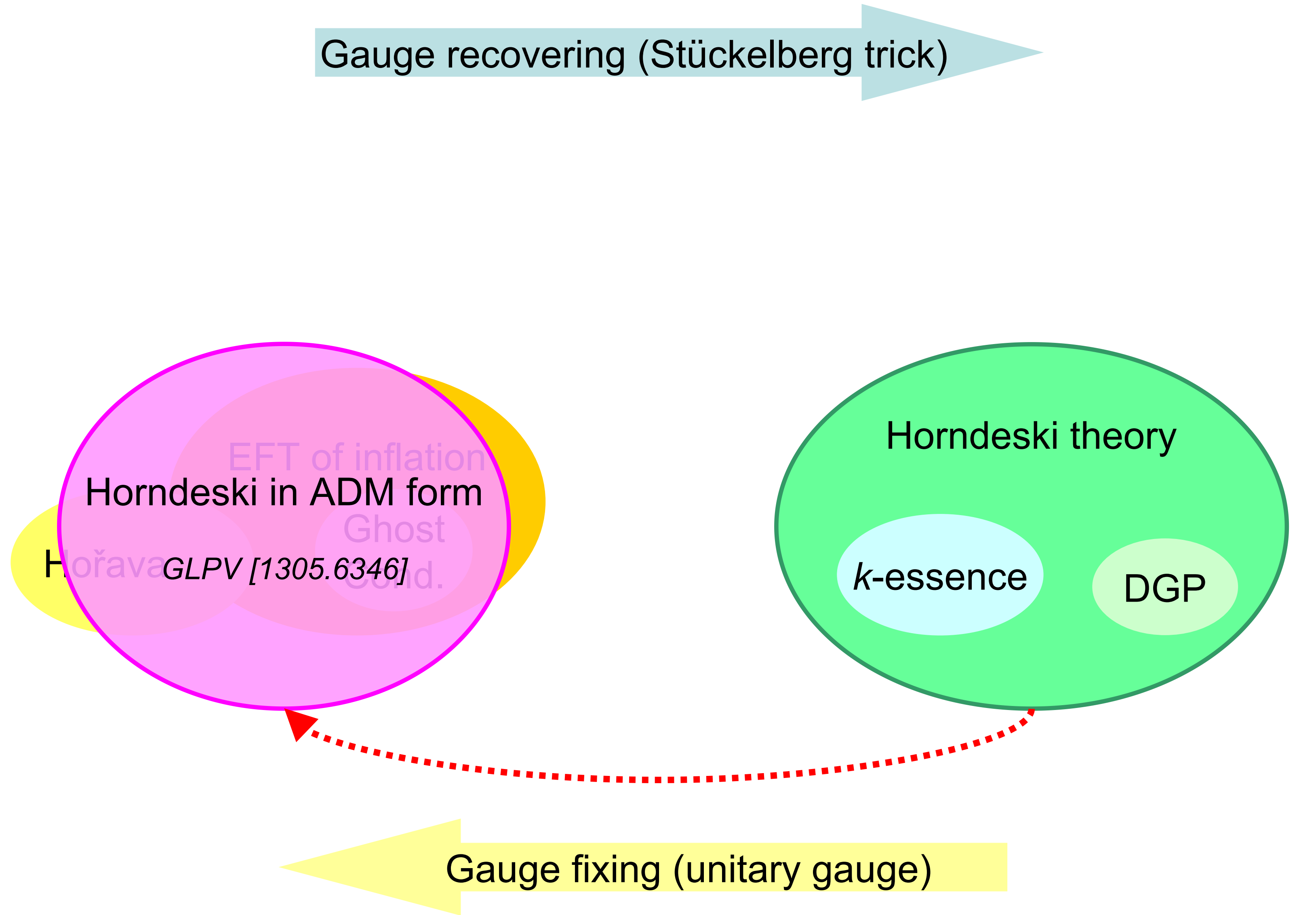

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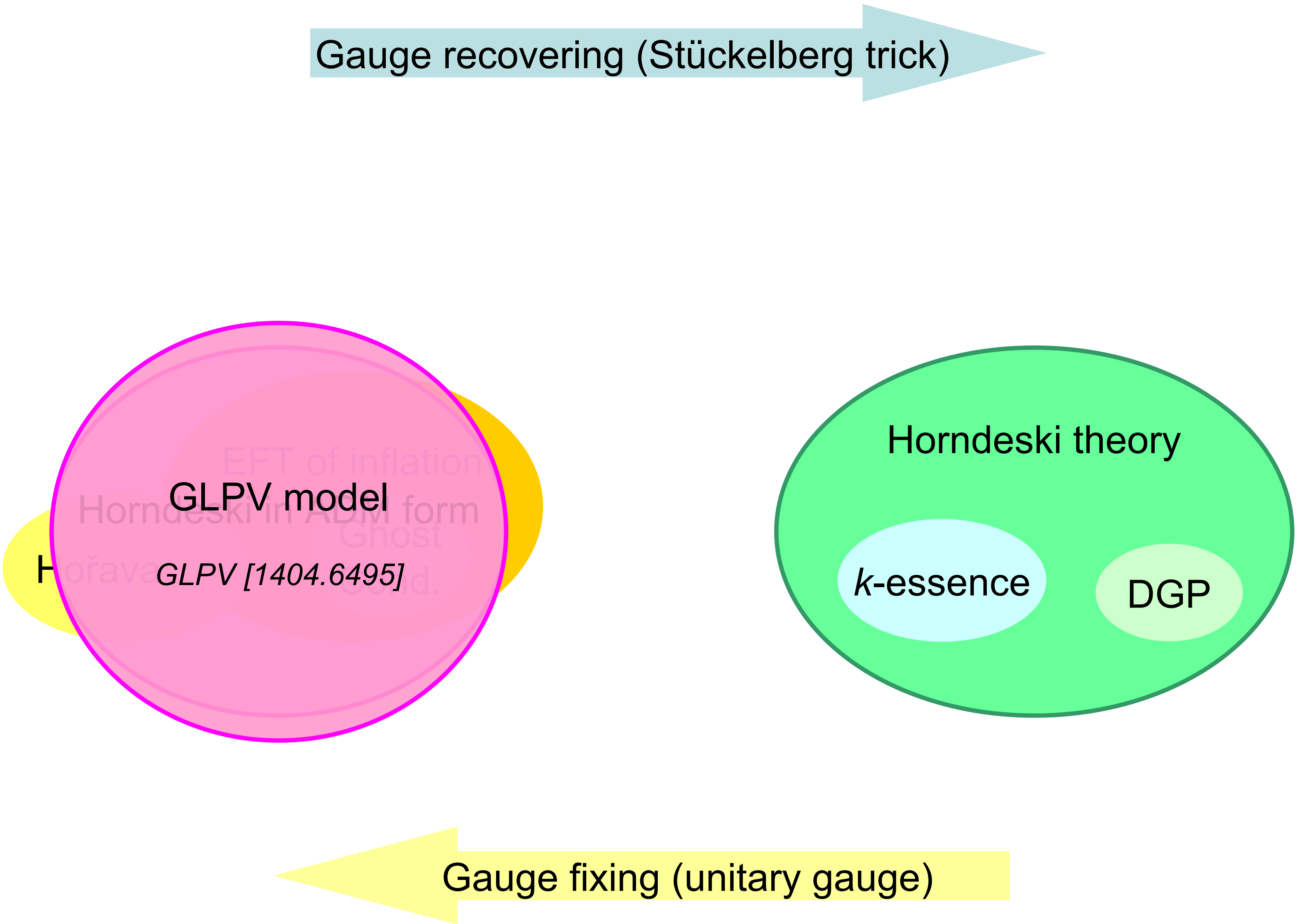

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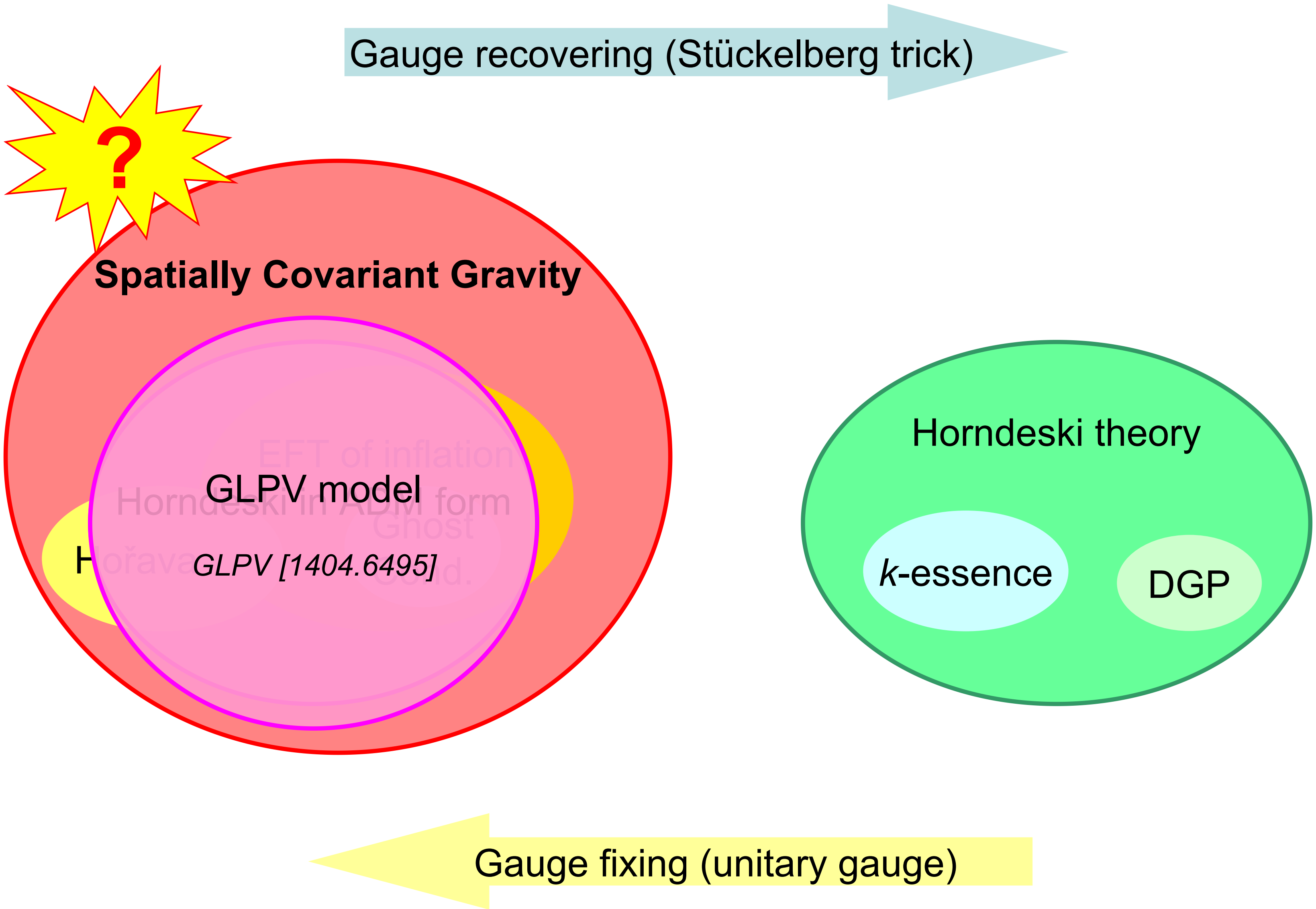

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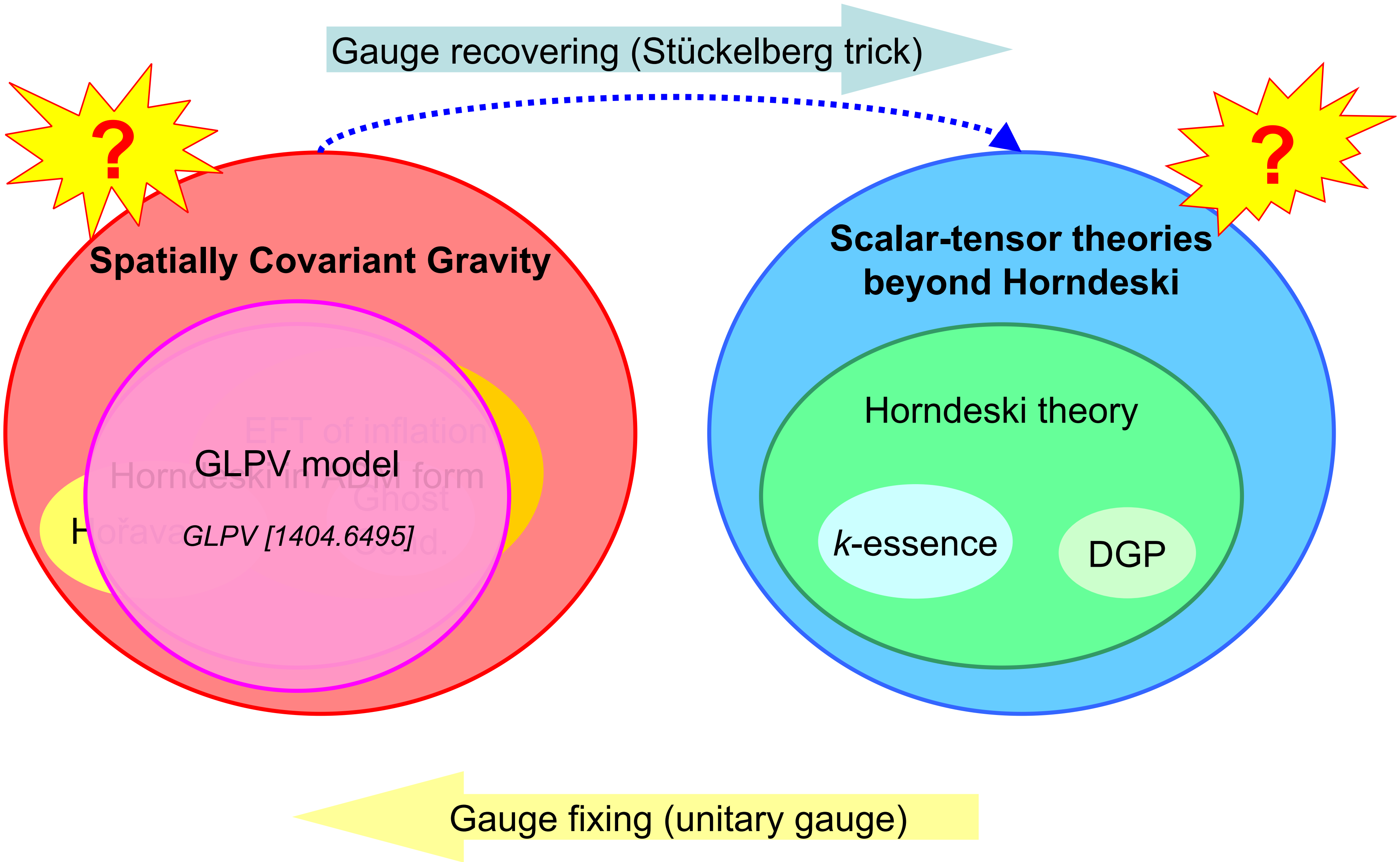

The general framework

A general class of Lagrangians that respects the spatial diffeomorphism:

$$\boxed{\sqrt{-g}\mathcal{L} = N\sqrt{h}\left(\sum_{n=1} \mathcal{G}_{(n)}^{i_1 j_1,\cdots,i_n j_n} K_{i_1 j_1}\cdots K_{i_n j_n} + \mathcal{V}\right)}$$

*[**XG** 1406.0822]*

where $\mathcal{V}, \mathcal{G}_{(n)}$'s are functions of

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"Translating" to the covariant language (Stueckelberg trick)

$$t \to \phi\left(t, \vec{x}\right), \qquad n_\mu \to -\frac{\nabla_\mu \phi}{\sqrt{-\left(\nabla\phi\right)^2}}$$

All terms can be written covariantly in terms of ϕ and its derivatives.

The general framework

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*[**XG** 1406.0822]*

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$$\left(t, N, {}^{(3)}h_{ij}, {}^{(3)}R_{ij}, \nabla_i\right)$$

"Translating" to the covariant language (Stueckelberg trick)

$$t \to \phi\left(t, \vec{x}\right), \qquad n_\mu \to -\frac{\nabla_\mu \phi}{\sqrt{-\left(\nabla\phi\right)^2}}$$

All terms can be written covariantly in terms of ϕ and its derivatives.

→ A more general class of scalar-tensor theory beyond the Horndeski theory, which propagates **2 tensor + 1 scalar** dofs, although the equations of motion are generally higher order.

Hamiltonian

A general class of Lagrangians that respects the spatial diffeomorphism:

$$\sqrt{-g}\mathcal{L} = N\sqrt{h}\left(\sum_{n=1} \mathcal{G}_{(n)}^{i_1 j_1,\cdots,i_n j_n} K_{i_1 j_1}\cdots K_{i_n j_n} + \mathcal{V}\right)$$

Conjugate momenta:

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$$

[Shinji's talk]

$$\pi^{ij} \equiv \frac{\partial\left(N\sqrt{h}\mathcal{L}\right)}{\partial \dot{h}_{ij}} = \frac{\sqrt{h}}{2}\left(\mathcal{G}_{(1)}^{ij} + 2\mathcal{G}_{(2)}^{ij,kl} K_{kl} + 3\mathcal{G}_{(3)}^{ij,k_1 l_1,k_2 l_2} K_{k_1 l_1} K_{k_2 l_2}\cdots\right)$$

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$$K_{ij} = \Gamma_{ij}^{(1)} + \frac{1}{\sqrt{h}}\Gamma_{ij,kl}^{(2)}\pi^{kl} + \frac{1}{h}\Gamma_{ij,k_1 l_1,k_2 l_2}^{(3)}\pi^{k_1 l_1}\pi^{k_2 l_2} + \cdots$$

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Canonical Hamiltonian:

$$\boxed{\mathcal{H}_{\mathrm{c}} \equiv \pi^{ij}\dot{h}_{ij} - N\sqrt{h}\mathcal{L} \simeq N\tilde{\mathcal{C}} + N_i\mathcal{C}^i}$$

$$\begin{aligned}\tilde{\mathcal{C}} &\equiv 2\pi^{ij}K_{ij} - \sqrt{h}\mathcal{L} = \sqrt{h}\left(C^{(0)} + \frac{1}{\sqrt{h}}C^{(1)}_{ij}\pi^{ij} + \frac{1}{h}C^{(2)}_{i_1j_1,i_2j_2}\pi^{i_1j_1}\pi^{i_2j_2} + \cdots\right)\\ &= \tilde{\mathcal{C}}\left(t, N, h_{ij}, R_{ij}, \nabla_i, \pi^{ij}\right),\\ \mathcal{C}^i &\equiv -2\sqrt{h}\nabla_j\left(\frac{\pi^{ij}}{\sqrt{h}}\right),\end{aligned}$$

Constraints

4 primary constraints:

$$\pi_N \equiv \frac{\partial\left(N\sqrt{h}\mathcal{L}\right)}{\partial \dot{N}} \approx 0, \qquad \pi_i \equiv \frac{\partial\left(N\sqrt{h}\mathcal{L}\right)}{\partial \dot{N}^i} \approx 0,$$

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Extended Hamiltonian: $H_{\mathrm{ex}} = \int d^3x \left(N\tilde{\mathcal{C}} + N_i\mathcal{C}^i + \lambda^N\pi_N + \lambda^i\pi_i\right)$

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N appears nonlinearly in the Hamiltonian, as the space-dependent time reparametrization invariance is broken.

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4 secondary constraints:

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_N = \{\pi_N, H_{\rm ex}\}_{\rm P} = -\mathcal{C}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\pi_i = \{\pi_i, H_{\rm ex}\}_{\rm P} = -\mathcal{C}_i$$

$$\mathcal{C} \equiv \tilde{\mathcal{C}} + \sqrt{h}\sum_{n=0}(-1)^n\nabla_{i_n}\cdots\nabla_{i_1}\left(\frac{N}{\sqrt{h}}\frac{\partial\tilde{\mathcal{C}}}{\partial\left(\nabla_{i_1}\cdots\nabla_{i_n}N\right)}\right)$$
$$= \mathcal{C}\left(t, N, h_{ij}, R_{ij}, \nabla_i, \pi^{ij}\right).$$

Degrees of freedom

Poisson brackets among all 8 constraints:

[Shinji's talk]

$\{\cdot,\cdot\}_{\mathrm{P}}$	π_N	π_j	$\mathcal{C}$	$\mathcal{C}_j$
π_N	0	0	$-\frac{\delta\mathcal{C}}{\delta N}$	0
π_i	0	0	0	0
$\mathcal{C}$	$\frac{\delta\mathcal{C}}{\delta N}$	0	0	$-\mathcal{E}_i$
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Eigenvalues: 6 zero, 2 non-zero:

$$\pm\sqrt{\left(\frac{\delta \mathcal{C}}{\delta N}\right)^2 + \mathcal{E}_i\mathcal{E}_i}$$

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[Shinji's talk]

$$
\begin{array}{c|cccc}
\{\cdot,\cdot\}_{\mathrm{P}} & \pi_N & \pi_j & \mathcal{C} & \mathcal{C}_j \\
\hline
\pi_N & 0 & 0 & -\frac{\delta \mathcal{C}}{\delta N} & 0 \\
\pi_i & 0 & 0 & 0 & 0 \\
\mathcal{C} & \frac{\delta \mathcal{C}}{\delta N} & 0 & 0 & -\mathcal{E}_i \\
\mathcal{C}_i & 0 & 0 & \mathcal{E}_i & 0
\end{array}
$$

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→ Among (linearly independent combinations of) 8 constraints:

6 are first class, 2 are second class

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→ Among (linearly independent combinations of) 8 constraints:
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→ Number of degrees of freedom:

$$\begin{aligned}\text{number of d.o.f.} &= \frac{1}{2}\big(2\times\text{number of canonical variables}-2\times\text{number of first class constraints}\\ &\quad -\text{number of second class constraints}\big)\\ &= \frac{1}{2}\left(2\times 10-2\times 6-2\right)=3.\end{aligned}$$

Linear perturbations

GLPV model contains very special combinations:

$$\begin{aligned}\mathcal{L}^{\mathrm{GLPV}} = & A_2(t,N) + B_4(t,N)\ {}^3R \\ & + \left(A_3(t,N)\, h^{ij} + B_5(t,N)\ {}^3G^{ij}\right) K_{ij} \\ & + A_4(t,N)\left(K^2 - K_{ij}K^{ij}\right) \\ & + A_5(t,N)\left(K^3 - 3KK_{ij}K^{ij} + 2K^i_jK^j_kK^k_i\right)\end{aligned}$$

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The equations of motion for linear perturbations stay at second order.

Within our general framework, we checked that at the cubic order in K_{ij}, the following combination also has this property:

$$\sim c(\phi, N)\left(3KK_{ij}K^{ij} - 5K^i_jk^j_kK^k_i\right)$$

*[**XG** 1406.0822]*

GLPV model is not unique nor that special.

Summary

• We propose a very large class of gravity theories, which respect to the spatial diffeomorphism.

• This class of gravity theories corresponds to single-field scalar-tensor theories, which generally possesses higher order equations of motion.

• How to prove the absence of ghost in a general gauge?
[Shinji's talk]

• Are all single-field scalar-tensor theory can be embedded in our formalism (e.g. f(R))?

• What is the most general spatially covariant gravity, with 3 d.o.f.s?

• Multi-field generalization?

Thank you for your attention!